TRACES, SIGNATURES AND INDEX THEOREMS COLLOQUIUM AT PENN. STATE, OCTOBER 24, 1996

RICHARD MELROSE

ABSTRACT. Starting from a discussion of the trace, signature and determinant of a matrix I will consider 'infinite dimensional' generalizations, with pseudodifferential operators on compact manifolds playing the role of infinite matrices. In particular I will describe joint work with Victor Nistor which proceeds from the realization of the index as a pairing in Hochschild homology and gives extensions to manifolds with boundary involving the eta invariant.

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1. Invertibility *vs* ellipticity

General questions. For $\Psi^{\mathbb{Z}}$ a filtered algebra when is an element filtered-invertible

$$A \in \Psi^m, \ \exists\ ?\ A^{-1} \in \Psi^{-m}.$$

Symbolic quotient. Under the quotient by the residual ideal

$$\Psi^{-\infty} = \bigcap_{m \in \mathbb{Z}} \Psi^m$$

$A \in \Psi^m$ filtered-invertible maps to a filtered-invertible $a = \sigma(A)$

$$0 \hookrightarrow \Psi^{-\infty} \xrightarrow{i} \Psi^{\mathbb{Z}} \xrightarrow{\sigma} \mathcal{A}^{\mathbb{Z}} \longrightarrow 0.$$

Ellipticity. We can say that $A \in \Psi^{\mathbb{Z}}$ is *elliptic* if $\sigma(A)$ is filtered-invertible; this is often easily determined.

Refined questions. Given an invertible $a \in \mathcal{A}^m$ when does there exists an invertible $A \in \Psi^m$ with $\sigma(A) = a$. What is the topology of this preimage?

Index. An obstruction to the invertibility of an elliptic element can reasonably be called an 'index'. Such obstructions can be found in K-theory, *Hochschild* or cylclic homology.

2. Hochschild homology

Hochschild complex of $\mathcal{B}$. Chains and differentials

$$\mathcal{C}_i(\mathcal{B}) = \mathcal{B}^{\hat{\otimes}(i+1)},\ b : \mathcal{C}_i(\mathcal{B}) \longrightarrow \mathcal{C}_{i-1}(\mathcal{B})$$
$$b(\beta_0 \otimes \beta_1 \otimes \cdots \otimes \beta_i) = b'(\beta_0 \otimes \beta_1 \otimes \cdots \otimes \beta_i) + (-1)^i \beta_0\beta_i \otimes \beta_1 \otimes \cdots \otimes \beta_{i-1}$$

where

$$b'(\beta_0 \otimes \beta_1 \otimes \cdots \otimes \beta_i) = \sum_{l=0}^{i-1} (-1)^l \beta_0 \otimes \cdots \otimes \beta_{l-1} \otimes \beta_{l+1}\beta_l \otimes \dots .$$

Then

$$b^2 = 0 = (b')^2.$$

One cycles. In particular

$$b(\beta_0 \otimes \beta_1) = \beta_1\beta_0 - \beta_0\beta_1 = [\beta_1, \beta_0].$$

Thus if $[\beta_1, \beta_0] = 0$ then $\beta_0 \otimes \beta_1$ is a 1-cycle. If β is invertible then $\beta^{-1} \otimes \beta$ is a 1-cycle.

Hochschild groups. $\mathrm{HH}_k(\mathcal{B})$ is the homology of

$$0 \xleftarrow{b} \mathcal{C}_0(\mathcal{B}) \xleftarrow{b} \mathcal{C}_1(\mathcal{B}) \xleftarrow{b} \mathcal{C}_2(\mathcal{B}) \dots .$$

Hochschild cohomology. Has chains spaces $\mathcal{C}^i(\mathcal{B})$ consisting of continuous $(i+1)$-multilinear functionals

$$F : \mathcal{B} \times \mathcal{B} \cdots \times \mathcal{B} \longrightarrow \mathbb{C}$$

and dual differential. The cohomology space $\mathrm{HH}^i(\mathcal{B})$ is dual to $\mathrm{HH}_i(\mathcal{B})$.

