8. Topic 2: Higher dimensional harmonic oscillator In place of lecture for Monday, 15 September

Carry through the discussion of the higher-dimensional isotropic smoothing operators, forming the algebra $\Psi^{-\infty}(\mathbb{R}^n)$, the associated group $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ and corresponding loop groups. Similarly, for any compact manifold X, for the moment without boundary, discuss $\Psi^{-\infty}(X)$, $G^{-\infty}(X)$ and $\tilde{G}^{-\infty}_{\text{sus}}(X)$ etc.

Here are some steps to help you along the way.

(1) Show that $\mathcal{S}(\mathbb{R}^{2n})$ becomes a non-commutative Fréchet algebra which will be denoted $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$, with continuous product given by operator composition as in the 1-dimensional case

$$a \circ b(z, z') = \int_{\mathbb{R}^n} a(z, z'')b(z'', z')dz''. \tag{8.1}$$

(2) Discuss the higher dimensional harmonic oscillator using the n creation and annihilation operators

$$\begin{aligned} &C_j = -\partial_{z_j} + z_j,\ A_j = C_j^* = \partial_{z_j} + z_j, \\ &\qquad H = H(n) = \sum_{j=1}^{n} C_j A_j + n,\ [A_j, C_j] = 2,\ j = 1, \dots, n. \end{aligned} \tag{8.2}$$

Show that H has eigenvalues $n + 2\mathbb{N}_0$ with the dimension of the eigenspace with eigenvalue $n + 2k$ equal to the dimension of the space of homogeneous polynomials of degree k in n variables.

(3) Compute the constants such that the functions

$$h_0 = c_0 \exp(-|z|^2/2),\ h_\alpha = c_\alpha C^\alpha h_0,\ \alpha \in \mathbb{N}_0^n \tag{8.3}$$

is orthonormal in $L^2(\mathbb{R}^n)$ and show that they form a complete orthonormal basis.

(4) Show that for any $u \in \mathcal{S}(\mathbb{R}^n)$ the Fourier-Bessel series

$$f = \sum_\alpha \langle f, h_\alpha \rangle h_\alpha \tag{8.4}$$

converges in $\mathcal{S}(\mathbb{R}^n)$ and that this gives an isomorphism

$$\mathcal{S}(\mathbb{R}^n) \longrightarrow \{\{c_\alpha\}; \sup_\alpha |\alpha|^N |c_\alpha| < \infty,\ \forall\ N \in \mathbb{N}\},\ |\alpha| = \sum_j \alpha_j. \tag{8.5}$$

(5) Show, either directly or by discussing the appropriate 'higher dimensional' versions of $\Psi^{-\infty}(\mathbb{N})$ based on sequences as in (8.5), that $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ is topologically isomorphic to the algebra $\Psi^{-\infty}(\mathbb{N})$.

(6) Briefly describe and discuss the group $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$.

(7) Introduce the (higher, pointed, flat) loop groups of $G^{-\infty}_{\text{sus}(k),\text{iso}}(\mathbb{R}^n)$.

(8) Show that

$$\text{tr}(a) = \int_{\mathbb{R}^n} a(z, z)dz \tag{8.6}$$

is the trace functional on $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$.

(9) Can you show that it is unique up to a constant multiple as a continuous linear functional which vanishes on commutators?

(10) See how everything else we have done so far looks in this setting!

(11) Extend these results further to any compact manifold, using the eigendecomposition for the Laplacian. I will come back to this and disuss it more seriously later.