42. Lecture 33: The B-field
Wednesday, 3 December, 2008

Reminder. *Last time I described the K-theory gerbe, without defining the class of objects of which this is an example, namely the notion of a* bundle gerbe. *Today I will finish the discussion of the B-field on the K-gerbe, then quickly show how the K-theory gerbe defines gerbe data – which I wrote down in the notes for yesterday but did not discuss – and use this to motivate the, or at least* a, *general bundle gerbe.*

The determinant bundle in (41.19) has connection given by (21.15). The curvature of this connection was computed in (24.20):-

$$\omega = -c\int_{\mathbb{R}} \operatorname{tr}(a^{-1}\frac{da}{dt}(a^{-1}da)^2)dt \text{ at } a \in G^{-\infty}_{\mathrm{sus,ind}=0}[\rho/\rho^2]. \tag{42.1}$$

In the discussion of the transgression of the Chern forms for the delooping sequence this form was 'lifted' to $\tilde{G}^{-\infty}_{\mathrm{sus}}$ simply by observing that the integral is still convergent – because one term has been differentiated with respect to t. This leads to

$$\tilde{\eta}_2 = \int_{\mathbb{R}} \operatorname{tr}(a^{-1}\frac{da}{dt}(a^{-1}da)^2)dt \text{ at } a \in \tilde{G}^{-\infty}_{\mathrm{sus}} \tag{42.2}$$

which therefore defines a form on the top part of the $*$-extended group and restricts to the subgroup defined by $\tilde{\eta} = \tilde{\eta}_0 = 0$.

The curvature of the pull-back of the determinant line bundle from $G^{-\infty}_{\mathrm{sus,ind}=0}$ to $\mathcal{G}$ in (41.22) is the pull-back of the curvature, so it is – up to a constant which I have lost but which is important – equal to

$$S^*\omega = \int_{\mathbb{R}} \operatorname{tr}(a^{-1}\frac{da}{dt}(a^{-1}da)^2)dt,\ a = h^{-1}g,\ (h,g) \in \mathcal{G}. \tag{42.3}$$

To compute this we need to expand out the differential, $d(h^{-1}g) = -h^{-1}dhh^{-1}g + h^{-1}dg$ and similarly for the derivative with respect to t. This gives a total of eight terms.

Lemma 39. *The pull-back of the curvature of the determinant line bundle is*

$$\begin{gathered} S^*\omega = \pi_R^*\tilde{\eta}_2 - \pi_L^*\tilde{\eta}_2 + d\alpha,\ \textit{where} \\ \alpha = \int_{\mathbb{R}} \operatorname{tr}\left(\frac{dg}{dt}g^{-1}(dh)h^{-1} - \frac{dh}{dt}h^{-1}(dg)g^{-1}\right)dt \textit{ and} \\ \pi_L,\ \pi_R : \mathcal{G} \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus},\tilde{\eta}=0} \end{gathered} \tag{42.4}$$

are the two projections.

Proof. After expanding out (42.3) as indicated above, the two 'pure' terms in which only one of h or g is differentiated are the two terms obtained by pull-back of $\tilde{\eta}_2$.

The other six can be combined to give

$$
\begin{aligned}
&\int_{\mathbb{R}} \operatorname{tr}((h^{-1}g)^{-1}\frac{d(h^{-1}g)}{dt}((h^{-1}g)^{-1}d(h^{-1}g))^2)dt \\
&= \int_{\mathbb{R}} \operatorname{tr}(g^{-1}\frac{dg}{dt}((g^{-1}dg)^2)dt - \int_{\mathbb{R}} \operatorname{tr}(\frac{dh}{dt}h^{-1}((dh)h^{-1}))^2)dt \\
&+ \int_{\mathbb{R}} \operatorname{tr}(g^{-1}\frac{dg}{dt}((h^{-1}g)^{-1}d(h^{-1}g))^2)dt - \int_{\mathbb{R}} \operatorname{tr}(g^{-1}\frac{dh}{dt}h^{-1}gd(h^{-1}g))^2)dt \\
&= \pi_R^*\tilde\eta_2 - \pi_L^*\tilde\eta_2 + d\left(\int_{\mathbb{R}} \operatorname{tr}\left(\frac{dg}{dt}g^{-1}(dh)h^{-1} - \frac{dh}{dt}h^{-1}(dg)g^{-1}\right)dt\right) \\
&\qquad + \int_{\mathbb{R}} \frac{d}{dt}\operatorname{tr}((dh)h^{-1}(dg)g^{-1}).
\end{aligned}
\tag{42.5}
$$

