

Eta forms and the odd index theorem[1]

Joint work with Frédéric Rochon
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Eta forms – what are they?

Odd index

Smoothing operators

Invertible perturbations

Delooping sequence

Chern and eta forms

Eta character

Universal eta transgression

Geometric Chern and eta forms

Geometric Chern forms

Eta form for an odd family

Index formula

Proof!

Extensions

Eta forms

- ▶ Talking about eta forms is easy, but they are rather like Leprechauns (it being St Patrick's Day). Many people have seen them but there is no general agreement as to what they are.
- ▶ Eta forms are regularizations of Chern forms or the Chern character
- ▶ They provide universal transgression forms from the Chern forms of even to odd (or of odd to even) K-theory
- ▶ I will illustrate this in the context of the delooping sequence and odd index theorem (of Atiyah and Singer)
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Odd index setup

- ▶ A fibration of compact manifolds

$$\begin{array}{ccc} Z & \text{---} & M \\ & & \downarrow \phi \\ & & B \end{array}$$

- ▶ A family of elliptic self-adjoint pseudodifferential operators

$$A \in \Psi^1(M/B; E), \quad A^* = A.$$

- ▶ Atiyah and Singer define (and compute) the odd index $\text{ind}(A) \in K^1(B)$ which is the full obstruction to the existence of a family of self-adjoint smoothing operators

$$p \in \Psi^{-\infty}(M/B; E), \quad p^* = p \text{ s.t. } (A + p)^{-1} \in \Psi^{-1}(M/B; E).$$

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Smoothing operators and odd K-theory

- ▶ For any compact manifold Z

$$G^{-\infty}(Z; E) = \{a \in \Psi^{-\infty}(Z; E) = \mathcal{C}^\infty(Z^2; \text{Hom}(E) \otimes \Omega_R); \\ (\text{Id} + a)^{-1} = \text{Id} + a', a' \in \Psi^{-\infty}(Z; E)\}$$

is a classifying group for odd K-theory

- ▶ For a compact fibration the homotopy classes of sections give the K-theory of the base $[B; G^{-\infty}(M/B; E)] = K^1(B)$.
- ▶ The corresponding flat loop group

$$G_{\text{SUS}}^{-\infty}(Z; E) = \{q \in \mathcal{S}(\mathbb{R}; \Psi^{-\infty}(Z; E)); \\ \text{Id} + q(t) \in G^{-\infty}(Z; E) \forall t \in \mathbb{R}\}$$

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Index bundle of a family

- ▶ For $b \in B$ set

$$\mathcal{A}_b = \{q \in \Psi_{\text{sus}}^{-\infty}(Z_b; E_b) = \mathcal{S}(\mathbb{R}; \Psi^{-\infty}(Z_b; E_b)); \\ (A_b + it + q(t))^{-1} \in \Psi^{-1}(Z_b; E_b) \forall t \in \mathbb{R}\}.$$

- ▶ It is a principal space for the action of $G_{\text{sus}}^{-\infty}(Z_b; E_b)$.
- ▶ These spaces form a ‘principal bundle’ (with varying structure groups)

$$G_{\text{sus}}^{-\infty}(M/B; E) \text{ — } \mathcal{A} \\ \downarrow \text{ } \exists \text{ section?} \\ B$$

which I will call the ‘index bundle’ since its non-triviality measures precisely the index.

Delooping sequence

- ▶ The ‘open flat loop group’

$$\begin{aligned} \tilde{G}_{\text{sus}}^{-\infty}(Z; E) = \\ \{a : \mathbb{R} \longrightarrow \Psi^{-\infty}(Z; E); \frac{da(t)}{dt} \in \mathcal{S}(\mathbb{R}; \Psi^{-\infty}(Z; E)), \\ \lim_{t \rightarrow -\infty} a(t) = 0, a(t) \in G^{-\infty}(Z; E) \forall t \in [-\infty, \infty]\} \end{aligned}$$

is contractible.

