Eta invariant on articulated manifolds
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I want to talk today about manifolds with corners. This may come as no great surprise to many of you, but I suspect that I have not talked enough about their basic geometry and analysis.

In this talk I will concentrate on incomplete metrics and the corresponding Dirac operators.

In fact I will start by (roughly) stating two related conjectures. That there should be such conjectures is well-known but perhaps they have not often been stated precisely (and maybe for good reason ...).

I want to at least show you that the tools now exist to check whether these are true or not.

Maybe someone here would like to take up the challenge.
Conjecture (Eta invariant)

Let $Y$ be an odd-dimensional articulated manifold without boundary and suppose $\partial_0$ is an articulated Dirac operator on a unitary Clifford module, $V_0$, with respect to a smooth incomplete metric then $\partial_0 : H^1(Y; V_0) \rightarrow L^2(Y; V_0)$ is self-adjoint with discrete spectrum and the associated eta function and eta invariant are well-defined.

For this to make any sense I need to describe what

- An articulated manifold $Y$ is
- An articulated Dirac operator on it is
- Why it might be true.

The case that I do assert that this is true is when $Y$ has articulation of codimension one.
Conjecture (APS boundary condition)

Let $X$ be an even-dimensional manifold (with corners) and suppose $\bar{\partial}$ is a Dirac operator on a unitary ($\mathbb{Z}_2$-graded) Clifford module, $V$, with respect to a smooth incomplete metric then $\bar{\partial}_+$ induces an articulated Dirac operator $\bar{\partial}_0$ on $V_0 = V|_{\partial X}$ and

$$\bar{\partial}_+ : \left\{ u \in H^3_2(X; V_+); \Pi_+(\bar{\partial}_0)(u|_{\partial X}) = 0 \right\} \longrightarrow H^1_2(X; V_-)$$

is Fredholm with index given by

$$\text{ind}(\bar{\partial}_+) = \int_X \hat{A} \text{Ch}'(V) + R - \eta(\bar{\partial}_0).$$

Here $R$ is supposed to be the sum of integrals of a local differential expressions on the boundary faces. I believe this to be true in codimension two as I will explain below.
Basics

Manifolds (with corners)

Here is an extrinsic definition, correct but bad. Of course this is really a theorem, a properly defined manifold (with corners) can always be embedded in this sense.

Definition

An embedded compact manifold (with corners) $X$ is a closed subset of a compact manifold without boundary $M$ of the form

$$X = \{ p \in M; \rho_i(p) \geq 0 \ \forall \ i \in \{1, \ldots, N\}\}$$

where $\rho_i \in C^\infty(M)$ are real-valued functions such that for any $I \subset \{1, \ldots, N\}$ and any $p \in M$

$$\rho_i(p) = 0 \ \forall \ i \in I \implies \rho_i(p) \text{ are independent in } T^*_p M, \ i \in I.$$ 

An (incomplete) metric on $X$ is then by definition the restriction to $X$ of a metric on $M$. The same is true for bundles, differential operators etc.
Articulated manifolds

Here is a similar, perhaps even worse definition.

Definition

A compact articulated manifold without boundary is a (finite union of) component(s) of the boundary of a compact manifold.

- Again this is really a theorem, that an intrinsically defined articulated manifold can be embedded in this way.
- So an articulated manifold is really a finite collection of compact manifolds (with corners of course) with their boundary hypersurfaces identified and consistently in higher codimension.
- The absence of boundary is a completeness condition – there are no unmatched hypersurfaces.
- The important point is that an articulated manifold is a wobbly thing – there are no angles between boundary hypersurfaces or anything like that.
50 years ago – Atiyah and Singer

- For an even-dimensional compact manifold without boundary, a Dirac operator \( \partial_+ : C^\infty(X; V_+) \rightarrow C^\infty(X; V_-) \) is an elliptic differential operator of first order, so Fredholm:

\[
\text{Nul}(\partial_+) \subset C^\infty(X; V_+), \quad \text{Nul}(\partial_-) = (\text{Ran}(\partial_+))^\perp
\]

are finite-dimensional.

