Problem 5.1. Let \( A \) be a self-adjoint (meaning \( A^* = A \)) compact operator on a separable Hilbert space \( H \). Recall from class that an eigenvalue of \( A \) is a complex number \( \lambda \) such that \( A - \lambda \text{Id} \) has non-trivial null space and that the eigenvalues of \( A \) (whether self-adjoint or not) form a discrete subset of \( \mathbb{C} \setminus \{0\} \) and that for each \( \lambda \) the space of associated generalized eigenvectors is finite dimensional.

1. Show that any eigenvalue of \( A \) is real.
2. Show that every generalized eigenvector, that is a solution of \( (A - \lambda \text{Id})^k u = 0 \) for some \( k \) and \( \lambda \neq 0 \), is actually an eigenvector. Hint:- Show that \( A \) acts on the generalized eigenspace \( E_\lambda \) corresponding to \( \lambda \) and is a self-adjoint matrix and then apply your knowledge of self-adjoint matrices.
3. Show that the non-zero eigenvalues of \( A^2 \) are positive and that \( t^2 > 0 \) is an eigenvalue of \( A^2 \) if and only if either \( t \) or \( -t \) is an eigenvalue of \( A \) and that the eigenspace of \( t^2 \) is the sum of the eigenspaces of \( A \) with eigenvalues \( \pm t \) (where the eigenspace of \( s \) is interpreted as \( \{0\} \) if \( s \) is not an eigenvalue).
4. Show that if \( A \) is not identically zero then \( A \) has an eigenvalue. Hint:- Look at the space of \( u \in H \) with \( \|u\| = 1 \) such that \( \|A^2 u\|^2 = \|A\|^2 \). Then choose a sequence \( u_n \) with \( \|u_n\| = 1 \) and \( \|A^2 u_n\| \to \|A\|^2 \). Show that \( u_n \) has a weakly convergent subsequence such that \( Au_{n_k} \) converges and check that the limit is in the desired space. Conclude that \( A \) has a non-zero eigenvalue.
5. Prove that the space \( N^\perp \), the orthocomplement in \( H \) of the null space of \( A \), has an orthonormal basis of eigenvectors of \( A \).

Problem 5.2. Let \( A \) be a self-adjoint Hilbert-Schmidt operator on a separable Hilbert space \( H \). Using the results of the previous problem, show that the non-zero eigenvalues \( \lambda_j \) of \( A \), repeated with the multiplicity (dimension of the associated eigenspace), are such that

\[
\sum_j \lambda_j^2 < \infty.
\]

Problem 5.3. Let \( T \) be a self-adjoint operator of trace class on a separable Hilbert space. Show that the eigenvalues, repeated with their multiplicities, satisfy

\[
\sum_j |\lambda_j| < \infty
\]

and that

\[
\text{Tr}(T) = \sum_j \lambda_j.
\]

Problem 5.4. Consider the operator on \( L^2(\mathbb{R}^n) \), depending on a parameter \( s > 0 \),

\[
 A_s : L^2(\mathbb{R}^n) \ni u \mapsto \mathcal{F}^{-1}(1 + |\xi|^2)^{-s/2} \hat{u} \in L^2(\mathbb{R}^n).
\]
(1) Show that if $s > n/2$ then $A_s$ can be written in the form

$$A_s u(x) = \int_{\mathbb{R}^n} K_s(x - y)u(y)dy, \quad K_s(z) \in L^2(\mathbb{R}^n).$$

(2) Show that, again for $s > n/2$, the operator on $L^2(B)$, with $B$ the unit ball in $\mathbb{R}^b$, given by

$$G_s u = \chi(A_s(\chi u)),$$

where $\chi$ is the characteristic function of $B$, is Hilbert-Schmidt.

**Problem 5.5.** Recall from class the operator $A$ which solves the Dirichlet problem in a bounded open domain $\Omega \subset \mathbb{R}^n$. It is the case that $A$ is compact and self-adjoint. Deduce that there is an orthonormal basis of $L^2(\Omega)$ composed of eigenvectors of the Dirichlet problem.