## 18.100B HOMEWORK 6, WAS DUE 9 MARCH 2004

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Problem 2. Since $\sqrt{n^2+n}-n=\frac{n^2+n-n^2}{\sqrt{n^2+n}+n}$ we can compute the limit as

$$\lim_{n\to\infty}\frac{1}{\sqrt{1+1/n}+1}=1. \tag{1}$$

Problem 7. By the Cauchy-Schwarz inequality,

$$\left(\sum_{n=1}^{N}\frac{\sqrt{a_n}}{n}\right)^2\le\sum_{n=1}^{N}\frac{1}{n^2}\sum_{n=1}^{N}a_n. \tag{2}$$

Both series on the right are convergent, hence the partial sums are bounded so the partial sum on the left is bounded, hence, being a series of non-negative terms, convergent.

Problem 12. (a) Since the $a_n>0$, $r_n$ is strictly decreasing as $n$ increases. Thus for $m<n$,

$$\frac{a_m}{r_m}+\cdots+\frac{a_n}{r_n}>\frac{1}{r_m}\left(a_m+\dots a_n\right)=\frac{r_n-r_m}{r_m}=1-\frac{r_n}{r_m}. \tag{3}$$

It follows that the series $\sum\limits_n\frac{a_n}{r_n}$ is not Cauchy since the right side tends to 1 as $n\to\infty$ for fixed $m$. Thus the series does not converge.

(b) Using the identity $(\sqrt{r_n}-\sqrt{r_{n+1}})(\sqrt{r_n}-\sqrt{r_{n+1}})=r_n-r_{n+1}=a_n$ and the fact that $r_n$ is strictly decreasing, we conclude that

$$a_n<2\sqrt{r_n}(\sqrt{r_n}-\sqrt{r_{n+1}}) \tag{4}$$

giving the desired estimate. From this inequality we find that

$$\sum_{n=1}^{q}\frac{a_n}{\sqrt{r_n}}<\sqrt{r_1}-\sqrt{r_{p+1}}<\sqrt{r_1} \tag{5}$$

so this series with positive terms is bounded and hence convergent.

Problem 16. (a) Proceeding inductively we can assume (since it is true for $n=1$) that $x_n>\sqrt{\alpha}$. Then $x_n^2-2\sqrt{\alpha}x_n+\alpha=(x-\sqrt{\alpha})^2>0$ so $x_n^2+\alpha>2x_n\sqrt{\alpha}$ and hence

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{\alpha}{x_n}\right)>\sqrt{\alpha}.$$

Since $\alpha/x_n<x_n$ also follows that $x_{n+1}<x_n$ so the sequence is strictly decreasing but always larger than $\sqrt{\alpha}$. Thus the limit $x_n\to x\ge\sqrt{\alpha}$ exists. Since $2x_nx_{n+1}=x_n^2-\alpha$ the limit must satisfy $2x^2=x^2-\alpha$, that is $x=\sqrt{\alpha}$.

(b)Defining $\epsilon_n=x_n-\sqrt{\alpha}$ we find that

$$\epsilon_{n+1}=\frac{1}{2x_n}\left(x_n^2-2x_n\sqrt{\alpha}+\alpha\right)=\frac{\epsilon_n^2}{2x_n}<\frac{\epsilon_n^2}{2\sqrt{\alpha}}.$$

Since this is true for all $n$, if we set $\gamma_n = \epsilon_n/\beta$, where $\beta = 2\sqrt{\alpha}$ then

$$\gamma_{n+1} < \gamma_n^2 \Longrightarrow \gamma_{n+1} < \gamma_1^{2^n},$$

so $\epsilon_{n+1} < \beta \left(\frac{\epsilon_1}{\beta}\right)^{2^n}$.

(c) If $\alpha = 3$ and $x_1 = 2$ then $1\frac{7}{10} < \sqrt{3} < 1\frac{8}{10}$ so $\epsilon_1 = 2 - \sqrt{3} < \frac{2}{10}$, $2\sqrt{3} > 2$ and $\epsilon_1/\beta < \frac{1}{10}$. Since $\beta < 4$, $\epsilon_5 < 4 \cdot 10^{-16}$ and $\epsilon_6 < 4\dot{1}0^{-32}$.

Problem 20. Suppose thet $\{p_n\}$ is a Cauchy sequence and some subsequence $\{p_{n(k)}\}$ converges to $p$. Then, given $\epsilon > 0$ there exists $N$ such that for $n, m \geq N$ $d(p_n, p_m) < \epsilon/2$ and there exists $N'$ such that $k > N'$ implies $d(p, p_{n(k)}) < \epsilon/2$. We can choose $k > N'$ so large that $n(k) > N$ and then

$$d(p, p_n) \leq d(p, p_{n(k)}) + d(p_n, p_{n(k)}) < \epsilon/2 + \epsilon/2 = \epsilon$$

provided only that $n \geq N$. Thus $p_n \to p$.

Problem 21. If $\{E_n\}$ is a decreasing sequence of non-empty closed sets in a metric space then there is a sequence $\{p_n\}$ with $p_n \in E_n$. The assumption that $\operatorname{diam} E_n \to 0$ means that given $\epsilon > 0$ there exists $N$ such that $n \geq N$ implies $d(p, q) < \epsilon$ if $p, q \in E_n$. Now, for $n \geq m \geq N$, $p_n \in E_n \subset E_m$ so $d(p_n, p_m) < \epsilon$. It follows that the sequence is Cauchy and hence, by the assumed completeness of $X$ that it converges to $p$. Since the sequence is in $E_n$ for $m \geq n$, $p \in E_n$ for all $n$ so $p \in \bigcap_n E_n$ as desired. Conversely there is only one point in this set since $q \in \bigcap_n E_n$ implies $d(p, q) \leq \operatorname{diam}(E_n) \to 0$ so $p = q$.