LECTURE 2 FOR 18.158 SPRING 2021
FINITE DIMENSIONAL MANIFOLDS

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1. Manifolds

Since we wish to consider paths, maps from an interval, and eventually 'iterated paths' as well as loops and iterated loops so we work from the beginning in the context of manifolds with corners. This is really for the domain space – the range space, which is what we are really interested in, will usually be a compact manifold without boundary.

A topological manifold with corners is a metrizable, separable, topological space with a covering by open subsets ('coordinate patches') equipped with homeomorphisms to (relatively) open subsets of $[0,\infty)^n$. Usually we either suppose the space to be connected, or require n to be fixed. Note that in the topological context this is the same as a manifold wih boundary, because $[0,\infty)^n$ is homeomorphic to $[0,\infty)\times\mathbb{R}^{n-1}$. The distinction appears when we require more regularity. [In that context I actually prefer a slightly stronger definition, but it is not very relevant here.]

Exercise 1. Show that this is equivalent to the standard notion – unless you allow non-separability (as in the long line).

We are basically interested in $\mathcal{C}^\infty$ manifolds. The $\mathcal{C}^\infty$ (or smooth) functions form a sheaf over $[0,\infty)^n$. For each open set $U \subset [0,\infty)^n$ we define $\mathcal{C}^k(U) \subset \mathcal{C}(U)$ by iterative regularity. Namely $\mathcal{C}^1(U) \subset \mathcal{C}(U)$ is the subspace of elements with continuous partial derivatives in $\mathcal{C}(U)$ and then $\mathcal{C}^k(U)$ is defined inductively by the condition that the partial derivatives lie in $\mathcal{C}^{k-1}(U)$.

Then

$$\mathcal{C}^\infty(U) = \bigcap_k \mathcal{C}^k(U). \tag{1}$$

These spaces form a fine sheaf of rings, where the fineness property comes from the existence of partitions of unity – for any collection of open subsets $\{U_a\}_{a\in A}$ there exist countably many $a_i \in A$ and $\rho_i \in \mathcal{C}^\infty(U_{a_i})$ of compact support $\operatorname{supp}(\rho_i) \Subset U_{a_i}$ such that only finitely many of the $\operatorname{supp}(\rho_i)$ meet any compact subset $K \subset \bigcup_a U_a = U$ and

$$\sum_i \rho_i = 1 \text{ on } U. \tag{2}$$

In particular of course the U_{a_i} must cover U. If you are unfamiliar with the behaviour at corners it is important to note that the two obvious definitions of smooth functions over (relatively of course) open subsets of $[0,\infty)^n$ are the same:

Proposition 1. *If $u \in \mathcal{C}(U)$ for $U \subset [0,\infty)^n$ open, then the existence of an open set $\tilde{U} \subset \mathbb{R}^n$ such that $\tilde{U} \cap [0,\infty)^n = U$ on which there exists $\tilde{u} \in \mathcal{C}^\infty(\tilde{U})$ with $u = \tilde{u}\big|_U$ is equivalent to the condition that $u' = u\big|_{\operatorname{int} U} \in \mathcal{C}^\infty(\operatorname{int} U)$, $\operatorname{int} U = U \cap (0,\infty)^n$ and the partial derivatives of all orders extend to $\mathcal{C}(U)$.*

Now of particular interest to us are the composition properties. If $U \subset [0,\infty)^n$ and $V \subset [0,\infty)^m$ are open and $f : U \longrightarrow V$ has components in $\mathcal{C}^\infty(U)$ then $g \circ f \in \mathcal{C}^\infty(U)$ for all $g \in \mathcal{C}^\infty(V)$. If we write $\mathcal{C}(U;V)$ for the space of maps $U \longrightarrow V$ with components in $\mathcal{C}^\infty(U)$ then $F \in \mathcal{C}^\infty(U;V)$ is a diffeomorphism iff and only if it has a two-sided inverse $F^{-1} \in \mathcal{C}^\infty(V;U)$; this implies $m = n$. Moreover the codimension of boundary points is preserved.

