43. Lecture 34: Dixmier-Douady invariant Friday, 5 December, 2008

Even though it is wandering further into Čech theory than I really wanted to go, I will discuss the Brylinzki-Hitchin definition of a gerbe (calling it 'gerbe data'), the derivation of the Dixmier-Douady class and show how the K-theory gerbe (and more generally any bundle gerbe) defines such gerbe data. If there is a little more time I will go through, at least in outline, the construction of a principal PU-bundle from gerbe data.

Let me start with the notion of a Čech type gerbe of Brylinski and later modified by Hitchin. For orientation, start with the 0-gerbe, the line bundle.

Definition 12. *Line bundle data* (to be considered as one word) on a manifold X – for convenience taken to be compact here – consists of the following:-

(1) A (finite) covering of X by open sets, U_i, $i \in N$.
(2) A $\mathcal{C}^\infty$ complex line bundle $L_i \longrightarrow U_i$ over each U_i.
(3) An isomorphism of complex line bundles for each i, j such that $U_{ij} = U_i \cap U_j \neq \emptyset$,
$$T_{ij} : L_i\big|_{U_{ij}} \longrightarrow L_j\big|_{U_{ij}} \text{ with } T_{ji} = T_{ij}^{-1}.$$
(4) The compatibility (cocylce) condition for each i, j, k such that $U_{ijk} = U_i \cap U_j \cap U_k \neq \emptyset$,

$$T_{ki}T_{jk}T_{ij} = \mathrm{Id} \text{ on } L_i\big|_{U_{ijk}}. \tag{43.1}$$

There are extreme cases of such vector bundle data. One possibility is that there is only one element in the open cover, $U_1 = Z$, and then $L \longrightarrow Z$ is simply a complex line bundle. Alternatively, all the line bundles could be trivial, $L_i = U_i \times \mathbb{C}$ and then we get what is the usual notion of a trivialization of a line bundle. Namely, the T_{ij} become maps $t_{ij} : U_{ij} \longrightarrow \mathbb{C}^*$ and the cocycle condition becomes

$$t_{ki}t_{jk}t_{ij} = 1. \tag{43.2}$$

In fact such line bundle data *always* defines a complex line bundle. Simply define the 1-dimensional complex vector space at each point by

$$L_p = \{(z_i) \in \bigoplus_{p\in U_i} (L_i)_p; T_{ij}z_i = z_j \ \forall\ i, j \text{ s.t. } p \in U_{ij}\}. \tag{43.3}$$

Then $L = \cup_p L_p$ is a complex line bundle and moreover there exist bundle isomorphisms

$$T_i : L\big|_{U_i} \longrightarrow L_i \text{ s.t. } T_{ij}T_i = T_j \text{ on } U_{ij}. \tag{43.4}$$

I do not want to follow all this Čech stuff to its logical conclusion, but observe that the converse of (43.4) is also true. If $\tilde{L}$ is a line bundle over X and there are bundle isomorphisms $\tilde{T}_i : L\big|_{U_i} \longrightarrow L_i$ for each i such that $T_{ij}\tilde{T}_i = \tilde{T}_j$ on U_{ij} then L and $\tilde{L}$ are globally isomorphic as vector bundles. Moreover, one can *refine* line bundle data given a refinement of the cover. That is, if U'_l, $l \in N'$, is another open cover with a map $I : N' \longrightarrow N$ such that $U'_l \subset U_{I(l)}$ for all $l \in N'$, then the $L'_l = L_{I(l)}\big|_{U'_l}$ carry 'obvious' induced line bundle data and the line bundle generated by this data is globally isomorphic to that generated by the original data.