- The index is computable:

\[
\text{ind}(\partial_+) = \dim \text{Nul}(\partial_+) - \dim \text{Nul}(\partial_-) = \int_X \hat{A} \text{Ch}'.
\]

- In fact in this form, with the twisting Chern character of the Clifford module, the index theorem is due to Berline, Getzler and Vergne[4].
35 years ago – Atiyah, Patodi and Singer

For a Dirac operator on an odd-dimensional compact manifold, the eta invariant, is well-defined in terms of the heat kernel by

$$
\eta(\bar{\partial}_0) = \frac{1}{\sqrt{\pi}} \int_0^\infty \text{Tr} \left( t^{-\frac{1}{2}} \bar{\partial}_0 \text{ext}(-it\bar{\partial}_0^2) \right) dt.
$$

A Dirac operator on a compact even-dimensional manifold with boundary induces a self-adjoint Dirac operator on the boundary; let $\Pi_+(\bar{\partial}_0)$ be the projection onto its positive part.

The operator with APS boundary condition

$$
\bar{\partial}_+ : \{ u \in C^\infty(X; V_+); \Pi_+(\bar{\partial}_+)(u|_{\partial X}) = 0 \} \longrightarrow C^\infty(X; V_-)
$$

is Fredholm with index

$$
\text{ind}_{\text{APS}}(\bar{\partial}_+) = \int \hat{A} \text{Ch'} + R - \eta(\bar{\partial}_0).
$$

If the operator is a product to first order at the boundary, $R = 0$. 
Calderón’s sequence

- The work of Calderón on boundary problems gives a very clean approach to understanding the APS theorem.

- Suppose given a linear, elliptic differential operator with smooth coefficients on a compact manifold with boundary

\[ D : C^\infty(X; V_+) \to C^\infty(X; V_-). \]

- I will assume that all bundles carry inner products and that a metric has been chosen.

- In particular, \( D \) has a formal adjoint

\[ D^* : C^\infty(X; V_-) \to C^\infty(X; V_+). \]

- Let \( \dot{C}^\infty(X; V) \subset C^\infty(X; V) \) be the closed subspace of elements which vanish in Taylor series at the boundary then

\[
\text{Nul}(D; C^\infty) \to C^\infty(X; V_+) \overset{D}{\to} C^\infty(X; V_-) \to \text{Nul}(D^*; \dot{C}^\infty)
\]

is exact with \( \text{Nul}(D^*; \dot{C}^\infty) = \text{Nul} \left( D^* : \dot{C}^\infty(X; V_-) \to \dot{C}^\infty(X; V_+) \right) \).
Calderón projector

- The null space of the restriction to the boundary of smooth solutions in the interior is finite dimensional
  \[ \text{Nul}(D; \mathcal{C}^\infty) \longrightarrow \text{Nul}(D; \mathcal{C}^\infty) \xrightarrow{|\partial X|} \mathcal{C}^\infty(\partial X; V_+). \]

Calderón showed that there is a projection precisely onto the range of this restriction which is a pseudodifferential operator

\[ \Pi_C \in \psi^0(\partial X; V_+), \quad \Pi_C : \mathcal{C}^\infty(\partial X; V_+) \longrightarrow \text{Nul}(D; \mathcal{C}^\infty)\big|_{\partial X}. \]

- For instance this is the case for the self-adjoint projection with respect to a choice of metrics and inner products.
- For any choice,

\[ \text{Ran}(\sigma_0(\Pi_C)) = \text{Ran}_+(\sigma_1(D_0)), \]

the range of the symbol is always the span of the generalized eigenvectors of the symbol of \( D_0 \) in the right half plane where

\[ D = N(\partial_x - iD_0) \at \partial X; \quad D_0 \in \text{Diff}^1(\partial X; V), \quad x = 0 \at \partial X. \]
Jumps formula – boundary case