The diffeomorphism between open subsets of $\mathbb{R}^n$ form a groupoid, as do the homeomorphism. Our definition of a topological manifold means that the transition maps defined by intersection of elements of the coordinate cover form a cocycle in the groupoid of homeomeorphisms of $[0,\infty)^n$. A $\mathcal{C}^\infty$ structure is determined by a choice of coordinate cover such that the transition cocycle takes values in the groupoid of $\mathcal{C}^\infty$ diffeomorphisms. It does not always exist. One can define structures with respect to any groupoid of homeomorphism in place of $\mathcal{C}^\infty$, but this is not really of interest here since interest to us since we really want to consider $\mathcal{C}^\infty$ manifolds.

2. $\mathcal{C}^\infty$ ALGEBRAS OF FUNCTIONS

We can abstract the properties of $\mathcal{C}^\infty$ functions and consider a general fine subsheaf of the continuous functions on $[0,\infty)^n$. Then it makes sense to say this is $\mathcal{C}^\infty$ *invariant* if composition with $\mathcal{C}^\infty$ maps on the right preserves the space and *a* $\mathcal{C}^\infty$ *algebra* if composition on the left with $\mathcal{C}^\infty$ maps preserves the space.

Proposition 2. *Any fine, $\mathcal{C}^\infty$-invariant, subsheaf of the continuous functions on $[0,\infty)^n$ defines a corresponding sheaf of any n-manifold.*

When it comes to maps between manifolds, we need the second condition as well.

Definition 1. A subsheaf $\mathcal{F}$ of the continous functions on $[0,\infty)^n$ which is $\mathcal{C}^\infty$-invariant and a $\mathcal{C}^\infty$-algebra defines a space of map $\mathcal{F}(Q,M)$ for $\mathcal{C}^\infty$-manifolds Q and M.

Having said all this abstractly, the examples we have most in mind are quite familiar ones.

Proposition 3. *For any $k \in \mathbb{N}_0$ and $0 < \alpha \le 1$ the Hölder spaces $C^{k,\alpha}$ form a fine $\mathcal{C}^\infty$-invariant sheaf of $\mathcal{C}^\infty$-algebras the same is true of the Sobolev spaces H^s for $s \ge n/2$.*

2.1. **Global spaces recalled.** At some points we will want to make some use of the negative Sobolev spaces as well, so for all $s \in \mathbb{R}$. These form a fine sheaf over $\mathbb{R}^n$ but we have to be a bit more careful about $[0, \infty)^n$ for $s < 0$. They are only sheaves of continuous functions for $s > n/2$ and we need some extension properties. I will add this later.

3. Tensor and general vector bundles

Will recall in the written notes. Should I say a bit more about principal bundles?

4. Riemann metrics

Let me also recall some of the basic statemnts about Riemann metrics, for the most part here I restrict attention to the boundaryless case.

4.1. **Levi-Civita connection.**

4.2. **Exponential map.** Geodesic on a Riemann manifold can be defined in several ways, most directly here as parallel curves for the Levi-Civita connection. The exponential map, defined by geodesics, is a smooth map for each p (which depends smootly on p as well)

$$\mathrm{Exp}_p : T_pM \longrightarrow M. \tag{1}$$

The main points we want are that Exp_p is a diffeomorphism from sufficiently small balls around $0 \in T_pM$ (in terms of the fibre metric) to their images, which are the metric balls in M around p of the same radius. One of the important properties is that for small open balls the exponential map around *any point* defines a contraction. This is really the statement that small balls are geodesically convex. In consequence

Proposition 4. *Small balls in a Riemann manifold (of fixed small radius if the manifold is compact) form a 'good' open cover, one in which all non-trival intersections are contractible.*

One way we want to use this is to discuss Čech cohomology. If we take an abelian topological group Z (we are really interested in the cases bbR, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Z}_p$) then we can consider the sheaf $\mathcal{C}(U;Z)$ for open subsets $A\subset M$ of a manifold. If U_a, $a\in A$ is an open cover then

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