Now, one can always find a *good open cover* which refines a given cover; it suffices to take a covering by sufficiently small balls with respect to some Riemannian structure on the manifold. The condition that an open cover be *good* is that all the non-trivial intersections of its elements be contractible. So, one can find a refinement to a good open cover still denoted U_i. In that case there is a trivialization of each $T_i : L_i \longrightarrow \tilde{L}_i = \mathbb{C} \times U_i$ over U_i. Then $\tilde{L}_i$ with $t_{ij} = \tilde{T}_j T_{ij} \tilde{T}_i^{-1} \in \mathcal{C}^\infty(U_i, \mathbb{C}^*)$ gives new line bundle data which also generates an isomorphic line bundle. Since U_{ij} is also contractible, one can choose logarithms

$$\gamma_{ij} \in \mathcal{C}^\infty(U_i, \mathbb{C}) \text{ s.t. } t_{ij} = \exp(2\pi i \gamma_{ij}),\ \gamma_{ji} = -\gamma_{ij}. \tag{43.5}$$

Now, on the triple overlaps

$$n_{ijk} = \gamma_{ij} + \gamma_{jk} + \gamma_{ki} \in \mathbb{Z} \text{ on } U_{ijk} \tag{43.6}$$

is constant and integral, since by the cocycle condition (43.2) $\exp(2\pi i n_{ijk}) = 1$ (and U_{ijk} is contractible). Moreover this satisfies the closure condition for Čech cocycles, that

$$n_{ijk} + n_{jkl} + n_{kli} + n_{lij} = 0 \text{ if } U_{ijkl} \neq \emptyset. \tag{43.7}$$

Thus the n_{ijk} fix a Čech 2-cocycle and hence a Čech cohomology class

$$\omega(L) \in \check{H}^2(X, \mathbb{Z}). \tag{43.8}$$

Of course, there is some work here to show that the Čech cohomology class is independent of the choice of good cover, etc.

Then one arrives at the well-known result:-

Theorem 16. *Two complex line bundles over a compact manifold are globally isomorphic if and only if they define the same class in $\check{H}^2(Z, \mathbb{Z})$ and every such class corresponds to an isomorphism class of line bundles.*

Proof. The main thing to see is the independence of the class $\omega(L)$ of the choice of good open cover – this really amounts to showing that the same class arises under refinement to another good open cover, since any two open covers have a common, good, refinement. The converse, that each class arises this way, follows by the fact that any Čech cocycle n_{ijk} with respect to some open cover, arises from γ_{ij}'s through (43.6). Namely one can just choose a partition of unity χ_i subordinate to the open cover and set

$$\gamma_{ij} = \sum_k \chi_k n_{ijk} \text{ on } U_{ij}. \tag{43.9}$$

Exponentiating the γ_{ij}'s gives line bundle data which in turn generates the original class n_{ijk}. □

So, why have I gone through all this standard Čechy stuff? Basically, I just wanted to prepare for the Čech version of a gerbe.

Definition 13. *Gerbe data* on a compact manifold X consists of

(1) A (finite) open cover U_i of Z.
(2) A $\mathcal{C}^\infty$ line bundle $L_{ij} \longrightarrow U_{ij}$ over each non-trivial $U_{ij} = U_i \cap U_j$ with $L_{ji} = L'_{ij}$ (the dual).
(3) For each non-trivial U_{ijk} a trivialization

$$T_{ijk} : L_{ij} \otimes L_{jk} \otimes L_{ki} \longrightarrow \mathbb{C} \text{ on } U_{ijk}. \tag{43.10}$$

(4) The cocycle condition that over each non-trivial U_{ijkl}

$$T_{ijk}T_{jkl}^{-1}T_{kli}T_{lij}^{-1} = 1 \tag{43.11}$$

where this makes sense because the tensor product of the four 3-fold tensor products, as in (43.10), is canonically trivial:

$$L_{ij} \otimes L_{jk} \otimes L_{ki} \otimes L'_{jk} \otimes L'_{kl} \otimes L'_{lj} \otimes L_{kl} \otimes L_{li} \otimes L_{ik} \otimes L'_{li} \otimes L'_{ij} \otimes L'_{jl} = \mathbb{C}. \tag{43.12}$$

