Addendum to Lecture 24

open bases

Recall that a metric space is said to be *separable* if it has a countable dense subset.

An open basis of a metric space X is a collection of open sets $\{U_a\}_{a\in A}$ with the property that for every open set $O \subset X$ and every point $p \in O$ there is some U_a in between, that is $p \in U_a \subset O$. Another way of saying that the U_a form an open basis is that any open set can be written as a union of them

$$O = \bigcup_{a\in B} U_a,\ B = \{b \in A; U_b \subset)\}.$$

lemma Any separable metric space has a *countable* open basis.

Proof Let $\{p_1, p_2, \ldots\}$ be the countable dense subset of X. Then the $\{V_n\}$ can be taken to be the balls $B(p_j, r)$ with r rational. This is a countable union of countable sets, so is countable and is clearly an open basis.

Now, what I was trying to extract from you was:

Proposition If a metric space X has a countable open basis then any open cover has countable subcover.

Proof Let $\{O_\alpha\}_{\alpha\in A}$ be an open cover. If $\{V_n\}$ is a countable basis of open sets then each O_α is a union of V_n's. Consider the subset $W_j = V_{n_j}$ which occur in at least one of these decompositions. Not only is this a countable collection of sets but it still covers X, since the O_α's cover it and they are unions of the W_j's. Now, for each j pick one of the O_α's, say O_{α_j}, which contains W_j. These O_{α_j}'s must still cover X. Thus we have shown that any open cover has a countable subcover.