- Consider the null space on extendible distributions on $M$
  \[
  \text{Nul}(D; \mathcal{C}^{-\infty}) = \{ u \in \mathcal{C}^{-\infty}(X; V_+); Du = 0 \},
  \mathcal{C}^{-\infty}(X; V_+) = \hat{\mathcal{C}}^{\infty}(X; V_+)'.
  \]

- Partial hypoellipticity up to the boundary implies that the restriction to the boundary is well-defined (as are higher normal derivatives),
  \[
  \text{Nul}(D; \mathcal{C}^{-\infty}) \ni u \mapsto Bu = u\big|_{\partial X} \in \mathcal{C}^{-\infty}(\partial X; V_+).
  \]

- The ‘jumps formula’ is also a consequence of this:- There is a unique $v \in \mathcal{C}^{-\infty}(X; V_+)$ such that
  \[
  v = 0 \text{ in } M \setminus x, \quad v = u \text{ on } X \setminus \partial X
  \]
  \[
  Pv = w\delta(\rho) \text{ and } w = -i\sigma(D)(d\rho)(Bu).
  \]
Now assume (for simplicity) that $D = \bar{\partial}_+$, is the restriction of a Dirac operator on the whole of $M \supset X$ and that $\bar{\partial} : C^\infty(M; V_+) \rightarrow C^\infty(M; V_-)$ is an isomorphism.

Then we get an explicit Calderón projector as

\[
C^\infty(\partial X; V_+) \ni v \xrightarrow{\Pi_C} \Pi_C v \in C^{-\infty}(\partial X; V_+)
\]

\[
-i\sigma(D)(d\rho)v \otimes \delta(\rho) \xrightarrow{\bar{\partial}^{-1}} C^{-\infty}(M; V_+) \xrightarrow{|x \setminus \partial X|} \text{Nul}_X(\bar{\partial}_+; C^{-\infty})
\]

In the general case one needs only do a little more work.
General case

I want to try to convince you of the existence of such a picture in the general case of a compact manifold with (non-trivial) corners.

The spaces $\mathcal{C}^\infty(X; V)$ with dual $\mathcal{C}^{-\infty}(X; V)$ and $\mathcal{C}^\infty(X; V)$ with dual $\mathcal{C}^{-\infty}(X; V)$ are well-defined (metrics everywhere) and in terms of an extension $X \subset M$

\[
\mathcal{C}^\infty[\text{resp } \mathcal{C}^{-\infty}](X; V) = \{ u \in \mathcal{C}^\infty[\text{resp } \mathcal{C}^{-\infty}](X; V); \text{supp}(u) \subset X \}
\]

\[
\mathcal{C}^\infty[\text{resp } \mathcal{C}^{-\infty}](X; V) = \mathcal{C}^\infty[\text{resp } \mathcal{C}^{-\infty}](M; V)|_{X \setminus \partial X}.
\]

So let $\partial_+: \mathcal{C}^\infty(X; V_+) \rightarrow \mathcal{C}^\infty(X; V_-)$ be a Dirac operator, this makes all the pesky finite-dimensional $\text{Nul}(\partial_\pm; \mathcal{C}^\infty)$ trivial.

In particular surjectivity holds

\[
\text{Nul}(\partial_+; \mathcal{C}^{-\infty}) \rightarrow \mathcal{C}^{-\infty}(X; V_+) \xrightarrow{D} \mathcal{C}^{-\infty}(X; V_-)
\]

So the whole issue is to define $B$ and $\Pi_C$. 
Although partial hypoellipticity fails we can still use a variant of the jumps formula to define $B$.

