# 18.100B HOMEWORK 6, WAS DUE 9 MARCH 2004

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Problem 2. Since $\sqrt{n^2+n}-n = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n}$ we can compute the limit as

$$\lim_{n\to\infty} \frac{1}{\sqrt{1+1/n}+1} = 1. \tag{1}$$

Problem 7. By the Cauchy-Schwarz inequality,

$$\left(\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n}\right)^2 \le \sum_{n=1}^{N} \frac{1}{n^2} \sum_{n=1}^{N} a_n. \tag{2}$$

Both series on the right are convergent, hence the partial sums are bounded so the partial sum on the left is bounded, hence, being a series of non-negative terms, convergent.

Problem 12. (a) Since the $a_n > 0$, $r_n$ is strictly decreasing as $n$ increases. Thus for $m < n$,

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{1}{r_m}(a_m + \ldots a_n) = \frac{r_n - r_m}{r_m} = 1 - \frac{r_n}{r_m}. \tag{3}$$

It follows that the series $\sum\limits_n \frac{a_n}{r_n}$ is not Cauchy since the right side tends to 1 as $n \to \infty$ for fixed $m$. Thus the series does not converge.

(b) Using the identity $(\sqrt{r_n} - \sqrt{r_{n+1}})(\sqrt{r_n} - \sqrt{r_{n+1}}) = r_n - r_{n+1} = a_n$ and the fact that $r_n$ is strictly decreasing, we conclude that

$$a_n < 2\sqrt{r_n}(\sqrt{r_n} - \sqrt{r_{n+1}}) \tag{4}$$

giving the desired estimate. From this inequality we find that

$$\sum_{n=1}^{q} \frac{a_n}{\sqrt{r_n}} < \sqrt{r_1} - \sqrt{r_{p+1}} < \sqrt{r_1} \tag{5}$$

so this series with positive terms is bounded and hence convergent.

Problem 16. (a) Proceeding inductively we can assume (since it is true for $n = 1$) that $x_n > \sqrt{\alpha}$. Then $x_n^2 - 2\sqrt{\alpha}x_n + \alpha = (x - \sqrt{\alpha})^2 > 0$ so $x_n^2 + \alpha > 2x_n\sqrt{\alpha}$ and hence

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) > \sqrt{\alpha}.$$

Since $\alpha/x_n < x_n$ also follows that $x_{n+1} < x_n$ so the sequence is strictly decreasing but always larger than $\sqrt{\alpha}$. Thus the limit $x_n \to x \ge \sqrt{\alpha}$ exists. Since $2x_nx_{n+1} = x_n^2 - \alpha$ the limit must satisfy $2x^2 = x^2 - \alpha$, that is $x = \sqrt{\alpha}$.

(b)Defining $\epsilon_n = x_n - \sqrt{\alpha}$ we find that

$$\epsilon_{n+1} = \frac{1}{2x_n}\left(x_n^2 - 2x_n\sqrt{\alpha} + \alpha\right) = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}.$$

Since this is true for all $n$, if we set $\gamma_n = \epsilon_n/\beta$, where $\beta = 2\sqrt{\alpha}$ then

$$\gamma_{n+1} < \gamma_n^2 \Longrightarrow \gamma_{n+1} < \gamma_1^{2^n},$$

so $\epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n}$.

(c) If $\alpha = 3$ and $x_1 = 2$ then $1\frac{7}{10} < \sqrt{3} < 1\frac{8}{10}$ so $\epsilon_1 = 2 - \sqrt{3} < \frac{2}{10}$, $2\sqrt{3} > 2$ and $\epsilon_1/\beta < \frac{1}{10}$. Since $\beta < 4$, $\epsilon_5 < 4 \cdot 10^{-16}$ and $\epsilon_6 < 4\dot{1}0^{-32}$.

Problem 20. Suppose thet $\{p_n\}$ is a Cauchy sequence and some subsequence $\{p_{n(k)}\}$ converges to $p$. Then, given $\epsilon > 0$ there exists $N$ such that for $n, m \geq N$ $d(p_n, p_m) < \epsilon/2$ and there exists $N'$ such that $k > N'$ implies $d(p, p_{n(k)}) < \epsilon/2$. We can choose $k > N'$ so large that $n(k) > N$ and then

$$d(p, p_n) \leq d(p, p_{n(k)}) + d(p_n, p_{n(k)}) < \epsilon/2 + \epsilon/2 = \epsilon$$

provided only that $n \geq N$. Thus $p_n \to p$.

Problem 21. If $\{E_n\}$ is a decreasing sequence of non-empty closed sets in a metric space then there is a sequence $\{p_n\}$ with $p_n \in E_n$. The assumption that $\operatorname{diam} E_n \to 0$ means that given $\epsilon > 0$ there exists $N$ such that $n \geq N$ implies $d(p, q) < \epsilon$ if $p, q \in E_n$. Now, for $n \geq m \geq N$, $p_n \in E_n \subset E_m$ so $d(p_n, p_m) < \epsilon$. It follows that the sequence is Cauchy and hence, by the assumed completeness of $X$ that it converges to $p$. Since the sequence is in $E_n$ for $m \geq n$, $p \in E_n$ for all $n$ so $p \in \bigcap_n E_n$ as desired. Conversely there is only one point in this set since $q \in \bigcap_n E_n$ implies $d(p, q) \leq \operatorname{diam}(E_n) \to 0$ so $p = q$.