5. Lecture 5: Harmonic oscillator Monday, 8 September, 2008

I have not really talked so far about the topology on the loop spaces. I hope to get to this today, or at least do the preparation for it, and also consider the first 'geometric form' of $G^{-\infty}$, namely the 'isotropic smoothing algebra', or Schwarz algebra, of operators on $\mathbb{R}$.

- Schwartz space
- Harmonic oscillator
- Creation and annihilation operators
- Eigenfunctions
- Hermite polynomials
- Completeness
- Convergence of eigenseries
- The algebra $\Psi^{-\infty}(\mathbb{R})$ and group $G^{-\infty}(\mathbb{R})$.
- Loop groups again.

I will assume that you are somewhat familiar with the Schwartz space $\mathcal{S}(\mathbb{R})$, but let me remind you of the definition and basic properties. In fact we might as well consider $\mathcal{S}(\mathbb{R}^n)$ for any n.

So, $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$ consists of all the (complex-valued) smooth functions of rapid decay, meaning that all the norms

$$\|u\|_{p,\infty} = \sup_{z\in\mathbb{R}^n,\ 0\le|\alpha|\le p} (1+|z|^2)^{\frac{p}{2}}|D^\alpha u(z)| < \infty \tag{5.1}$$

are finite. Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, so $\alpha_i \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$

$$D_z^\alpha u = i^{-|\alpha|}\frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}} u(z),\ |\alpha| = \alpha_1 + \cdots + \alpha_n \tag{5.2}$$

where the powers of i are there for reasons to do with formal self-adjointness. This sequence of norms is just like those considered on sequences above. Just as there, $\mathcal{S}(\mathbb{R}^n)$ is a complete metric space with the distance

$$d(u,v) = \sum_p 2^{-p}\frac{\|u\|_{p,\infty}}{1+\|u\|_{p,\infty}} \tag{5.3}$$

where convergence of a sequence with respect to this distance means exactly the same as convergence with respect to each of the norms $\|u\|_{p,\infty}$ (with no uniformity in p). The dual space, the space of continuous linear maps

$$U : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}, \tag{5.4}$$

is the space of tempered (or temperate) distributions, $\mathcal{S}'(\mathbb{R}^n)$. There is a natural inclusion, almost always treated as an identification

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n),\ u \longmapsto U_u : \mathcal{S}(\mathbb{R}^n) \ni f \longrightarrow \int_{\mathbb{R}^n} u(x)f(x)dx. \tag{5.5}$$

Since it is treated as an identification we normally write $U_u = u$.

Now consider the algebra

$$\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}) = \mathcal{S}(\mathbb{R}^2) \tag{5.6}$$

where the product is

$$ab(x,x') = \int_{\mathbb{R}} a(x,x'')b(x'',x')dx''. \tag{5.7}$$

These are the Schwartz smoothing operators on $\mathbb{R}$. They act on $\mathcal{S}(\mathbb{R})$ in the obvious way, as integral operators

$$a : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}),\ (au)(x) = \int_{\mathbb{R}} a(x,x')u(x')dx'. \tag{5.8}$$

Then (5.7) is operator composition.

The spectral theory of the harmonic oscillator

$$H = -\frac{d^2}{dx^2} + x^2 \tag{5.9}$$

on the line can be discussed in an essentially algebraic way. This is based on the two first order operators,

$$A = \frac{d}{dx} + x \text{ and } C = -\frac{d}{dx} + x, \tag{5.10}$$

respectively the annihilation and creation operator. The idenitites

$$H = CA + 1,\ [A,C] = 2[\frac{d}{dx}, x] = 2,\ Ae^{-\frac{x^2}{2}} = 0 \tag{5.11}$$

are easily checked. Since

$$\int_{\mathbb{R}} (e^{-\frac{x^2}{2}})^2 dx = \int e^{-x^2} dx = \sqrt{\pi} \tag{5.12}$$

the function

$$h_1 = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \tag{5.13}$$

has norm 1 in $L^2(\mathbb{R})$ and satisfies

$$Hh_1 = h_1. \tag{5.14}$$

This is the ground state of the harmonic oscillator. The higher eigenfunctions are obtained by applying the creation operator. Thus

(5.15)

$$C^k h_1(x) \text{ satisfies}$$
$$H(C^k h_1) = C^k h_1 + CAC^k h_1 = C^k h_1 + 2C^k h_1 + C^2 AC^{k-1} h_1 = (1+2k)C^k h_1$$

as follows from (5.11) by induction. Moreover it also follows inductively that

$$C^k h_1(x) = (2^k x^p + q_{k-1})(x))h_1(x) \tag{5.16}$$

where q_{k-1} is a polynomial of degree at most $k-1$. Certainly, $C^k h_1 \in \mathcal{S}(\mathbb{R})$. The L^2 norm can be computed by integration by parts using the fact that A and C are adjoints of each other

$$\int_{\mathbb{R}} (C^k h_1(x))^2 dx = \int_{\mathbb{R}} h_x(x) A^k C^k h_1(x) dx = 2^k k!. \tag{5.17}$$

Moreover $C^k h_1$ and $C^l h_1$ are orthogonal in L^2 by a similar argument and hence the

$$h_k = 2^{-\frac{k}{2}}(k!)^{\frac{1}{2}} C^k h_1,\ k = 0, 1, 2, \ldots \tag{5.18}$$

form an orthonormal sequence of eigenfunctions of H.

