Chapter 2: Problem 10

This is the 'discrete metric' on a set. Certainly $d : X \times X \longrightarrow [0, \infty)$ is well defined and $d(x, y) = 0$ iff $x = y$. Symmetry, $d(x, y) = d(y, x)$, is immediate from the definition and the triangle inequality

$$d(x, y) \leq d(x, z) + d(z, y) \;\forall\; x, y, x \in X \tag{1}$$

follows from the fact that the right hand side is always equal to 0, 1 or 2 and the LHS is 0 or 1 and if the LHS vanishes then $x = y = z$ and the RHS also vanishes.

All subsets are open, since if $E \subset X$ and $p \in E$ then $x \in X$ and $d(x, p) < 1$ implies $x = p$ and hence $x \in E$. Since the complements of open sets are closed it follows that all subsets are closed. The only compact subsets are finite. Indeed if $E \subset X$ is compact then the open balls of radius 1 with centers in E cover E and each contains only one point of E so the existence of a finite subcover implies that E itself is finite.

Chapter 2: Problem 12

We are to show that $K = \{1/n; n \in \mathbb{N}\} \cup \{0\}$ is compact as a subset of $\mathbb{R}$ directly from the definition of compactness. So, let U_a, $a \in A$, be an open cover of K. It follows that $0 \in U_{a_0}$ for some $a_0 \in A$. But since U_{a_0} is open it contains some ball of radius $1/n$ around 0. Thus all the points $1/m \in U_{a_0}$ for $m > n$. For each $m \leq n$ we can find some $a_m \in A$ such that $1/m \in U_{a_m}$, since the U_a cover K. Thus we have found a finite subcover

$$K \subset U_{a_0} \cup U_{a_1} \cup \cdots \cup U_{a_n} \tag{2}$$

and if follows that K is compact.

Chapter 2: Problem 16

Here $\mathbb{Q}$ is the metric space, with $d(p, q) = |p - q|$, the 'usual' metric. Set

$$E = \{p \in \mathbb{Q}; 2 < p^2 < 3\}. \tag{3}$$

Suppose x is a limit point of E as a subset of the rationals. Then we know that $(x - \epsilon, x + \epsilon) \cap E$ is infinite for each $\epsilon > 0$. Regarding x as a real number it follows that $x \in [2^{\frac{1}{2}}, 3^{\frac{1}{2}}]$. Since we know the end points are not rational and by assumption $x \in \mathbb{Q}$ it follows that $x \in E$. Thus E is closed. Certainly E is bounded since $p \in E$ implies $|p| < 3$.

To see that E is not compact, recall that if it were compact as a subset of $\mathbb{Q}$ it would be compact as a subset of $\mathbb{R}$ by Theorem 2.33. Since it is not closed as a subset of $\mathbb{R}$ it cannot be compact. Alternatively, for a direct proof of non-compactness, take the open cover given by the open sets $\{x \in \mathbb{Q}; |p - 2^{\frac{1}{2}}| > 1/n\}$. This can have no finite subcover since E contains points arbitrarily close to the real point $\sqrt{2}$.

Yes E is open in $\mathbb{Q}$ since it is of the form $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ where $G = (\sqrt{2}, \sqrt{3}) \subset \mathbb{R}$ is open, so Theorem 2.30 applies.

Chapter 2: Problem 22

We need to show that the set of rational points, $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. We can use the fact that $\mathbb{Q} \subset \mathbb{R}$ is dense. Thus, given $\epsilon > 0$ and $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ there exists $p_l \in \mathbb{Q}$ such that $|x_l - p_l| \leq \epsilon/k$ for each $l = 1, \dots, k$. Thus, as points in $\mathbb{R}^k$,

$$|x - p| \leq \sum_{l=1}^{k} |x_l - p_l| < \epsilon. \tag{4}$$

This shows that $\mathbb{R}^k$ is separable since we know that $\mathbb{Q}^k$ is countable.

Chapter 2: Problem 23

We are to show that a given separable metric space, X, has a countable base. The hint is to choose a countable dense subset $E \in X$ and then to consider the collection, $\mathcal{B}$, of all open subsets of X of the form $B(x, 1/n)$ where $x \in E$ and $n \in \mathbb{N}$. This is a countable union, over $\mathbb{N}$, of countable sets so is countable. Now, we need to show that this is a base. So, suppose $U \subset X$ is a given open set. If $x \in U$ then for some $m = m_x > 0$, $B(x, 1/m) \subset U$, since it is open. Also, by the density of E in X there exists some $e_x \in E$ with $|x - e_x| < 1/2m$ But then $y \in B(e_x, 1/2m)$ implies $d(x, y) < d(x, e_x) + d(e_x, y) < 1/2m + 1/2m = 1/m$. Thus $B(e_x, 1/2m) \subset U$. It follows that

$$U = \bigcup_{x \in U} B(e_x, 1/2m_x). \tag{5}$$

Thus U is written as a union of the elements of $\mathcal{B}$ which is therefore an open base.

Chapter 2: Problem 25

We wish to show that a given compact metric space K has a countable base. As the hint says, for each $n \in \mathbb{N}$ consider the balls of radius $1/n$ around each of the points of K :

$$K \subset \bigcup_{x \in K} B(x, 1/n) \tag{6}$$

since each $x \in K$ is in one of these balls at least. Now the compactness of K implies that there is a finite subcover, that is there is a finite subset $C_n \subset K$, for each n, such that

$$K \subset \bigcup_{p \in C_n} B(p, 1/n). \tag{7}$$

Now, set $E = \bigcup_{n \in \mathbb{N}} C_n$. This is countable, being a countable union of finite sets. Now it follows that C is dense in K. Indeed given $x \in K$ and $\epsilon > 0$ there exists $n \in \mathbb{N}$ with $1/n < \epsilon$ and from (7) a point in $p \in C_n \subset C$ with $|x - p| < 1/n < \epsilon$. Thus $K = \overline{C}$ and it follows that K is separable; from #23 it follows that K has a countable open base.

Alternatively one can see directly that the $B(p, 1/n)$, $p \in C_n$, form an open base.

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