

18.100B HOMEWORK 5, WAS DUE 2 MARCH 2004

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CHAPTER 2: PROBLEM 19

- a) If A and B are closed in X and disjoint then $\bar{A} = A$ and $\bar{B} = B$ so $\bar{A} \cap \bar{B} = \emptyset$ and $A \cap B = \emptyset$ follow from $A \cap B = \emptyset$ and hence the sets are separated.
- b) If A and B are open and disjoint then $A^c \supset B$ is closed so $\bar{B} \subset A^c$ which means $A \cap \bar{B} = \emptyset$; similarly $\bar{A} \cap B = \emptyset$ so they are separated.
- c) With $p \in X$ fixed and $\delta > 0$ put $A = \{q \in X; d(p, q) < \delta\}$ and $B = \{q \in X; d(p, q) > \delta\}$. If $x \in A$ is a limit point of A then given $\epsilon > 0$ it follows that there exists $y \in A$ with $d(x, y) < \epsilon$, hence $d(p, x) \leq d(p, y) + d(y, x) \leq \delta + \epsilon$. Since this holds for all $\epsilon > 0$, $d(p, x) \leq \delta$. Thus $\bar{A} \cap B = \emptyset$. Similarly if $x \in \bar{B}$ and $\epsilon > 0$ there exists $q \in B$ with $d(x, q) < \epsilon$ so $\delta < d(p, q) \leq d(p, x) + d(x, q)$ shows that $d(p, x) \geq \delta - \epsilon$ and so $d(p, x) \geq \delta$ if $x \in \bar{B}$ so $A \cap \bar{B} = \emptyset$ and A and B are separated.

Remark It *is not* in general the case that $\bar{A} = \{x \in X; d(p, x) \leq \delta\}$ and similarly for B . Make sure you understand why!

- d) If X is connected and contains at least two points, p and p' then $d(p, p') > 0$. For any $\delta \in (0, d(p, p'))$ there must exist at least one point in X with $d(p, x_\delta) = \delta$. Indeed if not, then $\{x \in X; d(p, x) = \delta\}$ is empty and $X = A \cup B$ with A and B as above. Since these sets are separated and $p \in A$, $p' \in B$ shows that neither is empty this would contradict the connectedness of X .

CHAPTER 2: PROBLEM 20

Suppose $C \subset X$ is connected and consider $\bar{C} = A \cup B$ where $\bar{A} \cap \bar{B} = \emptyset$ and $A \cap \bar{B} = \emptyset$. Then $C = (A \cap C) \cup (B \cap C)$ and $\overline{A \cap C} \subset \bar{A}$, $\overline{B \cap C} \subset \bar{B}$ so $A \cap C$ and $B \cap C$ are separated. Thus one must be empty by the connectedness of C ; changing names if necessary we can assume that it is A so $A \subset C'$ must consist only of limit points of C and necessarily $C \subset B$, but then $A \subset \bar{C}$ so $A = \emptyset$ and \bar{C} is indeed connected.

On the other hand the interior of a connected set need not be connected. Take the region $Q = \{(x, y) \in \mathbb{R}^2; xy \geq 0\}$. This is the union of the closed first and third quadrants. From the problem below, the two quadrants are themselves connected. Let us show that the union of two connected sets with non-empty intersection is connected. Thus suppose $C_i \subset X$, $i = 1, 2$, are connected sets and $C_1 \cap C_2 \neq \emptyset$. Then if $C_1 \cup C_2 = A \cup B$ with A and B separated it follows that $C_i = A_i \cup B_i$ with $A_i = A \cap C_i$, $B_i = B \cap C_i$. Furthermore, A_i and B_i are separated for $i = 1, 2$ (and the same i) since $\bar{A}_i \cap \bar{B}_i \subset \bar{A} \cap \bar{B} = \emptyset$ and $A_i \cap \bar{B}_i \subset A \cap \bar{B} = \emptyset$. From the connectedness of C_i we must have one of $A_1 = \emptyset$ or $B_1 = \emptyset$ and one of $A_2 = \emptyset$ or $B_2 = \emptyset$. Of course if we have both A_i empty then $A = \emptyset$ and similarly for the B_i 's. So, if necessary switching A and B the only danger is that $A_1 = \emptyset$ and $B_2 = \emptyset$ but