- ▶ The delooping sequence

$$G_{\text{sus}}^{-\infty}(Z; E) \longrightarrow \tilde{G}_{\text{sus}}^{-\infty}(Z; E) \xrightarrow{R_\infty} G^{-\infty}(Z; E)$$

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Chern forms

- ▶ The universal odd Chern character on $G^{-\infty}(Z; E)$, regarded as an open submanifold of $\Psi^{-\infty}(Z; E)$, is

$$\text{Ch}_{\text{odd}} = \frac{1}{2\pi i} \int_0^1 \text{Tr} \left((A^{-1}dA) \exp\left(\frac{1}{2\pi i} s(1-s)(A^{-1}dA)^2\right) \right), \quad A = \text{Id} + a.$$

- ▶ The universal even Chern character on $G_{\text{SUS}}^{-\infty}(Z; E)$ is

$$\text{Ch}_{\text{even}} = \int_{\mathbb{R}} p^* \text{Ch}_{\text{odd}}, \quad p: \mathbb{R} \times G_{\text{SUS}}^{-\infty}(Z; E) \longrightarrow G^{-\infty}(Z; E)$$

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Eta character

- ▶ The universal Chern forms induce the Chern character

$$K^*(Y) = [Y; G_{\text{SUS}}^{-\infty}(Z; E) \oplus G^{-\infty}(Z; E)] \ni [f] \\ \longmapsto [f^*(\text{Ch}_{\text{even}} \oplus \text{Ch}_{\text{odd}})] \in H^*(Y; \mathbb{C})$$

which is an isomorphism up to torsion.

- ▶ The universal (even) eta character on $\tilde{G}_{\text{SUS}}^{-\infty}(Z; E)$ is defined as for the even Chern character

$$\tilde{\eta}_{\text{even}} = \int_{\mathbb{R}} \tilde{p}^* \text{Ch}_{\text{odd}}, \quad \tilde{p} : \mathbb{R} \times \tilde{G}_{\text{SUS}}^{-\infty}(Z; E) \longrightarrow G^{-\infty}(Z; E).$$

- ▶ As opposed to the Chern forms $\tilde{\eta}$ is not closed; it is more like a connection form.

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Geometric Chern forms

- ▶ For the bundle of groups $G^{-\infty}(M/B; E)$ over B we need a connection for the fibration and bundle to write out explicit Chern forms

$$\text{Ch}_{\text{odd}} = \frac{1}{2\pi i} \int_0^1 \text{Tr} \left((L^{-1} \nabla L) e^{w(s)} \right) ds,$$

$$w(s) = (1-s)\omega + sL^{-1}\omega L + \frac{1}{2\pi i} s(1-s)(L^{-1}\nabla L)^2,$$

$$L = \text{Id} + q.$$

- ▶ These are closed forms on $G^{-\infty}(M/B; E)$ which pull-back under a section to represent the Chern class of that section.
- ▶ The even forms are similar.

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Eta form for A

- ▶ The total eta form form of the elliptic self-adjoint family $A \in \Psi^1(M/B; E)$ is the form on the total space of \mathcal{A} formally extending the even Chern form – now the connection acts on the original operator as well

$$\eta_A = \int_{\mathbb{R}} dt \frac{1}{2\pi i} \int_0^1 \text{Tr} \left((L^{-1} \nabla L) e^{w(s)} \right) ds,$$

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Index formula

► Theorem

The eta form is primitive for the action of sections, $s^\eta_A - \eta_A = d\alpha + \pi^* \text{Ch}_{\text{even}}(s)$, and the analogue of the transgression formula is that $d\eta_A$ is basic, i.e. descends to B , where it represents $\text{Ch}_{\text{odd}}(\text{ind}(A))$ in $H^{\text{odd}}(B; \mathbb{C})$.*



$$G_{\text{sus}}^{-\infty}(M/B; E) \longrightarrow \mathcal{A} \quad , \quad d\eta_A = \pi^*c, \quad c \in \mathcal{C}^\infty(B; \Lambda^{\text{odd}})$$

$$\downarrow \pi$$

$$B$$

where $[c] = \text{Ch}_{\text{odd}}(\text{ind}(A))$.

