## 4. Lecture 4: Delooping and Chern forms
## Friday, 5 September, 2008

(1) Contractibility of $\tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{N})$.
(2) Odd Chern forms.
(3) Even Chern forms – only started.
(4) Transgression under delooping; next time.

Last time I defined two versions of the loop group on $G^{-\infty}(\mathbb{N})$ and discussed the delooping sequences. The central group in the sequence consists of open loops. Identifying the circle with the quotient of the interval $[0,2\pi]$ there is in fact no contuity condition corresponding to $0=2\pi$ so this group can be written as

$$\tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{N}) = \{b : [0,2\pi]_s \longrightarrow G^{-\infty}(\mathbb{N}); b(0)=\mathrm{Id},\ \frac{d^kb}{dt^k}(0)=0,\ \frac{d^kb}{dt^k}(2\pi)=0\ \forall\ k\geq 1\}. \tag{4.1}$$

Thus, the value at the end, $s=2\pi$, of the loop is 'free' but the curve is required to be flat there and also to be flat as it approaches Id at $s=0$.

**Proposition 7.** *The is a smooth global retraction*

$$\begin{gathered} R : [0,1]_t \times \tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{N}) \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{N}), \\ R(1,b)=b,\ R(0,b)=\mathrm{Id}\ \ \forall\ b\in\tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{N}). \end{gathered} \tag{4.2}$$

*Proof.* The idea is to simply shorten the curves but we need to be a little careful in order to maintain the flatness conditions. First choose two smooth functions

$$\begin{gathered} \psi_i : [0,1] \longrightarrow [0,1], i=0,1 \text{ with} \\ \psi_1(1)=1 \text{ in } s>3/4,\ \psi_1(s)=0 \text{ in } s<1/2, \\ \psi_0(s)=0 \text{ in } s>3/4 \text{ and } s<1/4,\ \psi_0(s)+\psi(s)=1 \text{ in } s\geq 1/2. \end{gathered} \tag{4.3}$$

Then consider a smooth function $f:[0,2\pi]\longrightarrow[0,2\pi]$ with $f(s)=0$ in $s>1/20$, $f(s)=2\pi$ in $s>2\pi-1/20$ and define

$$\chi : [0,1]_t \times [0,2\pi]_s \longrightarrow [0,2\pi] \text{ by } \chi(t,s)=f(s)\psi_0(t)+s\psi_1(t). \tag{4.4}$$

Clearly $\chi(0,s)=0$, $\chi(1,s)=s$ for all $s\in[0,2\pi]$. Moreover, $\chi(t,0)=0$ forall $t$, $\chi(t,2\pi)=2\pi$, for $t\geq 1/2$ and for $t\leq 1/2$,

$$\frac{d^k\chi}{ds^k}(t,2\pi)=0\ \forall\ k>1. \tag{4.5}$$

Then the desired homotopy is given by

$$R(t,a)(s)=a(\chi(t,s))\in\tilde{G}^{-\infty}(\mathbb{N}) \tag{4.6}$$

where the flatness at $s=0$ follows from the flatness of $a$ at $s=0$ and the flatness at $s=2\pi$ follows from that of $a$ for $t\geq 1/2$ and from that of $\chi$ for $t\leq 1/2$. Thus (4.2) follows, proving the Proposition. □

Now, let me turn, or return, to the Chern forms. As in a Lie group, the canonical map $g : G^{-\infty}(\mathbb{N}) \longrightarrow \Psi^{-\infty}(\mathbb{N})$ which embeds the group as an open dense subset of the algebra trivializes the tangent bundle to the group, so we can identify

$$dg : TG^{-\infty} = G^{-\infty}\times\Psi^{-\infty}(\mathbb{N}). \tag{4.7}$$

The 'name' chosen for this identification, $dg$, is supposed to be suggestive but can be confusing. Really the '$g$' here just tells you at which point of the group you are supposed to be and the '$d$' indicates the identification of tangent spaces. However, it does give magical formulæ which fortunately are correct.

So, the higher tensor spaces, multi-tangent bundles, are just the formal tensor products. This means that if we want to have a cotensor at a point of $G^{-\infty}$ it will just be a continuous multilinear map

$$\Psi^{-\infty}(\mathbb{N}) \times \Psi^{-\infty}(\mathbb{N}) \cdots \Psi^{-\infty}(\mathbb{N}) \longrightarrow \mathbb{C}. \tag{4.8}$$

The continuity of such multilinear maps automatically generates a completed tensor product of the dual spaces, so we do not have to worry about formalizing this at the moment. In short then a k-form on $G^{-\infty}$ should be a smooth map

$$G^{-\infty}(\mathbb{N}) \times \Psi^{-\infty}(\mathbb{N}) \times \Psi^{-\infty}(\mathbb{N}) \cdots \Psi^{-\infty}(\mathbb{N}) \longrightarrow \mathbb{C} \tag{4.9}$$

which is linear in each of the last $k$ factors and which is totally antisymmetric in these. Before worrying too much about differentials etc, let's just check that we can manufacture some.

