18.155, PROBLEM SET 4

Prove the Paley-Wiener-Schwartz theorem.

Namely, if $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ is a distribution of compact support then the Fourier transform is the restriction to $\mathbb{R}^n$ of an entire function F (on $\mathbb{C}^n$) where there exists $A > 0$ and N such that

$$\sup_{\zeta \in \mathbb{C}^n} |F(\zeta)| \exp(-A|\operatorname{Im}\zeta|)(1+|\zeta|)^{-N} < \infty \tag{1}$$

and conversely every such entire function is the Fourier transform of some $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$.

You may choose to prove this however you wish but here is a sequence of smaller steps which will get you there and you might choose to follow this route; of course you then have to check each step (you can of course use anything from the notes and from class).

(1) I denote by $\mathcal{C}_c^{-\infty}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ the subspace of distributions of compact support. Recall the definition of the support of a distribution and show that to say that $u \in \mathcal{S}'(\mathbb{R}^n)$ has compact support is to say that there exists $\phi \in \mathcal{C}_c^{\infty}(\mathbb{R}^n)$ such that $\phi u = u$ in $\mathcal{S}'(\mathbb{R}^n)$.

(2) If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ then for any complex number, and with $\phi \in \mathcal{C}_c^{\infty}(\mathbb{R}^n)$ as above show that

$$F(\zeta) = u(\phi e^{-i\zeta\cdot x}),\ \zeta \in \mathbb{C}^n$$

(where x is the variable in $\mathbb{R}^n$) is well-defined by the right-hand side. Show that it is a smooth function on $\mathbb{C}^n$ and that it is entire. This means that it is holomorphic in each of the z_j variables and that in turn means just that it satisfies each of the differential equations

$$\frac{\partial F}{\partial \xi_j} + i\frac{\partial F}{\partial \eta_j} = 0 \text{ on } \mathbb{C}^n,\ \zeta_j = \xi_j + i\eta_j. \tag{2}$$

(3) Show that $F(\xi)$, that is the restricition of F to the real subspace $\mathbb{R}^n$, is the Fourier transform of u.

(4) Using the fact that $\phi u = u$ again and that $u \in H^{-M}(\mathbb{R}^n)$ for some $M \in \mathbb{N}$, by Schwartz representation theorem, show that

$$|F(\zeta)| \le C\|v_\zeta\|_{H^M} \le C \sum_{|\alpha|\le N} \sup_{|x|\le A} |D^\alpha v_\zeta(x)|,\ v_\zeta(x) = \phi(x)e^{-i\zeta\cdot x}. \tag{3}$$

(5) From this deduce (1).

(6) Just assuming that $F : \mathbb{C}^n \longrightarrow \mathbb{C}$ is a continuous function satisfying (1) and the equations (2) (this is what I mean by entire) show that it is a smooth function on $\mathbb{C}^n = \mathbb{R}^{2n}$ – In essence I did this in class. Work out the composite of the operator in (2) and its adjoint for each j and show that $F(z)$ is harmonic on $\mathbb{R}^{2n}$. Using results on elliptic operators from class deduce that it is smooth.

(7) Show that for each fixed $\eta \in \mathbb{R}^n$ $F(\xi + i\eta) \in \mathcal{S}'(\mathbb{R}^n_\xi)$.

(8) Define $u_\eta(x) \in \mathcal{S}'(\mathbb{R}^n)$, for each $\eta \in \mathbb{R}^n$ as the inverse Fourier transform of $F(\xi + i\eta)$.

(9) Show that

$$u_\eta(\psi) = u_0(e^{-x\cdot\eta}\psi). \tag{4}$$

(10) Show that if $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then for any $\eta \in \mathbb{R}^n$

$$u_0(\psi) = (2\pi)^{-n} \int F(\xi + i\eta)\widehat{\psi}(-\xi - i\eta)d\xi. \tag{5}$$

(11) (I will do this in class on Thursday). Show that if $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is supported in $x_j > R$ then its Fourier-Laplace transform (which is $F_\psi = \widehat{\psi}$) satisfies, for any N,

$$|\psi(\xi + \eta_j e_j)| \le C(1 + |\xi|)^{-N} \exp(-R\eta_j) \text{ in } \eta_j > 0. \tag{6}$$

Using the estimates on F and these estimates on $\widehat{\psi}$ show that if ϕ has support in any one of the regions $|x_j| > R$ for some fixed R (depending on the estimates on F) then the right side of (5) vanishes under some limit in η. [I am being intentionally vague here to leave the last step to you.]

(12) Conclude from this that the support of $u = u_0$ is compact and that $F(\zeta)$ is its Fourier-Laplace transform as in the first part.