3. Hochschild index

H-unitality. If $\mathcal{B}$ is unital, b' is acyclic. In general an algebra is H-unital if b' is acyclic.

Boundary map.

Proposition (Wodzicki)**.** *If $\Psi^{-\infty}$ is H-unital then there is a long exact sequence*

$$0 \longleftarrow \mathrm{HH}_0(\mathcal{A}^{\mathbb{Z}}) \xleftarrow{\sigma_*} \mathrm{HH}_0(\Psi^{\mathbb{Z}}) \xleftarrow{i_*} \mathrm{HH}_0(\Psi^{-\infty})$$

$$\xleftarrow{\partial} \mathrm{HH}_1(\mathcal{A}^{\mathbb{Z}}) \xleftarrow{\sigma_*} \mathrm{HH}_1(\Psi^{\mathbb{Z}}) \xleftarrow{i_*} \cdots$$

Index in Hochschild homology. So, suppose $\Psi^{-\infty}$ is H-unital and $A \in \Psi^{\mathbb{Z}}$ is elliptic, so $a = \sigma(A) \in \mathcal{A}^{\mathbb{Z}}$ is filtered-invertible then

$$\operatorname{ind}(A) = \operatorname{ind}(a) = \partial([a \otimes a^{-1}]) \in \mathrm{HH}_0(\Psi^{-\infty})$$

is the 'index'. If $\operatorname{ind}(A) \neq 0$ then A is not even stably invertible in the sense that $A + E$ is never invertible for $E \in \Psi^{-\infty}$.

4. Basic example

Here $\Psi^{\mathbb{Z}} = \Psi^{\mathbb{Z}}(X)$ is the algebra of pseudodifferential operators on a compact manifold without boundary; $\Psi^{-\infty}(X)$ is the ideal of smoothing operators and $\mathcal{A}^{\mathbb{Z}} = \mathcal{A}^{\mathbb{Z}}(X)$ is the 'full symbol algebra'.

Symbol algebra. A global quantization map identifies

$$\mathcal{A}^m \ni a \longmapsto (a_{m-j})_{j=0}^{\infty},$$
$$a_k \in \mathcal{C}^\infty(T^*X \setminus 0) \text{ homogeneous of degree } k.$$

The product is a (the proto-typical) 'star product'

$$a \star b = \sum_j P_j(a,b),\ \ P_0(a,b) = ab$$

with P_j bilinear differential and homogeneous of degree j.

Misleading notation. I am really considering matrix, or bundle-valued operators, but I will not acknowledge this.

Smoothing ideal. In this case the residual ideal is homologous to a matrix algebra

$$\mathrm{HH}_k(\Psi^{-\infty}(X)) = \begin{cases} \mathbb{C} & \text{if } k = 0 \\ \{0\} & \text{otherwise.} \end{cases}$$

Trace functional. The trace

$$\mathrm{Tr}(E) = \int E(x,x),\ \ \mathrm{Tr} : \Psi^{-\infty}(X) \longrightarrow \mathbb{C}$$

vanishes on commutators and generates $\mathrm{HH}^0(\Psi^{-\infty}(X))$.

5. Index

In this case the index becomes a number

$$\operatorname{ind}(A)=\operatorname{ind}(a)=\operatorname{Tr}(\partial[a^{-1}\otimes a]).$$

Index functional. Using the dual long exact sequence

$$\begin{tikzcd} & \mathbb{C} \arrow[d, equal] & \mathbb{C} \arrow[d, equal] & \mathbb{C} \arrow[d, equal] \\ 0 \arrow[r] & \mathrm{HH}^0(\mathcal{A}^{\mathbb{Z}}) \arrow[r, "\sigma_*"] & \mathrm{HH}^0(\Psi^{\mathbb{Z}}) \arrow[r, "i_*=0"] & \mathrm{HH}^0(\Psi^{-\infty}) \end{tikzcd}$$

$$\xrightarrow{\partial(\mathrm{Tr})=\mathrm{IF}} \mathrm{HH}^1(\mathcal{A}^{\mathbb{Z}}) \xrightarrow{\sigma_*} \mathrm{HH}^1(\Psi^{\mathbb{Z}}) \longrightarrow 0$$

the index can be rewritten as a pairing in Hochschild 1-homology

$$\operatorname{ind}(A)=\mathrm{IF}(a^{-1}\otimes a),\ \mathrm{IF}=\partial\operatorname{Tr}.$$