The last term evaluates to $\operatorname{tr}((da)a^{-1}(da)a^{-1}) = 0$ by symmetry, where $a = R_\infty(g) = R_\infty(h)$ is the common base-, or end-, point. Thus we arrive at (42.4). □

Theorem 15. *For the K-theory (principal) bundle gerbe*

$$
\begin{tikzcd}
 & \tilde{\mathcal{L}} = S^*\mathcal{L} \arrow[d] & \mathcal{L} \arrow[d] \\
\tilde G^{-\infty}_{\mathrm{sus},\tilde\eta=0} \arrow[dr, "R_\infty"'] & \mathcal{G} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "S"] \arrow[d] & G^{-\infty}_{\mathrm{sus},\mathrm{ind}=0}[\epsilon/\epsilon^2] \\
 & G^{-\infty}_{\det=1} &
\end{tikzcd}
\tag{42.6}
$$

the pulled-back determinant line bundle has a connection $\nabla_{\mathcal{G}}$ over $\mathcal{G}$ with curvature

$$
\omega_{\mathcal{G}} = \pi_R^*\tilde\eta_2 - \pi_L^*\tilde\eta_2 \tag{42.7}
$$

where the 'B-field' $\tilde\eta_2$ is a 2-form on $\tilde G^{-\infty}_{\mathrm{sus},\tilde\eta=0}$ with basic differential

$$
d\tilde\eta_2 = R_\infty^* c\operatorname{tr}((a^{-1}da)^3). \tag{42.8}
$$

It is important to track the constant, which I have not (yet) done.

Proof. This has all been done! The connection is obtained by adding α to the pulled-back connection and the formula for the differential of $\tilde\eta_2$ was worked out earlier. □

What does all this buy us? Or asked another way, are there any interesting examples? In fact there are plenty of examples!

One such is to consider the group $\mathrm{SU}(N)$ of unitary $N\times N$ matrices of determinant one. This Lie group is connected and simply connected – this we have really already used. Now, we can certainly embed it into the stabilized group

$$
i_N : \mathrm{SU}(N) \longrightarrow G^{-\infty}_{\det=1}, \tag{42.9}
$$

say in the isotropic model by making it act on the first N eigenfunctions of the harmonic oscillator (and stabilizing by the identity of course). Here we use the consistency of the usual and the Fredholm determinant. Thus, we can pull the K-theory gerbe back to $\mathrm{SU}(N)$ and we have an induced 'gerbe'

(42.10)
$$\begin{tikzcd} & & \tilde{\mathcal{L}} \arrow[d] \\ i_N^* \tilde{G}^{-\infty}_{\mathrm{sus},\tilde\eta=0} \arrow[r, equal] & \mathcal{E} \arrow[dr] & \mathcal{E}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[d] \\ & & \mathrm{SU}(N). \end{tikzcd}$$

Here $\mathcal{E}^{[2]}$ is the fibre-product of $\mathcal{E}$ with itself – which is to say it is the pull-back of $\mathcal{G}$ which is just the fibre-product of $\tilde{G}^{-\infty}_{\mathrm{sus},\tilde\eta=0}$ with itself over $G^{-\infty}_{\det=1}$. Moreover, the set up (42.10), with $\mathcal{L}$ the pulled-back line bundle over $\mathcal{E}^{[2]}$, comes equipped with a connection on $\mathcal{L}$ and a B-field on the total space of the bundle with 'curvature' a multiple of the 3-form $\mathrm{tr}((g^{-1}dg)^3)$ on $\mathrm{SU}(N)$. How cool is that? Is this the gerbe of Meinrenken – [6] or is the 'curvature' a multiple of the minimal integral form $\frac{1}{24}\,\mathrm{tr}((g^{-1}dg)^3)$. This needs to be checked!

