Paths, loops and fusion
Seminar on Loops and Gerbes

Richard Melrose
Department of Mathematics
Massachusetts Institute of Technology

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I will briefly describe the loop space of a manifold, concentrating on finite-energy ($H^1$) paths and loops.
Interval and circle

- Consider as basic parameter space the interval $[0, \pi]$.
- We can realize the circle as the ‘fusion’ of two intervals $\mathbb{S} = [0, \pi] \cup [0, \pi]'/\{0 = 0', \pi = \pi'\}$.
- Here we think of the second variable as reversed so $[\pi, 2\pi] \ni t \mapsto t' = 2\pi - t \in [0, \pi]'$.
- This recovers the circle $\mathbb{S} = [0, 2\pi]/(0 = 2\pi)$ as a Lipschitz manifold – not quite smooth (although smoothable)!

‘Fusion’ of two intervals
Consider the Hilbert spaces of ‘finite-energy’ (real-valued) functions

$$L^2[0, \pi] \supset H^1[0, \pi], \quad L^2(\mathbb{S}) \supset H^1(\mathbb{S}), \quad 0 \leq s \leq 1.$$  \hspace{1cm} (1)

For periodic functions these are determined by their Fourier expansions

$$\mathbf{v} = \sum_{k \in \mathbb{Z}} u_k e^{ikx} \text{ in } L^2(\mathbb{S}), \quad \mathbf{v} \in H^1 \iff \sum_k (1 + |k|^2) |u_k|^2 < \infty,$$  \hspace{1cm} (2)

$$C^0(\mathbb{S}) \supset H^1(\mathbb{S}) \text{ is a } C^\infty \text{ ring.}$$

We can define $H^1[0, \pi]$ as the image under restriction.

Then fusion of the intervals gives

$$H^1(\mathbb{S}) = \{(u, u') \in (H^1[0, \pi])^2; \ u(0) = u'(\pi), \ u(\pi) = u'(0)\}$$  \hspace{1cm} (3)
Geodesic balls in the manifold

- Let $M$ be a compact manifold; choose a Riemann metric on it.
- For $0 < \epsilon$ below the injectivity radius the balls
  \[ \{ B(m, \epsilon) \}_{m \in M} \]
  give a good cover of $M$ with a finite subcover.
- The exponential map is a diffeomorphism onto the geodesic ball
  \[ \exp_m \{ \mathbf{w} \in T_m M; |\mathbf{w}|_g < \epsilon \} \rightarrow B(m, \epsilon) \]
- The transition maps for this cover are diffeomorphisms of open subsets of vector spaces $\exp_m \exp_{m'}^{-1}$. 

Richard Melrose (Department of Mathematics, Massachusetts Institute of Technology)
The finite-energy path space of a compact manifold is

\[ \mathcal{P}_E M = \mathcal{P} M = \{ \lambda : [0, \pi] \rightarrow M; \lambda \text{ is in } H^1 \text{ in local coordinates.} \} \]

Pulling back \( \lambda^* TM \) is an \( H^1 \) bundle (trivial for oriented \( M \)) on \([0, \pi] \).

The geodesic tubes give a good coordinate cover of \( \mathcal{P} M \):

\[ U(\lambda) = \{ \lambda' \in \mathcal{P} M; \lambda'(t) \in B(\lambda(t), \epsilon), \ \forall \ t \in [0, \pi] \} \]

\[ \exp_{\lambda(*)}^{-1} \left\{ \tau \in H^1([0, \pi]; \lambda^* TM; \sup |\tau| < \epsilon) \right\} \]

The coordinate transition maps \( \exp_{\lambda_2(*)}^{-1} \circ \exp_{\lambda_1(*)} \) are then smooth making \( \mathcal{P} M \) into a \( C^\infty \) Hilbert manifold (it is better than this).
Path fibration

If $\{t_i\}_{i=1}^N \subset [0, \pi]$ is a finite set of distinct points then

$$PM \longrightarrow M^N, \lambda \longmapsto \{\lambda(t_i)\}$$

is a fibre bundle.

In particular it is open and surjective and maps tubes to products of balls.
The most important case of this fibration is to the two end-points, \( \mathcal{P}M \to M^2 \), by default \( \mathcal{P}M \) is interpreted as this bundle.

The importance comes from the basic ‘fusion’ construction identifying the fibre product with loop space

\[
\mathcal{P}^{[2]}M = \mathcal{L}_EM = \mathcal{L}M
\]  

(4)

using the earlier identification of the circle with two paths (the second reversed). We write \( \gamma = j(\lambda_1, \lambda_2) \).

The discussion above of tubes as a coordinate patches for \( \mathcal{P}M \) extends to \( \mathcal{L}M \) (and higher fibre products) again giving good open covers.

Abstractly the tubes on \( \mathcal{L}M \) are fibre products of those on \( \mathcal{P}M \).
The fundamental notion of the fusion product was introduced by Stolz and Teichner.

Fusion arises from the three ‘simplicial maps’ (dropping a factor)

\[ \mathcal{P}^3 M \xrightarrow{p_i} \mathcal{P}^2 M = \mathcal{L} M, \quad i = 1, 2, 3. \] (5)
These maps generate a differential on functions, for instance

\[ f : \mathcal{L}M \to U(1) \implies \delta^* f = (\delta_1^* f)(\delta_2^* f)^{-1}(\delta_3^* f) : \mathcal{P}^3 M \to U(1). \]

There are higher differentials with \( \delta^2 = 1 \).

