## 41. Lecture 32: The K-theory gerbe
## Monday, 1 December, 2008

First let me apologize for not having been able to keep up with the notes while I was away. With any luck I will catch up a bit with what I had meant to put in about the Chern character etc.

Today I want to describe the K-theory gerbe in one of its forms. Rather than define what a gerbe is – in the widest sense the term is used for any geometrical object which is classified by, or at least realizes all, integral 3-cohomology – I will describe it and then try to explain the salient features. In brief the *universal* K-theory gerbe is a 'geometric invariant' associated with, in the first instance a bundle with some structure over, a (reduced) classifying space for odd K-theory which 'captures' the primitive three-dimensional cohomology class.

However, first let me recall the 'geometric invariants' – in degrees 0, 1 and 2, that we have already introduced, since the gerbe is analogous to these:-

(1) The index.
(2) The determinant.
(3) The determinant line bundle.

Of course the first two of these don't look very geometric but that is what happens in low degree.

**The index.** We have two basic 'series' of classifying spaces the loop groups of (a) $G^{-\infty}$ and the loop spaces of the space of involutions $\mathcal{H}^{-\infty}$. The index is most easily seen as the map

$$\mathcal{H}^{-\infty} \ni \{I = \gamma_\infty + \gamma; \gamma \in \Psi^{-\infty} \otimes M(2,\mathbb{C}); I^2 = \mathrm{Id}\} \ni I \longmapsto \frac{1}{2}\operatorname{tr}(\gamma) \in \mathbb{Z}. \tag{41.1}$$

We have usually taken $\gamma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The index labels the components, i.e. induces an isomorphism

$$\mathrm{ind} : \Pi_0(\mathcal{H}^{-\infty}) \longrightarrow \mathbb{Z} \tag{41.2}$$

which is additive (under compression to finite rank and direct sum) and as such is unique up to sign (which needs to be worried about).

The (flat-pointed) loop group on $G^{-\infty}$, $G^{-\infty}_{\mathrm{sus}}$ is also a classifying space for even K-theory and we showed that the index can be transferred to it. Namely the map

$$\mathrm{cl}_{\mathrm{eo}} : \mathcal{H}^{-\infty} \longrightarrow G^{-\infty}_{\mathrm{sus}}(\cdot;\mathbb{C}^2) \tag{41.3}$$

is an homotopy equivalence and under it

$$\mathrm{ind} = \frac{1}{2}\operatorname{tr} = {\mathrm{cl}_{\mathrm{eo}}}^* \mathrm{ind}_{\mathrm{sus}},\ \mathrm{ind}_{\mathrm{sus}}(g) = \frac{1}{2\pi 1}\int_{\mathbb{R}} \operatorname{tr}\left(g^{-1}(t)\frac{dg(t)}{dt}\right) dt. \tag{41.4}$$

So, it is reasonable just to write $\mathrm{ind}_{\mathrm{sus}} : G^{-\infty}_{\mathrm{sus}} \longmapsto \mathbb{Z}$ as 'ind' and take (41.4) as a natural identification; however I will still use the notation $\mathrm{ind}_{\mathrm{sus}}$ where this seems helpful.[6]

Now, the index on $G^{-\infty}_{\mathrm{sus}}$ can be recognized as the functional induced by the 1-form

$$\mathrm{Ch}_1^{\mathrm{odd}} = \frac{1}{2\pi i}\operatorname{tr}\left(g^{-1}dg\right) \text{ on } G^{-\infty}. \tag{41.5}$$

[6]The index functional on the higher loop groups $G^{-\infty}_{\mathrm{sus}(2k+1)}$ was supposed to have been discussed in the write-ups while I was away – this may still appear.

Namely, under the evaluation map and projection

$$\begin{tikzcd} \mathbb{R} \times G^{\infty}_{\mathrm{sus}} \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi_2"] & G^{-\infty} \\ G^{-\infty}_{\mathrm{sus}} \end{tikzcd}, \ \mathrm{ind}_{\mathrm{sus}} = (\pi_2)_*(\mathrm{ev}^* \mathrm{Ch}^{\mathrm{odd}}_1). \tag{41.6}$$

**The determinant.** This, meand $\mathrm{ind}_{\mathrm{sus}}$, was the basis of the (second) construction of the Fredholm determinant. Recall the delooping sequence:-

$$G^{-\infty}_{\mathrm{sus}} \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus}} \xrightarrow{R} G^{-\infty}. \tag{41.7}$$