Other 'obvious' examples come more directly from index theory and I will describe these below. First let me try to abstract from the K-theory gerbe to get the notion of a 'bundle gerbe' which is due to Michael Murray [8].

So, abstractly, consider a setup as in (42.10)

(42.11)
$$\begin{tikzcd} & \mathcal{L} \arrow[d, "p"] \\ \mathcal{E} \arrow[dr, "\pi"'] & \mathcal{E}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[d, "\pi^{[2]}"] \\ & X \end{tikzcd}$$

where X can be a finite dimensional (say compact) smooth manifold and we are no longer assuming that $\mathcal{E}$ is pulled back from somewhere else. We will need to specify what sort of bundle $\mathcal{E}$ is over X. Since we can expect, in general, that $\mathcal{E}$ will be infinite dimensional we will need to specify the structure group. Let me just gloss over this for the moment to get the formal set up clear first. So, just pretend everything is finite-dimensional (which it could be) and then what makes the discussion above, relating the K-theory gerbe to Čech gerbe data, work? What we have used is:-

(1) $\mathcal{E}$ is a fibre bundle over X.
(2) $\mathcal{E}^{[2]}$ is the fibre product of $\mathcal{E}$ with itself – meaning it is the fibre diagonal in $\mathcal{E}\times\mathcal{E}$.
(3) $\mathcal{L}$ is a line bundle over $\mathcal{E}^{[2]}$.
(4) $\mathcal{L}$ has a *primitivity property* – if we consider $\mathcal{E}^{[3]}$, the triple fibre product and the three projections

(42.12)
$$\pi_O : \mathcal{E}^{[3]} \longrightarrow \mathcal{E}^{[2]},\ O = F, S, C$$

where π_F is projection onto the rightmost two factors, π_S onto the leftmost two factors and π_C onto the outer two factors[7] then there is a given trivialization

$$
\begin{aligned}
T &: \pi_S^*\mathcal{L} \otimes \pi_F^*\mathcal{L} \xrightarrow{\simeq} \pi_C^*\mathcal{L} \text{ over } \mathcal{E}^{[3]} \text{ or} \\
\tilde{T} &: \pi_S^*\mathcal{L} \otimes \pi_F^*\mathcal{L} \otimes \pi_C^*\mathcal{L}' \xrightarrow{\simeq} \mathbb{C}.
\end{aligned}
\tag{42.13}
$$

(5) Finally we need this trivialization to be natural, in an appropriate sense. Namely if we go up to $\mathcal{E}^{[4]}$ then there are four versions of T from the four ways of mapping from $\mathcal{E}^{[4]}$ back to $\mathcal{E}^{[3]}$ by dropping one of the factors. Then the tensor product of the four pulled-back line bundles as in the second version of (42.13) is canonically trivial and we require that the product of the four $\tilde{T}$'s should reduce to the identity.

What is triviality for such a bundle gerbe? It is the condition that there is a line bundle K over E such that there is an isomorphism of line bundles

$$
\mathcal{L} \xrightarrow{\simeq} \pi_R^*K \otimes \pi_L^*K'. \tag{42.14}
$$

Definition 10. A bundle gerbe with connection is a bundle as in (42.11) satisfying 1–5 where $\mathcal{E} \longrightarrow X$ is a smooth Fréchet fibre bunle, $\mathcal{L}$ is a smooth line bundle over $\mathcal{E}^{[2]}$ with smooth connection ∇, the diffeomorphism T is smooth and under (42.13) the connection pulls back to the product connection.