Analytic index. If $A\in\Psi^m(X)$ is elliptic then it defines a Fredholm operator on $\mathcal{C}^\infty(X)$ and its analytic index is by definition

$$\operatorname{ind}_{\mathrm{a}}(A)=\dim(A)-\operatorname{codim}(A\mathcal{C}^\infty(X)).$$

(This is alwas zero if $\dim X\ge 2$ which is why I should be talking about matrix-valued operators).

Analytic=topological index. That $\operatorname{ind}_{\mathrm{a}}(A)=\operatorname{ind}(A)$ follows from the definition of the boundary map. Namely if $B\in\Psi^{-m}(X)$ is a parametrix, $\sigma(B)=\sigma(A)^{-1}$ then $[A,B]\in\Psi^{-\infty}(X)$ and

$$\operatorname{ind}(A)=\operatorname{Tr}([A,B])=\operatorname{ind}_{\mathrm{a}}(A).$$

6. Index functional

Hochschild homology of $\mathcal{A}^{\mathbb{Z}}(X)$. Wodzicki computed this, then Brylinski and Getzler gave a clear discussion and generalization. The main result is

$$\mathrm{HH}_k(\mathcal{A}^{\mathbb{Z}}(X)) \simeq H^{2n-k}(\mathbb{S}^*X \times \mathbb{S}^1_\theta).$$

Here $\mathbb{S}^*X = (T^*X \setminus 0)/\mathbb{R}^+$ is the cosphere bundle of X.

Wodzicki's residue trace. Is a generator $\mathrm{Tr}_{\mathrm{R}} \in \mathrm{HH}^0(\mathcal{A}^{\mathbb{Z}}(X))$ which acts on $\mathrm{HH}_0(\mathcal{A}^{\mathbb{Z}}(X))$ by integration over $\mathbb{S}^*X \times \mathbb{S}$.

Derivations. A derivation D on an algebra $\mathcal{B}$ induces an operation on Hochschild homology, at the chain level given by

$$D(\beta_0 \otimes \beta_1 \otimes \cdots \otimes \beta_i) = D(\beta_0)\beta_1 \otimes \cdots \otimes \beta_i.$$

An interior derivation acts trivially but $\Psi^{\mathbb{Z}}(X)$ has an exterior derivation defined from any positive elliptic invertible element of order 1, for example

$$Q = (\Delta + 1)^{\frac{1}{2}},$$

by

$$D_Q(A) = \frac{d}{dz}(Q^z A Q^{-z})\Big|_{z=0} = [\log Q, A].$$

Radial derivation. The action on homology of D_Q is independent of the choice of Q and acts as contraction with ∂_θ on $\mathbb{S}^1_\theta$. Furthermore

$$\mathrm{IF} = \partial\,\mathrm{Tr} = D_Q\,\mathrm{Tr}_{\mathrm{R}}\,.$$

7. Atiyah-Singer formula (in part)

With the index functional identified in its action on $\mathrm{HH}_1(\mathcal{A}^{\mathbb{Z}}(X))$ as

$$\begin{aligned}\mathrm{IF} : \mathrm{HH}_1(\mathcal{A}^{\mathbb{Z}}(X)) &\longrightarrow H^{2n-1}(\mathbb{S}^*X \times \mathbb{S}^1_\theta)\\ &\longrightarrow H^{2n-1}(\mathbb{S}^*X) \xrightarrow{\int_{\mathbb{S}^*X}} \mathbb{C}.\end{aligned}$$

the 'shape' of the Atiyah-Singer formula follows:

$$\operatorname{ind}(A) = \int_{\mathbb{S}^*X} \mathrm{AS}(a)$$

where $\mathrm{AS}(a)$ is a form on $\mathbb{S}^*X$ determined by $a = \sigma(A)$ as the image of $[a^{-1} \otimes a]$ in $H^{2n-1}(\mathbb{S}^*X)$. To find it 'explicitly' still requires a detour through K-theory.