In fact this is a complete orthonormal basis of $L^2(\mathbb{R})$. To see this observe from (5.16) that the span of the first k elements consists of all the products $q(x)e^{-\frac{x^2}{2}}$ where q is a polynomial of degree at most k. In particular if $u \in L^2(\mathbb{R})$ then

$$\int u(x)h_k(x)dx = 0 \ \forall \ k \Longleftrightarrow \int u(x)x^k e^{-\frac{x^2}{2}}dx = 0 \ \forall \ k. \tag{5.19}$$

Taking the Fourier transform and using Plancherel's formula and the fact that the Fouier transform of h_1 is a multiple of itself, (5.19) is equivalent to

$$\frac{d^k}{d\tau^k}v(0) = 0 \ \forall \ k, \ v(\tau) = \int e^{-ix\tau}u(x)\exp(-\frac{x^2}{2})dx. \tag{5.20}$$

Now v is entire, since the integral defining it is absolutely convergent for all $\tau \in \mathbb{C}$. It follows that $v \equiv 0$ and hence $u \equiv 0$ by Fourier inversion. This shows that the h_k form a complete orthonormal basis of $L^2(\mathbb{R})$.

Lemma 6. *If $u \in \mathcal{S}(\mathbb{R})$ it follows that the Fourier-Bessel expansion of u in terms of the h_k converges in $\mathcal{S}(\mathbb{R})$:*

$$u(x) = \sum_{k=1}^{\infty} c_k h_k, \ c_k = \int h_k(x)u(x) \Longrightarrow \sum_k k^p|c_k| < \infty \ \forall \ p. \tag{5.21}$$

Proof. This follows from Stirling's formula

$$k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k. \tag{5.22}$$

which implies the existence of positive constants R, c and C such that

$$cR^k k^{k+\frac{1}{2}} \le 2^k k! \le CR^k k^{k+\frac{1}{2}}. \tag{5.23}$$

Fix $p \in \mathbb{N}$. Then $h_{p+k} = \mu(k,p)C^p h_k$ so integrating by parts from the definition of the coefficients in (5.21),

$$c_{p+k} = \mu(k,p)\int_{\mathbb{R}} h_k(x)A^p u(x)dx. \tag{5.24}$$

Now, in terms of the seminorms on $\mathcal{S}(\mathbb{R})$,

$$|A^p u(x)| \le 2^p(1+|x|)^{-2}\|u\|_{p+2,\infty} \tag{5.25}$$

where the extra factor is to ensure integrability. Thus

$$|c_{p+k}| \le \mu(k,p)2^p\|u\|_{p+2,\infty}. \tag{5.26}$$

Combining (5.23) and (5.26) it follows that

$$|c_{p+k}| \le C_p k^{-p/2}\|u\|_{p,\infty}. \tag{5.27}$$

Thus the coefficients decrease rapidly.

Estimating directly it also follows that

$$\|h_k\|_{p,\infty} \le C_p k^{p/2+1} \tag{5.28}$$

so the sequence does indeed converge in $\mathcal{S}(\mathbb{R})$. $\square$

Proposition 9. *The map*

$$\Psi^{-\infty}(\mathbb{N}) \ni a_{ij} \longmapsto \sum_{ij} a_{ij}h_i(x)h_j(x') \in \Psi^{-\infty}(\mathbb{R}) \tag{5.29}$$

is an isomorphism.

Proof. This requires the same sort of argument as in the previous proof, but now applied in both variables. □

So everything I have said for $\Psi^{-\infty}(\mathbb{N})$ carries over to $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R})$ and we can define the group $G^{-\infty}_{\mathrm{iso}}(\mathbb{R})$ which is similarly isomorphic, as a topological group and by a smooth isomorphism, to $G^{-\infty}(\mathbb{N})$. The trace functional is the integral over the diagonal

$$\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}) \ni a \longmapsto \mathrm{tr}(a) = \int_{\mathbb{R}} a(x,x)dx. \tag{5.30}$$

Thus the Chern character forms look the same as before but now involve lots of integrals instead of sums.

Now, if I get this far, the loop group on $\Psi^{-\infty}(\mathbb{N})$ can also be written is 'Schwartz form'. Namely we can take an isomorphism

$$(0,2\pi) \longrightarrow \mathbb{R},\ T(\theta) = \arctan((\theta-\pi)/2) \tag{5.31}$$

which identifies smooth functions on $[0,2\pi]$ which vanish with all their derivatives at the end points with $\mathcal{S}(\mathbb{R})$. Basically only the 'polynomial' behaviour of T at 0 and 2π (and the fact that it is a diffeomorphism of the open sets of course) is important here.