Here, the middle group is the half-open flat loops,

(41.8)

$$g : \mathbb{R} \longrightarrow \Psi^{-\infty} \text{ s.t. } \frac{dg}{dt} \in \mathcal{S}(\mathbb{R}; \Psi^{-\infty}),\ g(t) \in G^{-\infty}\ \forall\ t \in \mathbb{R} \text{ and } \lim_{t \to -\infty} g(t) = 0.$$

This central group is contractible and the construction (41.6) extends to it to define

$$\begin{tikzcd} \mathbb{R} \times \tilde{G}^{\infty}_{\mathrm{sus}} \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi_2"] & G^{-\infty} \\ \tilde{G}^{-\infty}_{\mathrm{sus}} \end{tikzcd}, \ \tilde{\eta} = (\pi_2)_*(\mathrm{ev}^* \mathrm{Ch}^{\mathrm{odd}}_1) : \tilde{G}^{-\infty}_{\mathrm{sus}} \longrightarrow \mathbb{C}. \tag{41.9}$$

This 'eta function' has the properties that it restricts to $\mathrm{ind}_{\mathrm{sus}}$ on the subgroup $G^{-\infty}_{\mathrm{sus}}$ and is log-additive, so the exponentiated function

$$\det = \exp(2\pi i \tilde{\eta}) : G^{-\infty} \longrightarrow \mathbb{C}^* \tag{41.10}$$

is well-defined, multiplicative and restricts to the usual determinant on $\mathrm{GL}(N, \mathbb{C})$ included into $G^{-\infty}$ by stabilization. Moreover, we know that

$$d\tilde{\eta} = R^* \mathrm{Ch}^{\mathrm{odd}}_1. \tag{41.11}$$

It follows that as a map (41.10), det represents a generating class for $H^1(G^{-\infty}, \mathbb{Z})$.

*Exercise* 35. If this is not 'geometric' enough for you, the picture can be expanded a little. Namely consider the possible values of 'log det' at a point of $G^{-\infty}$ – there should be a $\mathbb{Z}$ of them at each point. To do this explicitly, take $\tilde{G}^{-\infty} \times \mathbb{C}$ and then identify all pairs $(\tilde{g}_1, z_1)$ and $(\tilde{g}_2, z_2)$ if $R(\tilde{g}_1) = R(\tilde{g}_2)$ and $z_1 - z_2 = 2\pi i \,\mathrm{ind}(\tilde{g}_2 \circ (\tilde{g}_1)^{-1})$. Show that this results in a principal bundle

$$\begin{tikzcd} \mathbb{Z} \arrow[r, dash] & Z \arrow[r, "\tilde{\eta}"] \arrow[d] & \mathbb{C} \\ & G^{-\infty}. \end{tikzcd} \tag{41.12}$$

over $G^{-\infty}$ with structure group $\mathbb{Z}$ on which $\tilde{\eta}$ is a 'connection' in the sense that it is a well-defined function on the total space of the bundle which shifts by $n$ under the action of $n \in \mathbb{Z}$.

**Determinant line bundle.** The determinant bundle was constructed over the group $G^{-\infty}_{\mathrm{sus,ind}=0}[[\rho]]$, the component of the identity in $G^{-\infty}_{\mathrm{sus}}[[\rho]]$ using the quantization sequence. Here $G^{-\infty}_{\mathrm{sus}}[[\rho]]$ is a $*$ extension of the group $G^{-\infty}_{\mathrm{sus}}$. Namely as a space it is consists of formal power series in $\rho$ – which is just another way of saying sequences –

$$h=\sum_{j=0}^{\infty}h_j\rho^j,\ h_0\in G^{-\infty}_{\mathrm{sus}},\ h_j\in\Psi^{-\infty}_{\mathrm{sus}},\ p=h\circ k=\sum_j^{\infty}B_j(h,k)\rho^j \tag{41.13}$$

where the product is associative and the $B_j$ are differential operators (acting only in the suspension variable):

$$B_0(h,k)=hk,\ B_j(h,k)=\sum_{l+l'+p+p'=j}c_{l,l',p,p'}\frac{d^ph_l}{dt^p}\frac{d^{p'}k_{l'}}{dt^{p'}} \tag{41.14}$$

where the product on the right is in $\Psi^{-\infty}$.

For the quantization sequence, the product just comes from the formula for the composition of isotropic pseudodifferential operators on $\mathbb{R}$ – sometimes called the Moyal product.