Exercise 36. Show, if only formally, that under the triviality condition the B-field can be taken to be the curvature of K and hence the 3-form which is its derivative vanishes. Going a little further, show that the Dixmier-Douady invariant, in $H^3(X;\mathbb{Z})$ vanishes in this case.

There are finite dimensional examples. Recall that in $\mathrm{SU}(N)$ there are still non-trivial multiples of the identity, at least if $N > 1$. Namely $\tau\,\mathrm{Id} \in \mathrm{SU}(N)$ if $\tau^N = 1$. These Nth roots of unity form a normal subgroup and the quotient is the smaller group $\mathrm{PU}(N)$:

$$
\{\tau \in \mathbb{C}; \tau^N = 1\} \longrightarrow \mathrm{SU}(N) \longrightarrow \mathrm{PU}(N). \tag{42.15}
$$

Proposition 58 (At least mainly due to Serre.). *Let E be a principal $\mathrm{PU}(N)$ bundle over a compact manifold X then the central extension (42.15) induces a primitive, flat, line bundle, L_N, over $\mathrm{PU}(N)$ which defines a bundle gerbe*

$$
\begin{tikzcd}
 & \mathcal{L}_N \arrow[d, "p"] & L_N \arrow[d] \\
E \arrow[dr, "\pi"'] & E^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "S"] \arrow[d, "\pi^{[2]}"] & \mathrm{PU}(N) \\
 & X &
\end{tikzcd}
\tag{42.16}
$$

the Dixmier-Douady invariant for which is a torsion element of $H^3(X;\mathbb{Z})$ and all such elements arise this way.

[7]These letter stand for **F**irst, **S**econd and **C**omposte, coming from the composition of operators.

Exercise 37. Suppose $E \longrightarrow X$ is a principal bundle for a group G where G has a central extension by the circle (or $\mathbb{C}^*$) – meaning there is a short exact sequence of groups

$$\mathrm{U}(1) \longrightarrow \hat{G} \longrightarrow G. \tag{42.17}$$

Show that E fixes a bundle gerbe over X (assuming appropriate regularity especially if the setup is infinite dimensional).

Here is another example taken from the recent preprint [5]. Let X be a compact manifold and suppose that L is a complex line bundle over X and $f : X \longrightarrow \mathbb{C}^*$ is a smooth function. The former defines an element of $H^2(X, \mathbb{Z})$ and the latter an element of $H^1(X; \mathbb{Z})$. Together this gives an integral 3-class, how can we construct a bundle gerbe out of this data? Choose an Hermitian inner product on the fibres of L, so that the circle bundle

$$\hat{L} = \{p \in L; \|p\| = 1\} \xrightarrow{p} X \tag{42.18}$$

is well-defined. It is indeed a principal $\mathrm{U}(1)$ bundle over X. Thus if we take the fibre product $\hat{L}^{[2]}$ over X then we have the usual S map

$$\hat{L}^{[2]} \xrightarrow{s} \mathrm{U}(1). \tag{42.19}$$

This map is itself 'primitive' (sometimes called a groupoid character), meaning that the three versions of it over $\hat{L}^{[3]}$ satisfy

$$\pi_S^* s \cdot \pi_F^* s = \pi_C^* s. \tag{42.20}$$

Next think about the map $f : X \longrightarrow \mathrm{U}(1)$. Together with (42.19) this leads to a map to the 2-torus:

$$s \times \pi^{[2]} * f : \hat{L}^{[2]} \longrightarrow \mathbb{T}^2. \tag{42.21}$$

Over the torus there is a line bundle, corresponding to the fundamental, volume, class in $H^2(\mathbb{T}^2; \mathbb{Z})$. This line bundle can be pulled back to $\hat{L}^{[2]}$ giving at least the basic setup of a bundle gerbe.