then $B = B_1 = C_1$ and $A = A_2 = C_2$ but then $A \cap B = C_1 \cap C_2 \neq \emptyset$ contradicting the assumption. Thus $C = C_1 \cup C_2$ is connected and in particular this applies to our union of two quadrants.

The interior is the union of the two open quadrants and these are separated, since the closure of each is the corresponding closed quadrant so the union is not connected.

CHAPTER 2: PROBLEM 21

- a) Certainly $A_0 \cap B_0 = \emptyset$ since $A \cap B = \emptyset$ and $p^{-1}(A) \cap p^{-1}(B) = p^{-1}(A \cap B)$. If $s \in A'_0$ then it is for $\epsilon > 0$ $(t - \epsilon, t + \epsilon) \cap A_0$ is infinite which means that

$$(1) \quad B((1-s)a + sb, \epsilon) \cap A$$

is infinite, so $p(s)$ a limit point of A , hence $p(s) \notin B$ (since A and B are separated) so $s \notin B_0$ which shows that $\overline{A_0} \cap B_0 = \emptyset$. The same argument shows that $A_0 \cap \overline{B_0} = \emptyset$.

- b) Put $s = \sup\{t \in A_0; t < 1\}$. Since $0 \in A_0$ this set is non-empty and clearly bounded above, so s exists. Moreover, $0 \leq s < 1$ since $s = 1$ would imply that $\overline{A_0} \cap B_0 \neq \emptyset$. Now set $s' = \inf\{t \in B_0; t > s\}$. Certainly $s' \leq 1$ and $s' \geq s$. If $s' = s$ then $s \notin A_0$ and $s' = s \notin B_0$, since otherwise they are not separated so take $t_0 = s = s'$. If $s < s'$ then $(s, s') \cap A_0 = \emptyset$ and $(s, s') \cap B_0 = \emptyset$. In any case we have found $t_0 \in (0, 1)$, $t_0 \notin A_0 \cup B_0$ which implies $p(t_0) \notin A \cup B$.
- c) If G is convex, then by definition for $a, b \in G$, $p(t) = (1-t)a + tb \in G$ for $t \in [0, 1]$. From the argument above, if $G = A \cup B$ where A and B are separated then we have a contradiction to the fact that both are non-empty, since then taking $a \in A$ and $b \in B$ we have found $p(t_0) \notin A \cup B$ but $p(t_0) \in G$ by convexity. Thus any convex subset of \mathbb{R}^k is connected.

CHAPTER 2: PROBLEM 24

Remark: Note that separable and separated are rather unrelated concepts.

We assume that X is a metric space in which every infinite subset has a limit point (sequentially compact is what I called these in lecture). We are to show that X is separable. For each n choose successively points $x_j \in X$ for $j = 1, 2, \dots$, such that $d(x_j, x_k) \geq 1/n$ for $k < j$. Either at some point no further choice is possible, or else we can choose this way an infinite set $F \subset X$ with $d(x, y) \geq 1/n$ for all $x, y \in F$ distinct. By assumption on X this set F must have a limit point, call it x^* . Since $B(x^*, \frac{\delta}{4}) \cap F$ must be infinite, it must contain at least two distinct points x, y which have

$$d(x, y) \leq d(x, x^*) + d(x^*, y) \leq \delta/2$$

which is a contradiction. Thus any such procedure must lead to a finite set; call one such choice C_n . Then consider $C = \bigcup_n C_n$. This is a countable subset of X and by construction for any $x \in X$ and any n there exists $y_n \in C_n \subset C$ such that $d(x, y_n) \leq 1/n$ (since otherwise we could increase C_n) Thus x is in the closure of F , i.e. $\overline{F} = X$ or F is dense. Hence X is separable.