Proof of index formula

Diagramme chase!

$$\begin{array}{ccccc}
 G_{\text{sus}}^{-\infty}(M/B; E) & \longrightarrow & \tilde{G}_{\text{sus}}^{-\infty}(M/B; E) & \longrightarrow & G^{-\infty}(M/B; E) \\
 \downarrow & & \downarrow & \nearrow & \nearrow \\
 \mathcal{A} & \longrightarrow & \tilde{\mathcal{A}} & \nearrow & \\
 & \searrow & \downarrow & \nearrow & \\
 & & B & \nearrow & \\
 & & & \text{ind}(A) &
 \end{array}$$

The diagram shows a commutative diagram with nodes $G_{\text{sus}}^{-\infty}(M/B; E)$, $\tilde{G}_{\text{sus}}^{-\infty}(M/B; E)$, $G^{-\infty}(M/B; E)$, \mathcal{A} , $\tilde{\mathcal{A}}$, and B . Arrows indicate relationships between these objects, including a section \tilde{A} from $\tilde{\mathcal{A}}$ to B and an arrow labeled $\text{ind}(A)$ from $\tilde{\mathcal{A}}$ to $G^{-\infty}(M/B; E)$.

Here $\tilde{\mathcal{A}}$ is the extension of the index bundle to have contractible structure group, hence has a section \tilde{A} .

Proof of index formula

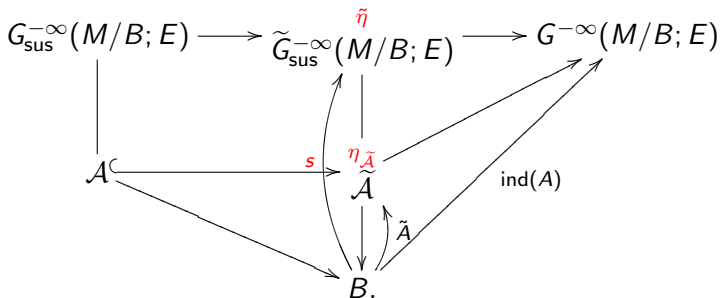
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$$\begin{array}{ccccc}
 G_{\text{sus}}^{-\infty} \overset{\text{Ch}_{\text{even}}}{(M/B; E)} & \longrightarrow & \tilde{G}_{\text{sus}}^{-\infty} \overset{\tilde{\eta}}{(M/B; E)} & \xrightarrow{R_{\infty}} & G^{-\infty} \overset{\text{Ch}_{\text{odd}}}{(M/B; E)} \\
 \downarrow & & \downarrow & \nearrow & \nearrow \\
 A^{\mathbb{C}} & \longrightarrow & \tilde{A} & & \\
 \downarrow & \searrow & \downarrow & \nearrow & \nearrow \\
 & & B & \nearrow & \\
 & & & \text{ind}(A) &
 \end{array}$$

We need to construct the geometric analogue of the Chern-eta sequence for the bundle case, $d\tilde{\eta} = R_{\infty}^* \text{Ch}_{\text{odd}}$.

Proof of index formula

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Then extend the geometric eta form to \tilde{A} and check that for the action of a section of the structure group $s^* \eta_{\tilde{A}} - \eta_{\tilde{A}} = d\beta + s^* \tilde{\eta}$.

Proof of index formula

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$$\begin{array}{ccccc}
 G_{\text{sus}}^{-\infty}(M/B; E) & \longrightarrow & \tilde{G}_{\text{sus}}^{-\infty}(M/B; E) & \longrightarrow & G^{-\infty}(M/B; E) \\
 \downarrow & & \downarrow & \nearrow & \text{Ch}_{\text{odd}} \\
 \mathcal{A}^c & \longrightarrow & \tilde{\mathcal{A}} \, d\eta_{\tilde{\mathcal{A}}} & & \\
 \searrow & & \downarrow & \nearrow & \\
 & & B & & \\
 & & \uparrow & \nearrow & \\
 & & \tilde{\mathcal{A}}^* & \text{ind}(A)^* &
 \end{array}$$

From this it follows that $d\tilde{\mathcal{A}}^*\eta_{\tilde{\mathcal{A}}} = d\eta_{\mathcal{A}} + \text{ind}(A)^* \text{Ch}_{\text{odd}}$ on B .

Extensions

- ▶ There is also a looping sequence which involves Bott periodicity.
- ▶ This looping sequence can be used to give a similar construction for the standard, even, case of the index theorem.
- ▶ We believe these constructions give an effective basis for a 'smooth K-theory' based on pseudodifferential operators on fibrations.

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Richard Melrose and Frédéric Rochon, *Eta forms and the odd pseudodifferential index*, In preparation, March 2008.