Trace-derivation formula. I shall not do this, rather I want to consider generalizations of the formula for the index in Hochschild homology obtained above

$$\boxed{\operatorname{ind}(a) = \operatorname{Tr}_{\mathrm{R}}(a^{-1}D(a)).}$$

8. Determinant and signature

To save energy I will regard E as either a matrix of an element of $\Psi^{-\infty}(X)$ (a 'big' matrix). The trace on matrices is

$$\operatorname{tr}(E) = \sum_i E_{ii},\ \operatorname{Tr}(E) = \int_X E(x,x).$$

Determinant. The determinant of a matrix satisfies the variational formula

$$\frac{d}{ds}\det(E(s)) = \det(E(s))\operatorname{tr}\left(E(s)^{-1}\frac{d}{ds}E(s)\right).$$

As a formula for the variation of $\log\det(E(s))$ this is of the trace-derivation type. This formula can be used to *define* the Fredholm determinant on the algebra $\operatorname{Id}+\Psi^{-\infty}(X)$.

Signature. If E is a self-adjoint matrix its signature is

$$\begin{aligned}\operatorname{sgn}(E) = \#\{&\text{Positive eigenvalues}\}\\ &- \#\{\text{Negative eigenvalues}\}\end{aligned}$$

counted with multiplicity. This can also be expressed in terms of the determinant by

$$\operatorname{sgn}(E) = \frac{i}{\pi}\lim_{T\to\infty}\int_{-T}^{T}\frac{d}{d\tau}\log\det(iE-\tau)d\tau.$$

9. Second example: Suspended matrix algebra

Consider the algebra

$$\Psi_{\text{sus}} = \Big\{ u : \mathbb{R} \longrightarrow M_N(\mathbb{C}); \mathcal{C}^\infty \text{ with expansions} \\ u(\tau) \simeq \sum_{j=0}^{\infty} u_j^{\pm} \tau^{m-j},\ \tau \to \pm\infty \Big\}.$$

Here, m is the order.

Ellipticity and Invertibility. The ellipticity of u, of order m is the invertibility of $\sigma(u) \in \mathcal{E}^m_{\text{sus}}$, just the invertibility of $u^{\pm}_m$.

$$0 \longrightarrow \Psi^{-\infty}_{\text{sus}} \xrightarrow{i} \Psi^{\mathbb{Z}}_{\text{sus}} \xrightarrow{\sigma} \mathcal{E}^{\mathbb{Z}}_{\text{sus}} \longrightarrow 0.$$

Then u is filtered-invertible if $u(t)$ is invertible for all t.

Lemma. *The filtered invertible set is open and dense in the elliptic set for each m.*

Thus there is *NO INDEX.*

Average trace. The functional

$$\overline{\text{Tr}}(A) = \frac{1}{2\pi} \operatorname*{LIM}_{T\to\infty} \int_{-T}^{T} \operatorname{tr}\big(A(\tau)\big) d\tau$$

(where $\operatorname*{LIM}_{T\to\infty}$ means take the coefficient of T^0 in the expansion) vanishes on commutators and so defines an element of $\text{HH}^0(\Psi_{\text{sus}})$.

10. η INVARIANT

Generalizing the trace-derivation formula for the signature set

$$\eta(A) = 2i\,\overline{\mathrm{Tr}}(A^{-1}\frac{d}{d\tau}A)$$

for any element $A \in \Psi^m_{\mathrm{sus}}$ with inverse in Ψ^{-m}_{sus}. Thus, if $A(\tau) = iE - \tau$ then $\eta(A) = \mathrm{sgn}(E)$. In general

$$\eta(AB) = \eta(A) + \eta(B).$$

Residue trace. This algebra also has an analogue of the residue trace,

$$\mathrm{Tr}_{\mathrm{R}}(u) = \frac{1}{2\pi}\big(\,\mathrm{tr}(u_0^+) - \mathrm{tr}(u_0^-)\big).$$

Variation. Unlike the signature, $\eta(u)$ is not homotopy invariant. In fact

$$\frac{d}{ds}\eta(u_s) = 2i\,\mathrm{Tr}_{\mathrm{R}}(u_s^{-1}, \frac{d}{ds}u_s).$$

The important point to note here is that the right side is defined if u_s is simply elliptic.