CHAPTER 2: PROBLEM 26

Suppose X is sequentially compact in the sense that every infinite subset of X has a limit point. We already know from Exercises 23 and 24 that X has a countable basis of open sets. That is, there is a countable collection of open sets \mathcal{B} such that any open set is a union of elements of \mathcal{B} . That is, given $U \subset X$ open define $S = \{B \in \mathcal{B}; B \subset U\}$ then $U = \bigcup_{B \in S} B$. Suppose

$$(2) \quad X = \bigcup_{a \in A} U_a$$

is an arbitrary cover of X by open sets. Since each U_a is a union of the sets from \mathcal{B} if we define

$$(3) \quad D = \{B \in \mathcal{B}; B \subset U_a \text{ for some } a \in A\}$$

we must have a subset of \mathcal{B} , hence countable (possibly finite of course). For each $B \in D$ we can choose an $a \in A$ such that $B \subset U_a$. Let $A' \subset A$ be a set chosen in this way, there is a surjective map from D to A' so A' is countable and by definition of D ,

$$(4) \quad X = \bigcup_{a \in A'} U_a.$$

Thus we have found a countable subcover of the original cover.

Now, replace A' by a labelling by integers, so we can write

$$(5) \quad X = \bigcup_{i=1}^{\infty} U_{a_i} \iff \bigcap_{i=1}^{\infty} (U_{a_i})^c = \emptyset.$$

Of course if A' is finite we are finished already. Otherwise, suppose that for each N

$$(6) \quad X \neq \bigcup_{i=1}^N U_{a_i} \iff F_N = \bigcap_{i=1}^N (U_{a_i})^c \neq \emptyset.$$

The F_N form a decreasing sequence of closed subsets of X . If they are all nonempty then we can choose a point from each, forming a set F . Either F is finite or else infinite. In the first case there is one point in F which is in F_N for arbitrarily large N hence in all the F_N since they are decreasing. In the second case F must have a limit point, x^* . Since, for each N , all but a finite subset of F is contained in F_N , x^* must be a limit point of each F_N , hence in each F_N (since they are closed). In either case this gives a point in $\bigcap_N F_N = \emptyset$ because of (5) this is a contradiction. Thus the F_N must be empty from some point onwards, giving us a finite subcover.

Remark: In lecture I did not go through the step of extracting the countable subcover, just used the cover given by D directly.

CHAPTER 2: PROBLEM 29

Suppose $O \subset \mathbb{R}$ is open. Since \mathbb{R} is separable, it contains a countable dense subset, for instance \mathbb{Q} . Take a surjection $\mathbb{N} \rightarrow O \cap \mathbb{Q}$ and write p_i for the image of i . Since O is open if $x \in O$ then $(x - \delta, x + \delta) \subset O$ for some $\delta > 0$ must contain at least one of the p_i . Consider successively each of the p_i . Again, there is an interval $(p_i - \epsilon, p_i + \epsilon) \subset O$. Now, if $[p_i, \infty) \not\subset O$ consider

$$(7) \quad A_i = \sup\{t \in \mathbb{R}; [p_i, t) \subset O\}.$$

Similarly if $(-\infty, p_i] \not\subset O$ consider

$$(8) \quad B_i = \sup\{t \in \mathbb{R}; [p_i, p_i - t) \subset O\}.$$

In all four cases we obtain an open interval $(p_i - B_i, p_i + A_i) \subset O$, possibly infinite in one direction or the other. By definition this interval is *maximal* in O and containing p_i . Another way of thinking about this interval is as the union of all intervals in O containing p_i and noting that the union of any collection of open intervals with a fixed point in common is an open interval. Now, drop the p_i 's from $O \cap \mathbb{Q}$ which are contained in one of the previous intervals and we have a countable (possibly finite) collection of disjoint intervals with union O (since each point of O is in an interval containing one of the p_i).