Take a smooth \( U(1) \) principal bundle with connection \( \nabla \) over \( M \):

\[ \begin{array}{ccc}
U(1) & \xrightarrow{L} & \mathcal{L}M \\
\downarrow & & \downarrow \\
M & & \end{array} \]

The holonomy of \( L, \nabla \) is a function \( f : \mathcal{L}M \to U(1) \) with \( \delta^* f = 1 \).

This is the basic example of (geometric) transgression.
The three-point fibration, including both end-points and central point will be denoted \( P^\tau M \to M^3 \).

There are again three simplicial maps (which involve more reparametrization) forming ‘figure-of-eight’ configurations

\[
P^{\tau [2]} M \xrightarrow{q_i} \mathcal{L}M
\] (8)
Again there is a differential (but no higher ones it seems) on functions on

\[ f \mapsto \delta_8 f = (q_a^*f)(q_b^*f)(q_c^*f)^{-1} : \mathcal{P}^2 M \to U(1). \]  

Again the holonomy of a circle bundle satisfies \( \delta_8 f = 1 \).

In fact this characterizes holonomy functions up to homotopy (through such functions).

Theorem (Kottke-M.)

The abelian group of homotopy classes of continuous functions \( f : \mathcal{L}M \to U(1) \) which are fusive in the sense that \( \delta^* f = 1, \delta_8 f = 1 \) is naturally \( H^2(M; \mathbb{Z}) \).
To complete the proof of this result one needs a construction of a circle bundle from $f$ as ‘descent data’.

Take the trivial bundle $\mathcal{P}M \times U(1)$.

Identify points by $(\lambda_1, z_1) \sim (\lambda_2, z_2)$ if $z_1 = f(\gamma(\lambda_1, \lambda_2))z_2$.

Fusion makes this an equivalence relation, giving a principal $U(1)$ bundle $\tilde{L}_f$ over $M^2$.

The figure-or-eight condition shows that under the simplicial maps $\pi_i : M^3 \rightarrow M^2$, $\pi_1^* \tilde{L}_f \otimes (\pi_2^* \tilde{L}_f)^{-1} \otimes \pi_3^* \tilde{L}_f \simeq U(1)$ is trivial over $M^3$.

It follows that $\tilde{L}_f = \pi_1^* L_f \otimes (\pi_2^* L_f)^{-1}$ where $L_f$ is defined up to equivalence.

Of course, the claim is that trangression and regression are inverses of each other.
Transgression/Regression pairs

- Chris will talk about (iterated) transgression of integral cohomology in the form of Čech cohomology.
- These trasgressed objects realize higher gerbes – Vishesh will at least start a discussion of this.
- In particular bundle gerbes (in the sense of Murray) transgress to fusive circle bundles over $\mathcal{L}M$.
- Stolz and Teicher introduced fusion (figure-of-eight is nominally older!) to complete a program started by Atiyah and Witten.

Theorem (Stolz-Teichner)

Spin structures on (oriented) manifolds, $M$, correspond precisely to ‘fusion loop-orientations’ on $\mathcal{L}M$.

- Loop-orientation here corresponds to continuous, i.e. locally constant, maps $e : \mathcal{L}TM \rightarrow \mathbb{Z}_2$ which take both signs on each fibre of $\mathcal{L}TM \rightarrow \mathcal{L}M$. Fusion implies the figure-of-eight condition.
The ‘higher’ version of this theorem is relevant for the Dirac-Ramond operator:

**Theorem (Waldorf, Kottke-M.)**

*String structures on a spin manifold $M$ are in 1-1 correspondence with Loop-spin structures on $\mathcal{L}M$.***

This can be explained in due course.

The upshot is that the string condition, $\frac{1}{2}p_1(M) = 0$, where $\frac{1}{2}p_1 \in H^4(M; \mathbb{Z})$ is the Spin-Pontryagin class, is equivalent to the existence of a principal bundle with structure group the central extension of the loop group on Spin$(n)$ over $\mathcal{L}M$ which is *fusive* in an appropriate sense.
There are two important, indeed I believe fundamental, notions that I have left out so far. One is equivariance under reparameterization.

If $\text{Dff}^+(S)$ is the group of orientation-preserving diffeomorphisms of the circle then there is an action of $\text{Dff}^+(S)$ on the loop space(s) by reparameterization.

For finite energy loops this extends to an action of the group of orientation-preserving Lipschitz diffeomorphisms.

For loop groups and principal bundles with structure group which is a central extension of a loop group one expect the action of the central extension of $\text{Dff}^+$, the Bott-Virasoro group.
Litheness corresponds to ‘pointwise’ regularity of derivatives. The tangent space $T_{\gamma} \mathcal{LM} \simeq (H^1(\mathbb{S}))^n$, essentially the linear model. Riesz’ Representation Theorem then identifies the derivative of a function with an element $df(\gamma) \in T^*_{\gamma} \mathcal{LM} = (H^1(\mathbb{S}))^n$.

However this identification uses the inner product on $H^1$ and in terms of the integral ($L^2$) pairing this means $df(\gamma) \in (H^{-1}(\mathbb{S}))^n$.

This is an essential part of the difficulty in defining the Dirac-Ramond operator.

Litheness (valid for $\mathcal{LM}$ but meaningless for general Hilbert manifolds) means that the condition $df(\gamma) \in (H^1(\mathbb{S}))^n$ with respect to the $L^2$ pairing is meaningful (not automatic, hence the notion of lithic structures).

This also plays an essential role in resolving the conflict arising in constructions using the fibre-product to regain reparameterization-invariance.