There is a surjective restriction map

$$\hat{\mathcal{C}}^{-\infty}(M; V) \longrightarrow \mathcal{C}^{-\infty}(M; V)$$

with null space the distributions supported by the boundary; $u \in \text{Nul}(\hat{\partial}_+; C^{-\infty})$ can be extended to $M$ to vanish outside $X$.

In fact there is always such a ‘zero extension’ $v \in \hat{\mathcal{C}}^{-\infty}(X; V_+)$ with

$$\hat{\partial}_+(v) = \sum_H v_H \otimes \delta(\rho_H), \quad v_H \in \hat{\mathcal{C}}^{-\infty}(H; V_-)$$  \hspace{1cm} (1)

Here, each boundary hypersurface $H$ has a defining function $\rho_H$ and the space on the right is a well-defined in $\hat{\mathcal{C}}^{-\infty}(X; V_-)$.

However, there are two problems, the zero extension – even with this property – is not unique and nor are the ‘boundary values’ $v_H$ (even fixing the $\rho_H$ which we can. So the presentation (1) is also not unique; the crucial question is just how non-unique.
Formal boundary data

To answer this we now switch to the ‘formal smooth theory’.

Think of $\partial X$ as an articulated manifold – the union of the boundary hypersurfaces with only their boundaries identified in the obvious way. Then the ‘smooth’ sections of a bundle over $\partial X$ are

$$C^\infty(\partial X; V) = \left\{ u_i \in C^\infty(H_i; V); u_i|_{H_i \cap H_j} = u_j|_{H_i \cap H_j} \right\} = C^\infty(M; V)|_{\partial X}.$$ 

As remarked above, this space is ‘too big’ in the sense that there are no compatibility conditions for the normal derivatives at intersections of boundary faces.

However, a first order elliptic differential operator, gives rise to much smaller subspace of ‘compatible’ sections

$$C_D^\infty(\partial X; V_+) = \left\{ u \in C^\infty(X; V_+); Du \in \dot{C}^\infty(X; V_+) \right\}|_{\partial X} \subset C^\infty(Y; V_+).$$
Properties of $C_D^\infty$. 

Lemma

For an elliptic differential operator on a compact manifold (with corners) $D \in \text{Diff}^1(X; V_+, V_-)$ restriction to any one of the of the boundary hypersurfaces defines a surjective map

$$C_D^\infty(\partial X; V_+) \xrightarrow[H]{} C^\infty(H; V_+), \ H \in \mathcal{M}_1(M),$$

and there is a natural extension giving an injective map

$$\bigoplus_{H \in \mathcal{M}_1(M)} \hat{C}^\infty(H; V_+) \hookrightarrow C_D^\infty(\partial X; V_+). \quad (2)$$
Note that $\partial X$ can be ‘smoothed’ (more like annealed!) to a compact manifold without boundary

$$\tilde{H} = \{ p \in X; \prod_{\mathcal{H}} \rho_{\mathcal{H}} = \epsilon \}, \ \epsilon > 0 \ small.$$  

Then $C^\infty_D(\partial X; V_+)$ ‘looks’ like $C^\infty(\tilde{H}; V)$ in the sense that the Taylor series at any boundary point coming from one boundary hypersurface determines the Taylor series at any others.

This new space is not a module of $C^\infty(\partial X)$.

On the other hand, it does have a topology very similar to that of $C^\infty(\tilde{H}; V)$ such that the maps in (2) are continuous.

The dual space $C^{-\infty}(\partial X; V_+)$ is similar to $C^{-\infty}(\tilde{H}; V_+)$. 
Properties of $\mathcal{C}_D^{-\infty}$.

