18.155, PROBLEM SET 5

Problem 5.1. Let A be a self-adjoint (meaning $A^* = A$) compact operator on a separable Hilbert space H. Recall from class that an eigenvalue of A is a complex number λ such that $A - \lambda\,\mathrm{Id}$ has non-trivial null space and that the eigenvalues of A (whether self-adjoint or not) form a discrete subset of $\mathbb{C} \setminus \{0\}$ and that for each λ the space of associated generalized eigenvectors is finite dimensional.

(1) Show that any eigenvalue of A is real.
(2) Show that every generalized eigenvector, that is a solution of $(A-\lambda\,\mathrm{Id})^k u = 0$ for some k and $\lambda \neq 0$, is actually an eigenvector. Hint:- Show that A acts on the generalized eigenspace E_λ corresponding to λ and is a self-adjoint matrix and then apply your knowledge of self-adjoint matrices.
(3) Show that the non-zero eigenvalues of A^2 are positive and that $t^2 > 0$ is an eigenvalue of A^2 if and only if either t or $-t$ is an eigenvalue of A and that the eigenspace of t^2 is the sum of the eigenspaces of A with eigenvalues $\pm t$ (where the eigenspace of s is interpreted as $\{0\}$ if s is not an eigenvalue).
(4) Show that if A is not identically zero then A has an eigenvalue. Hint:- Look at the space of $u \in H$ with $\|u\| = 1$ such that $\|A^2u\|^2 = \|A^2\|$. Then choose a sequence u_n with $\|u_n\| = 1$ and $\|A^2u_n\| \to \|A^2\|$. Show that u_n has a weakly convergent subsequence such that Au_{n_k} converges and check that the limit is in the desired space. Conclude that A has a non-zero eigenvalue.
(5) Prove that the space $N^\perp$, the orthocomplement in H of the null space of A, has an orthonormal basis of eigenvectors of A.

Problem 5.2. Let A be a self-adjoint Hilbert-Schmidt operator on a separable Hilbert space H. Using the results of the previous problem, show that the non-zero eigenvalues λ_j of A, repeated with the multiplicity (dimension of the associated eigenspace), are such that

$$\sum_j \lambda_j^2 < \infty. \tag{1}$$

Problem 5.3. Let T be a self-adjoint operator of trace class on a separable Hilbert space. Show that the eigenvalues, repeated with their multiplicities, satisfy

$$\sum_j |\lambda_j| < \infty \tag{2}$$

and that

$$\mathrm{Tr}(T) = \sum_j \lambda_j. \tag{3}$$

Problem 5.4. Consider the operator on $L^2(\mathbb{R}^n)$, depending on a parameter $s > 0$,

$$A_s : L^2(\mathbb{R}^n) \ni u \longrightarrow \mathcal{F}^{-1}(1 + |\xi|^2)^{-s/2}\hat{u} \in L^2(\mathbb{R}^n). \tag{4}$$

(1) Show that if $s > n/2$ then A_s can be written in the form

$$A_s u(x) = \int_{\mathbb{R}^n} K_s(x-y)u(y)dy,\ K_s(z) \in L^2(\mathbb{R}^n). \tag{5}$$

(2) Show that, again for $s > n/2$, the operator on $L^2(B)$, with B the unit ball in $\mathbb{R}^b$, given by

$$G_s u = \chi(A_s(\chi u)), \tag{6}$$

where χ is the characteristic function of B, is Hilbert-Schmidt.

Problem 5.5. Recall from class the operator A which solves the Dirichlet problem in a bounded open domain $\Omega \subset \mathbb{R}^n$. It is the case that A is compact and self-adjoint. Deduce that there is an orthonormal basis of $L^2(\Omega)$ composed of eigenvectors of the Dirichlet problem.