Divisor flow. If $[0,1] \longmapsto u_s$ is a curve of elliptic elements of some fixed order with invertible endpoints then

$$\begin{aligned}\mathrm{DF}(u_1, u_0) = \frac{1}{2}\big(\eta(u_1) - \frac{1}{2}\eta(u_0)\big)& \\ - i\int_0^1 \mathrm{Tr}_{\mathrm{R}}\,\big(u_s^{-1}, \frac{d}{ds}u_s\big)ds&\end{aligned}$$

is an integer which is zero if and only if u_0 and u_1 are in the same component of the elliptic set.

11. Suspended pseudodifferential algebra

Again with X a compact $\mathcal{C}^\infty$ manifold without boundary we combine these two examples and examine the filtered algebra

$$\Psi^{\mathbb{Z}}_{\mathrm{sus}}(X) \subset \Psi^{\mathbb{Z}}(\mathbb{R} \times X).$$

This consists of those translation-invariant elements

$$A : \mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R} \times X) \longrightarrow \mathcal{C}^\infty(\mathbb{R} \times X),\ T_t A = A T_t.$$

with convolution kernels rapidly decreasing at infinity. It is an algebra of operators on Schwartz functions

$$A : \mathcal{S}(\mathbb{R} \times X) \longrightarrow \mathcal{S}(\mathbb{R} \times X).$$

Trivial case. If X is a point this reduces to the previous case after Fourier transform. The symbolic quotient is similar

$$0 \longrightarrow \Psi^{-\infty}_{\mathrm{sus}}(X) \xrightarrow{\ i\ } \Psi^{\mathbb{Z}}_{\mathrm{sus}}(X) \xrightarrow{\ \sigma\ } \mathcal{A}^{\mathbb{Z}}_{\mathrm{sus}}(X) \longrightarrow 0.$$

The set of filtered-invertible elements is open and dense in the set of elliptic elements, so *NO INDEX*.

Example. If ð is a first oder self-adjoint elliptic differential operator on X (for instance the Dirac operator on an odd-dimensional spin manifold) then ð $+ \frac{\partial}{\partial t} \in \Psi^1_{\mathrm{sus}}(X)$ is elliptic and it is invertible with ð.

12. Traces and eta

Both the traces Tr_{R} on $\mathcal{A}^{\mathbb{Z}}_{\mathrm{sus}}(X)$ and $\overline{\mathrm{Tr}}$ on $\Psi^{\mathbb{Z}}_{\mathrm{sus}}(X)$ exist as before.

eta. The same definition

$$\eta(A) = 2\,\overline{\mathrm{Tr}}(A^{-1}[t, A])$$

gives a multiplicative functional generalizing the eta invariant of Atiyah, Patodi and Singer

$$\eta(AB) = \eta(A) + \eta(B),$$
$$\frac{d}{ds}\eta(A_s) = 2i\,\mathrm{Tr}_{\mathrm{R}}(A_s^{-1}, \frac{d}{ds}A_s).$$

Divisor flow. Hence a divisor flow on invertible elements in each component of the elliptic set can be defined by

$$\mathrm{DF}(A_1, A_0) = \frac{1}{2}\big(\eta(A_1) - \frac{1}{2}\eta(A_0)\big) \\ - i\int_0^1 \mathrm{Tr}_{\mathrm{R}}\left(A_s^{-1}, \frac{d}{ds}A_s\right)ds.$$

It is again an integer vanishing precisely when the elements are in the same component.