Lemma

The topological dual $\mathcal{C}_D^{-\infty}(\partial X; V_+)$ comes equipped with a natural surjection to extendible distributions on the boundary hypersurfaces

$$\mathcal{C}_D^{-\infty}(\partial X; V_+) \longrightarrow \bigoplus_{H \in \mathcal{M}_1(X)} \mathcal{C}^{-\infty}(H; V_+)$$

and injections on supported distributions for each $H \in \mathcal{M}_1(M)$

$$\hat{\mathcal{C}}^{-\infty}(H; V_+) \hookrightarrow \mathcal{C}_D^{-\infty}(\partial X; V_+)$$

such that the collective map is surjective

$$[\cdot] : \bigoplus_{H \in \mathcal{M}_1(X)} \hat{\mathcal{C}}^{-\infty}(H; V_+) \longrightarrow \mathcal{C}_D^{-\infty}(\partial X; V_+).$$
This space answers the question of just how well-defined the boundary data for the null space of an elliptic operator on a compact manifold with corners is where now we have a boundary pairing which gives

\[ \mathcal{C}_D^{-\infty}(\partial X; V_+) = (\mathcal{C}_D^{\infty}(\partial X; V_-))'. \]

**Theorem**

*With the global hypotheses above on the first order elliptic differential operator \( D \), there is a well-defined injective boundary map \( B \) giving a commutative diagram*

\[
\begin{align*}
\text{Nul}(D; \mathcal{C}^{-\infty}) & \xrightarrow{B} \mathcal{C}_D^{-\infty}(\partial X; V_+) \\
\left\{ v \in \dot{\mathcal{C}}^{-\infty}(X; V_+), \quad \bar{\partial}_+ v = \sum_H -i\sigma(D)(d\rho_J)w_H \otimes \delta(\rho_H) \right\} & \mapsto [w_H]
\end{align*}
\]
Calderón projector, corners case

This in turn allows us to define the Calderón projector as in the case of a manifold with boundary except for the extra algebraic overhead

\[ \Pi_C : \mathcal{C}^{-\infty}_D(\partial X; V_+) \to \mathcal{C}^{-\infty}_D(\partial X; V_+) \text{ by } \]

\[ \Pi_C([w_H]) = B \left( D^{-1} \left( \sum_H -i\sigma(\partial \rho_H)w_H\delta(\rho_H) \right) \big|_X \right). \]

**Theorem**

The Calderón projector is a continuous projection on \( \mathcal{C}^{-\infty}_D(\partial X; V_+) \) and has range precisely equal to the range of \( B \) which maps \( \text{Nul}(D; \mathcal{C}^{-\infty}) \) injectively into \( \mathcal{C}^{-\infty}_D(\partial X; V_+) \).
This Calderón projector is as close to being a pseudodifferential operator as one could expect on an articulated manifold. Namely, it consists of pseudodifferential operators on each of the hypersurfaces plus ‘Poisson’ type operators between them. In particular, it preserves $C^\infty_D(\partial X; V_+)$, even though the pseudodifferential pieces do not satisfy the transmission condition. The singularities are cancelled by the Poisson pieces. These results should extend to the general case where $D$ is not assumed to either have the extension property or the unique continuation property. The extension to higher order systems would be a more serious pain!
Continuing under the global assumptions, observe that for $t \in \mathbb{R}$, $|t| < \frac{1}{2}$, and on any compact manifold with corners, the extendible and supported Sobolev spaces are identified

$$\dot{H}^t(H; V) = (H^{-t}(H; V))' \equiv H^t(H; V), \quad -\frac{1}{2} < t < \frac{1}{2}.$$ 

That is, each element of these Sobolev spaces has a unique zero extension with the same regularity (with which it can therefore be identified).

