18.103, SPRING 2004
PRACTICE FINAL

This exam is closed book, no books, papers or recording devices permitted. You may use theorems from class, or the book, provided you can recall them correctly.

Problem 1

Suppose $f \in L^1([0,1])$ and $\int_{[0,1]} f\chi dx = 0$ for all simple measurable functions χ on $[0,1]$. Show thatt $f = 0$ almost everywhere with respect to Lebesgue measure.

Problem 2

Suppose A is a compact operator on a Hilbert space H, and that A^*A has no positive eigenvalues, show that $A = 0$.

Problem 3

Give an example of a function $u \in L^2(\mathbb{R})$ which is continuous but is such that its Fourier transform $\hat{u} \notin L^1(\mathbb{R})$.

Problem 4

Suppose $u \in L^2([-\pi,\pi])$ and there exists $v \in L^2([-\pi,\pi])$ such that

$$\int_{[-\pi,\pi]} u(x)\frac{d}{dx}\phi(x)dx = \int_{[-\pi,pi]} v(x)\phi(x)$$

for all smooth 2π-periodic functions, ϕ, on the real line. Show that u has a continuous representative in $L^2([-\pi,\pi])$.

Problem 5

Suppose $f \in \mathcal{S}(\mathbb{R})$ has Fourier transform satisfying $\hat{f}(\xi) = 0$ in $|\xi| < 1$. Show that there exists $g \in \mathcal{S}(\mathbb{R})$ such that $f(x) = \frac{d^2}{dx^2}g(x)$ for all $x \in \mathbb{R}$.

Problem 6

Show that there is no element of $L^1([-\pi,\pi])$ satisfying

$$\int_{[-\pi,\pi]} f(x)e^{ik^3x} = 1 \;\forall\; k \in \mathbb{N}.$$

Problem 7

Suppose $f \in L^2([-\pi,\pi]))$ had Fourier coefficients c_j, $j \in \mathbb{Z}$ satisfying

$$\sum k \in \mathbb{Z}k|c_k| < \infty.$$

Show that there exists a function $g \in L^2([-\pi,\pi])$ such that

$$\int_{[-\pi,pi]} g(x)\phi(x)dx = \int_{[-\pi,pi]} f(x)\left(\frac{d}{dx}\phi(x) + \cos(x)\phi(x)\right)dx$$

for all smooth 2π-periodic functions ϕ on the real line.

Problem 8

If $f \in \mathcal{C}([0,1])$, show that

$$(Au)(x) = \int_x^1 f(t)u(t)dt,\ \ x \in [0,1], \tag{1}$$

defines a compact operator $A : L^2([0,1]) \longrightarrow L^2([0,1])$.

Problem 9

Show that there is an infinite orthonormal sequence $u_j \in L^2(\mathbb{R})$ with each element satifying

$$\widehat{u_j} = c_j u_j,\ \ c_j \in \mathbb{C}.$$