The simplest forms one could think of would be those 'independent' of the first factor in (4.9) – although such independence is illusionary since the trivialization of the tangent bundle introduces a degree of twisting. Thus, since we have the product in the algebra and the trace functional at our disposal we can just consider (4.10)

$$\Psi^{-\infty}(\mathbb{N}) \times \Psi^{-\infty}(\mathbb{N}) \cdots \Psi^{-\infty}(\mathbb{N}) \ni (b_1, \dots, b_k) \longmapsto \sum_{\Sigma_k \ni i} \operatorname{sgn}(i) \operatorname{tr}(b_{i_1} b_{i_2} \dots b_{i_k}).$$

Here I have explicitly introduced the exterior product by antisymmetrizing in all the variables. So, at the identity of the group this can be written

$$\operatorname{tr}(dg \wedge dg \wedge \dots dg)(b_1, \dots, b_k) \text{ at } g = \mathrm{Id} \in G^{-\infty}(\mathbb{N}). \tag{4.11}$$

This does indeed define a global form on $G^{-\infty}$ but not a very interesting one as it turns out. Rather we need to introduce factors of $g^{-1}$ to map everything back to the identity. So we consider the $k$-form

$$\operatorname{tr}(g^{-1}dg \wedge \dots g^{-1}dg) = \operatorname{tr}\left((g^{-1}dg)^k\right). \tag{4.12}$$

Written out at any point on the group it just looks like (4.10) with $g^{-1}$'s inserted between the factors and then antisymmetrized in the tangent variables. Clearly (4.12) is a rather simpler formula, especially when we suppress the wedge product as well!

Now, from antisymmetry alone this form vanishes identically if $k$ is even. You can think of this as 'moving' the first factor to last – which is okay because of the properties of the trace – but in doing so one has to pass over an odd number of terms each of which reverses the sign, so overall it is equal to its negative. Thus we only consider the odd case and write

$$\mathrm{Ch}^{\mathrm{odd}}_{2k+1} = \mathrm{Ch}_{2k+1} = \operatorname{tr}\left((g^{-1}dg)^{2k+1}\right), \ k = 0, 1, 2, \dots. \tag{4.13}$$

The 'odd' here is redundant, since the forms are only in odd degree anyway.

Note that the insertion of the factors of $g^{-1}$ makes this form left-invariant. That is, consider the map from $G^{-\infty}$ to itself given by multiplication on the left by $h \in G^{-\infty}(\mathbb{N})$, fixed but arbitrary. Ther is of course a similar right multiplication map, which conventionally has an inverse inserted

$$\begin{aligned} L_h &: G^{-\infty}(\mathbb{N}) \longrightarrow G^{-\infty}(\mathbb{N}), \ g \longmapsto hg \\ R_h &: G^{-\infty}(\mathbb{N}) \longrightarrow G^{-\infty}(\mathbb{N}), \ g \longmapsto gh^{-1} \end{aligned} \tag{4.14}$$

are both global diffeomorphisms.

**Lemma 4.** *All the (odd) Chern forms in* (4.13) *are are bi-invariant.*

*Proof.* Trivial enough. Namely $(hg)^{-1} = g^{-1}h^{-1}$ and $d(hg) = hdg$. Thus even the product $g^{-1}dg = (hg)^{-1}d(hg)$ is left-invariant. On the other hand under the right action, $R_h^*(g^{-1}dg) = h(g^{-1}dg)h^{-1}$. Thus the forms are obviously left-invariant and right-invariance follows from the invariance properties of the trace:

$$R_h^* \operatorname{Ch}_{2k+1} = \operatorname{tr}\left(h(g^{-1}dg)^{2k+1}h^{-1}\right) = \operatorname{Ch}_{2k+1}. \tag{4.15}$$

□

Most importantly of all, of course, is that

**Lemma 5.** *The (odd) Chern forms are closed.*

*Proof.* The operator $d$ is perfectly well-defined as in the finite-dimensional case. Let me just leave this as an exercise for the moment! In fact $d$ makes good sense on smooth forms valued in any vector space, such as $\Psi^{-\infty}(\mathbb{N})$. Thus we see,

$$g^{-1} : G^{-\infty}(\mathbb{N}) \longleftrightarrow G^{-\infty}(\mathbb{N}),\ dg^{-1} = -g^{-1}dgg^{-1}. \tag{4.16}$$