Dirac operators. If $\eth$ is an (admissible) Dirac operator on an odd-dimensional manifold then

$$\eta(\eth + \frac{\partial}{\partial t}) = \eta(\eth)$$

where on the right is the APS eta invariant:

$$\eta(\eth) = \frac{1}{\sqrt{\pi}}\int_0^\infty s^{-\frac{1}{2}}\,\mathrm{Tr}(\eth e^{-s\eth^2})ds.$$

13. Cusp calculus

If X is a compact manifold with boundary there an algebra $\Psi^{\mathbb{Z}}_{\mathrm{c}}(X)$ generalizating that in the boundaryless case; it was discussed initially by Livingston and more completely by Mazzeo and myself. It is an algebra of 'asymptotically normally translation-invariant' operators on $\mathcal{C}^\infty(X)$. There is a symbol map much as before

$$0 \hookrightarrow \Psi^{-\infty}_{\mathrm{c}}(X) \xrightarrow{i} \Psi^{\mathbb{Z}}_{\mathrm{c}}(X) \xrightarrow{\sigma} \mathcal{A}^{\mathbb{Z}}_{\mathrm{c}}(X) \longrightarrow 0.$$

However this does not capture the Fredholm property.

Normal operator. In fact there is a second, non-commutative, morphism

$$0 \hookrightarrow x^\infty\Psi^{\mathbb{Z}}_{\mathrm{c}}(X) \xrightarrow{i} \Psi^{\mathbb{Z}}_{\mathrm{c}}(X) \xrightarrow{\mathrm{In}} \Psi^{\mathbb{Z}}_{\mathrm{sus}}(\partial X)[x] \longrightarrow 0.$$

Here $x \in \mathcal{C}^\infty(X)$ defines the boundary.

Joint symbol. Together these give a 'joint symbol' which does capture the Fredholm property:

$$0 \hookrightarrow x^\infty\Psi^{-\infty}_{\mathrm{c}}(X) \xrightarrow{i} \Psi^{\mathbb{Z}}_{\mathrm{c}}(X) \xrightarrow{j} \mathcal{J}^{\mathbb{Z}}(X) \longrightarrow 0$$

$$\mathcal{J}^{\mathbb{Z}}(X) = \Big\{(a, B) \in \mathcal{A}^{\mathbb{Z}}_{\mathrm{c}}(X) \oplus \Psi^{\mathbb{Z}}_{\mathrm{sus}}(\partial X)[x]; \\ \sigma(B) \equiv a \text{ at } \partial X\Big\}.$$

Then we can say that A is elliptic if $j(A)$ is invertible.

14. Index theorem for manifolds with boundary

Theorem. *(with V. Nistor) If $A \in \Psi^{\mathbb{Z}}_c(X)$ is totally elliptic,* $\mathrm{In}(A)$ *only has a constant term in x and this is a normal operator then*

$$\mathrm{ind}_{\mathfrak{a}}(A) = \mathrm{Tr}_{\mathrm{R}}(\sigma(A)^{-1}[\log Q, \sigma(A)]) - \frac{1}{2}\eta(\mathrm{In}(A)).$$

Here Q is essentially as in the boundaryless case, i.e. an elliptic self-adjoint symbol. This formula is a direct pseudodifferential generalization of the Atiyah-Patodi-Singer index theorem for Dirac operators.

Proof. Diligently follow the ideas outlined above! The ideal $x^\infty \Psi^{-\infty}_{\mathrm{c}}(X)$ is homologous to a matrix algera, with Tr generating HH^0. If A is totally elliptic and B is a parametrix then the analytic index is again given by

$$\begin{aligned}\mathrm{ind}_{\mathrm{a}}(A) &= \mathrm{Tr}([A, B]) \\ &= \mathrm{Tr}(b(B \otimes A)) = \mathrm{IF}(j(a)^{-1} \otimes j(a)).\end{aligned}$$

Thus the problem is to compute the form of the index functional, $\mathrm{IF} = \partial\,\mathrm{Tr}$, in terms of other concrete functionals. See [2]. □

15. Further extensions

The K-theory of the cusp algebra (even on manifolds with corners) is discussed in [3]. Higher suspensions, higher eta invariants and higher indexes (the degree) are discussed in [1]. We plan to analyze the index formula on a compact manifold with corners tomorrow!

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Department of Mathematics, Massachusetts Institute of Technology

E-mail address: `rbm@math.mit.edu`