Proposition 59. *Any gerbe data defines a class (the Dixmier-Douady class)* $\mathrm{DD} \in H^3(X, \mathbb{Z})$ *which is constant under refinement and any two collections of gerbe data with the same Dixmier-Douady class are isomorphic after refiment (to a common good open cover).*

Proof. The definition of the Dixmier-Douady class follows the same idea as for line bundle data above. Namely, first refine to a good open cover (of course one has to define this process and check that it does indeed give new gerbe data). Then all the L_{ij} are trivial, with trivializations $\tilde{T}_{ij}$. The T_{ijk} now become maps $t_{ijk} : U_{ijk} \longrightarrow \mathbb{C}^*$ and so have logarithms, γ_{ijk}. These generate integers

$$n_{ijkl} = \gamma_{ijk} - \gamma_{jkl} + \gamma_{kli} - \gamma_{lij} \text{ on } U_{ijkl} \tag{43.13}$$

and these form a Čech 3-cocycle and hence class $[n] \in \check{H}^3(X; \mathbb{Z})$.

So, now the checking begins! I leave it to you (after consulting Brylinski's book, [2], if you prefer) to show that this class is well-defined, i.e. does not change under refinement and determines the gerbe data up to the natural notion of isomorphism after sufficient refinement. Moreover, every integral Čech 3-class arises this way. □

Theorem 17. *Čech gerbe data in the sense of Definition 13 defines a principal* PU *bundle over* X, *where* $\mathrm{PU} = \mathrm{U} / \mathrm{U}(1)$ *is the quotient of the group of unitary operators on a separable, infinite-dimensional, Hilbert space by the multiples of the identity, all* PU *bundles (up to isomorphism) arise this way and two principal* PU *bundles are isomorphic if and only if the Dixmier-Douady invariants of their gerbe data are equal.*

Proof. Not very likely. □

Now, let me check that we can extract 'gerbe data' from the K-theory gerbe as just described. To do this, consider the pull-back of the gerbe under some map from a finite dimensional manifold $X \longrightarrow G^{-\infty}_{\det=1} \longrightarrow G^{-\infty}$ which therefore represents an odd K-class on X. Let $\mathcal{E}$ be the pull-back of the bundle $\tilde{G}^{-\infty}_{\mathrm{sus},\tilde{\eta}=0}$. The first thing to note is that we can find local sections of $\mathcal{E}$, meaning it is locally trivial. Indeed, without the restriction to $\tilde{\eta} = 0$ this was discussed earlier. Since $\tilde{\eta}$ exponentiates to $\det \circ R_\infty$, it is enough to recall that R_∞ is surjective, since on a local section of $\tilde{G}^{-\infty}_{\mathrm{sus}}$ on which $\det \circ R_\infty = 0$ the funtion $\tilde{\eta}$ is necessarily constant. Thus, there is an open cover U_i of X on the elements of which $\mathcal{E}$ has a section (and as a principal bundle is then trivial). On the overlaps U_{ij} there are two sections, and hence a section of $\mathcal{E}^{[2]}$. Using this the determinant line bundle may be pulled back to define a line bundle L_{ij} over U_{ij}. It only remains to check the properties required of gerbe data in Definition 13. That L_{ji} is the dual of L_{ij} follows from the primitivity of the determinant line bundle and the fact that it is canonically trivial over the diagonal. Similarly the existence of a trivialization of the triple tensor product in (43.10) over any U_{ijk} follows from the primitivity of $\mathcal{L}$, as does the naturality (43.11).

Thus the K-theory gerbe does define Čech gerbe data.

Exercise 42 (I will do this eventually). Show that the Dixmier Douady invariant of the pull-back of the K-theory gerbe to X is (a multiple of) the second odd Chern class of the element $\mathrm{K}^1(X)$ which the map defines.