Exercise 38 (Maybe for me). Check that if L is equipped with an Hermitian connection then this defines a connection $d + \gamma$ on the (trivial) pull-back of L to $\hat{L}$. Then show that the structure above is a bundle gerbe in the sense of Definition 10 and with B-field on $\hat{L}$ $cd \log f \wedge \gamma$ (including working out the constant) and with curvature 3-form

$$\frac{c'}{2\pi i} \omega \wedge d \log f \text{ on } X. \tag{42.22}$$

Addenda to Lecture 33 The notion of equivalence of a bundle gerbes needs to be addressed, corresponding to a weakening of the notion of triviality.

First we can say that two gerbes over the same base, are isomorphic if there is a fibre-preserving Fréchét isomorphism between the corresponding bundles $\mathcal{E}_i$, $i = 1, 2$ such that under the induced isomorphisms of the $\mathcal{E}_i^{[2]}$ the bundles $\mathcal{L}_i$ become isomorphic and that under the induced isomorphism of the $\mathcal{E}_i^{[3]}$ the primitivity isomorphism T_i are intertwined.

This corresponds to the ability to pull back gerbes. Thus suppose Γ is a gerbe as in (42.11) and $\Phi : \mathcal{E}_1 \longrightarrow \mathcal{E}$ is a smooth bundle-preserving map, where $\mathcal{E}_1 \longrightarrow X$ is a locally trivial Fréchét fibre bundle. Then $\Phi^*\Gamma$ is the gerbe with line bundle $(\Phi^{[2]})^*\mathcal{L}$ over $\mathcal{E}_1$.

Exercise 39. You should check that all the conditions hold for the pull-back and that if ∇ is a primitive connection on $\mathcal{L}$ then the pull-back is a primitive connection on $\Phi^*\mathcal{L}$. Check that Γ and $\Phi^*\Gamma$ have the *same* Dixmier-Douady invariant.

Now we can say that one gerbe Γ_i *extends* another, Γ_2, if there is a fibre smooth map $\Phi : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ such that $\Gamma_1 \equiv \Phi^*\Gamma_1$. Two gerbes are *equivalent* if each extends the other.

Next we can consider the 'tensor product' of two bundle gerbes Γ_i with Fréchét fibrations $\mathcal{E}_i$ and primitive line bundles with connection $\mathcal{L}_i$, over the same base X. The *tensor product* $\Gamma_1 \otimes \Gamma_2$ (maybe it should be written as an exterior tensor product, $\Gamma_1 \boxtimes \Gamma_2$) is just obtained by taking the fibre product of the bundles $\mathcal{E} = (\mathcal{E}_1) \times_X (\mathcal{E}_2)$ and the exterior tensor product of the primitive line bundles. Alternatively one can think of this in two steps. First define the tensor product when the fibrations are the same – just as the tensor product of the two line bundles and connections. Then define the general case as the tensor product in this sense of the two pull-backs – of Γ_i to $\mathcal{E}$ under the two projections $p_i : \mathcal{E} \longrightarrow \mathcal{E}_i$.

Exercise 40. Check it all – that the required conditions hold for these operations to be well-defined and most importantly that the Dixmier-Douady invariant of the tensor product is the sum of the Dixmier-Douady invariants.

Exercise 41. Make sure that you can see that duality also works – the dual of a gerbe is just the gerbe with the dual bundle and dual connection and that this process reverses the sign of the Dixmier-Douady invariant. Observer that the tensor product of a gerbe and its dual is isomorphic to a trivial gerbe.

Definition 11. Let $\mathcal{F} \longrightarrow X$ be a Fréchét bundle over a manifold X, then a bundle gerbe Γ over $\mathcal{F}^{[2]}$ (with Fréchét bundle $\mathcal{E}$ and primitive line bundle $\mathcal{L}$) is *primitive* if there is a smooth Fréchét bundle map (as bundles over $\mathcal{F}^{[3]}$)

$$\pi_S^*\mathcal{E} \times_{\mathcal{F}^{[3]}} \pi_F^*\mathcal{E} \longmapsto \pi_C^*\mathcal{E} \tag{42.23}$$