As usual, this just follows by differntiating the identity $g^{-1}g = \operatorname{Id}$. Thus the product $g^{-1}dgg^{-1}$ is closed, in fact is exact. Similarly of course, $dg$ is closed – being the differential of a linear map. So,

$$\operatorname{tr}\left((g^{-1}dg)^{2k+1}\right) = \operatorname{tr}\left(g^{-1}dg(g^{-1}dgg^{-1}dg)^k\right). \tag{4.17}$$

Since tr is a continuous linear functional $d\operatorname{tr}(F) = \operatorname{tr}(dF)$ so we see that

$$d\operatorname{Ch}_{2k+1} = \operatorname{tr}\left((dg^{-1})dg(g^{-1}dg)^k\right) = -\operatorname{tr}\left(g^{-1}dgg^{-1}dg(g^{-1}dg)^k\right) = 0 \tag{4.18}$$

since we are back to the case of an even number of factors. □

Where does this lead us? Each of these Chern forms defines a cohology class on $G^{-\infty}(\mathbb{N})$ – of course we have not yet checked that they are non-zero. In fact they are and so it is interesting to consider the pull-backs:

**Proposition 8.** *The odd Chern forms define maps for each* $k$

$$K^{-1}(X) \longrightarrow H^{2k+1}(X;\mathbb{C}) \tag{4.19}$$

*for any compact manifold.*

*Proof.* By definition an odd K-class is defined by a smooth map $f : X \longrightarrow G^{-\infty}(\mathbb{N})$. Thus we can simply pull the forms back to get

$$f^* \operatorname{Ch}_{2k+1} = \operatorname{tr}\left((f^{-1}df)^{2k+1}\right) \tag{4.20}$$

where now we can think of $f$ as a map into $\Psi^{-\infty}({}^{\mathrm{b}}N)$ (which happens to map into $G^{-\infty}(\mathbb{N})$ of course). So, we only need to show that the cohomology class defined by this form is the same for homotopic $f$'s. Given a homotopy $F : [0,1]_t \times X \longrightarrow G^{-\infty}$ the Chern form pulls back to $\gamma = F^* \operatorname{Ch}_{2k}$ which is a smooth closed form on $[0,1] \times X$. Then if $f_0$ and $f_1$ are the restrictions to $t = 0$ and $t = 1$ it follows that

$$f_1^* \operatorname{Ch}_{2k} - f_0^* \operatorname{Ch}_{2k} = d\tau \tag{4.21}$$

for a smooth form $\tau$. Indeed $\gamma = dt \wedge v + v'$ where $v$, $v'$ are forms on $X$ which depend on $t$ as a parameter. That $\gamma$ is closed means that

$$\frac{\partial v'}{\partial t} - d_X v = 0,\ d_X v' = 0. \tag{4.22}$$

Hence

$$f_1^* \operatorname{Ch}_{2k} - f_0^* \operatorname{Ch}_{2k} = v'(1) - v'(0) = \int_0^1 \frac{\partial v'}{\partial t} dt = d_X \tau, \ \tau = \int_0^1 v(t) dt. \tag{4.23}$$

□

In fact it is usual to sum these forms up, to give the Chern character mapping from odd K-theory to odd cohomology – this involves questions of normalization which I will postpone for a little while.

So, next consider the even analogue of these forms. Of course there are no even forms on $G^{-\infty}(\mathbb{N})$ but these are forms on $G^{-\infty}_{\text{sus}}(\mathbb{N})$. In fact these are induced by the forms we already have on $G^{-\infty}(\mathbb{N})$ through the evaluation map

$$\operatorname{ev} : \mathbb{S} \times G^{-\infty}_{\text{sus}}(\mathbb{N}) \longrightarrow G^{-\infty}, \ (\theta, g) \longrightarrow g(\theta). \tag{4.24}$$

Since this map is smooth, we can pull the forms $\operatorname{Ch}_{2k+1}$ back to the product and then we can push-forward to the suspended group by integrating over the circle. Thus reduces the degree by one, so we define

$$\operatorname{Ch}^{\text{even}}_{2k} = \int_{\mathbb{S}} \operatorname{ev}^* \operatorname{Ch}_{2k+1} \ \text{ on } G^{-\infty}_{\text{sus}}(\mathbb{N}). \tag{4.25}$$

Now, it is straighforward to write this form down in terms of $dg$, the same map on $G^{-\infty}(\mathbb{N})$ and the parameter $\theta \in \mathbb{S}$ :

$$\operatorname{Ch}_{2k} = \int_{\mathbb{S}} \operatorname{tr} \left( g^{-1} \frac{\partial g}{\partial \theta} (g^{-1} dg)^{2k} \right). \tag{4.26}$$