In view of the properties of the spaces discussed above it follows that

$$\bigoplus_{H \in \mathcal{M}_1(X)} H^t(H; V_-) \subset \mathcal{C}^{-\infty}(\partial X; V_+), \quad -\frac{1}{2} < t < \frac{1}{2}$$

are well-defined subspaces for any elliptic first-order $D$. 
The regularity properties of $D^{-1}$ show that that

$$\Pi C \text{ acts on } \bigoplus_{H \in M_1(M)} H^t(H; V_+), \quad -\frac{1}{2} < t < \frac{1}{2},$$

with range precisely the boundary restrictions of

$$\text{Nul}_s(D) = \{ u \in H^s(X; V_+); Du = 0 \}, \quad s = t + \frac{1}{2}.$$

Thus, for instance, for $\frac{1}{2} < s < 1$ there is a short exact sequence

$$\{ U \in H^{s-\frac{1}{2}}(\partial X; V_+); \Pi C U = U \} \longrightarrow H^s(X; V_+) \overset{D}{\longrightarrow} H^{s-1}(H; V_-).$$

where the first map is a Poisson operator.
For Dirac operators ‘restriction’ to a boundary hypersurface is functorial - giving a Dirac operator $\bar{\partial}_H$ on each $H \in \mathcal{M}_1(X)$.

This involves the product decomposition near a hypersurface in terms of the distance, in which the metric decomposes as

$$g = dx^2 + x^2 h(x), \ h(x) \text{ a family of metric on } H.$$

There is no (simple) analogue of this in codimension two.

Nevertheless the double restriction, from $\bar{\partial}_+$ on $X$ to a boundary face of codimension two is consistent (with change of orientation)

$$(\bar{\partial}_H)_{H \cap G} + (\bar{\partial}_G)_{H \cap G} = 0. \quad (1)$$

This is what is meant above by a Dirac operator on an articulated manifold – on each boundary hypersurface there is a Dirac operator $\bar{\partial}_H$ associated to a metric and a Clifford module (and unitary Clifford connection). The bundles and metrics must be consistent on the intersection faces of codimension two – from either side one gets the same restriction – and the Clifford modules are consistent in the sense of (1).
This is enough to give sense to the ‘Eta invariant’ conjecture.

**Conjecture (Eta invariant)**

Let $Y$ be an odd-dimensional articulated manifold without boundary and suppose $\partial_0$ is an articulated Dirac operator on a unitary Clifford module, $V_0$, with respect to a smooth incomplete metric then $\partial_0 : H^1(Y; V_0) \to L^2(Y; V_0)$ is self-adjoint with discrete spectrum and the associated eta function and eta invariant are well-defined.

- I claim this is true for an articulated manifold with intersection faces only of codimension one – this is close to the boundary case.
- One can get a parametrix, in the sense of an inverse modulo compact errors by summing the generalized inverse of the APS problem on each boundary hypersurface (there is an odd/even switch here).
- In particular the projection onto the positive part makes sense.
Conjecture (APS boundary condition)

Let $X$ be an even-dimensional manifold (with corners) and suppose $\bar{\partial}$ is a Dirac operator on a unitary ($\mathbb{Z}_2$-graded) Clifford module, $V$, with respect to a smooth incomplete metric then $\bar{\partial}_+$ induces an articulated Dirac operator $\bar{\partial}_0$ on $V_0 = V|_{\partial X}$ and

$$\bar{\partial}_+ : \left\{ u \in H^3_2(X; V_+); \Pi_+(\bar{\partial}_0)(u|_{\partial X}) = 0 \right\} \rightarrow H^1_2(X; V_-)$$

is Fredholm with index given by

$$\text{ind}(\bar{\partial}_+) = \int_X \hat{\text{A}} \text{Ch}'(V) + R - \eta(\bar{\partial}_0).$$

The existence of $\Pi_+$ follows from the discussion above in case $X$ has boundary of codimension two. The Fredholm property should follow from a symbolic analysis of the two projections, Calderón and APS.
Final remarks

- A lot of this is conjectural, but the case of $X$ of codimension two is surely within reach.
- There is the possibility of induction over boundary codimension.
- If this is all too easy for you, try the ‘annealing limit’ as $\epsilon \downarrow 0$, passing from a manifold with boundary to the general case.
- I have not given references but there is a large literature related to this subject – but not the Calderón projector as far as I know.