Now, the subject of today's lecture is the next step, the K-theory gerbe. To construct this again consider the delooping sequence, but now it nees to be both restricted and expanded. The basic delooping sequence is (41.7) above. The restriction is to kill off the determinant – so consider the subgroups

$$\begin{tikzcd} G^{-\infty}_{\mathrm{sus,ind}=0} \arrow[r] \arrow[d, hook] & \tilde G^{-\infty}_{\mathrm{sus},\tilde\eta=0} \arrow[r, "R"] \arrow[d, hook] & G^{-\infty}_{\det=1} \arrow[d, hook] \\ G^{-\infty}_{\mathrm{sus}} \arrow[r] & \tilde G^{-\infty}_{\mathrm{sus}} \arrow[r, "R"] & G^{-\infty}. \end{tikzcd} \tag{41.15}$$

From the earlier discussion, the top row is exact.

The expansion is to consider the $*$ product. Thus, consider just the case $\epsilon^2=0$, meaing pairs

$$h=h_0+\epsilon h_1,\ h_0\in G^{-\infty}_{\det=1},\ h_1\in\Psi^{-\infty} \tag{41.16}$$

with the projected $*$ product

(41.17)

$$h\circ k=(h_0k_0)+\epsilon\big(h_0k_1+h_1k_0+B(h_0,k_0)\big),\ B(h_0,k_0)=\frac{1}{2i}\left(\frac{dh_0}{dt}k_0-h_0\frac{dk_0}{dt}\right).$$

Then if we take the restricted groups

$$\begin{gathered} G^{-\infty}_{\mathrm{sus,ind}=0}[\epsilon/\epsilon^2]=G^{-\infty}_{\mathrm{sus,ind}=0}+\epsilon\Psi^{-\infty}_{\mathrm{sus}}, \\ \tilde G^{-\infty}_{\mathrm{sus},\tilde\eta=0}[\epsilon/\epsilon^2]=\tilde G^{-\infty}_{\mathrm{sus},\tilde\eta=0}+\epsilon\Psi^{-\infty}_{\mathrm{sus}} \end{gathered} \tag{41.18}$$

where there are no restrictions on the lower order terms, we get a new short exact sequence in place of (41.7):

$$\begin{tikzcd} \mathcal{L} \arrow[d] & & \\ G^{-\infty}_{\mathrm{sus,ind}=0}[\epsilon/\epsilon^2] \arrow[r] & \tilde G^{-\infty}_{\mathrm{sus},\tilde\eta=0}[\epsilon/\epsilon^2] \arrow[r, "R"] & G^{-\infty}_{\det=1}. \end{tikzcd} \tag{41.19}$$

Here I have included the fact that deterimant bundle is well-defined over the 'dressed' group $[\epsilon/\epsilon^2]$ – it also comes equipped with a connection.

The basic 'bundle gerbe' construction (the idea is due to Michael Murray) is to take the fibre product of this thought of as a fibration. That is, consider

$$\mathcal{G} = \left\{(g,g') \in \tilde{G}^{-\infty}_{\mathrm{sus},\tilde{\eta}=0}[\epsilon/\epsilon^2]; R(g) = R(g')\right\} = \left(\tilde{G}^{-\infty}_{\mathrm{sus},\tilde{\eta}=0}[\epsilon/\epsilon^2]\right)^{[2]} \tag{41.20}$$

which is the 'fibre diagonal' in the full product of the central (contractible) group with itself. This is, by construction, a bundle over $G^{-\infty}_{\det=1}$. Moreover, there is a map back to the (dressed) flat-pointed loop group:

$$\begin{tikzcd} \mathcal{G} \arrow[r, "S"] \arrow[d] & G^{-\infty}_{\mathrm{sus},\mathrm{ind}=0}[\epsilon/\epsilon^2] \\ G^{-\infty}_{\det=1} & \end{tikzcd} \tag{41.21}$$

Here $S(g,g') = h$ if and only if $g' = hg$ – since $R(g') = R(g)$ the composite $g^{-1}g' = h$ is flat to the identity at both ends, and hence is an element of $G^{-\infty}_{\mathrm{sus},\mathrm{ind}=0}[\epsilon/\epsilon^2]$. We can use $S$ to pull back the determinant line bundle and so get a tower

$$\begin{tikzcd} \tilde{\mathcal{L}} = S^*\mathcal{L} \arrow[d] & \mathcal{L} \arrow[d] \\ \mathcal{G} \arrow[r, "S"] \arrow[d] & G^{-\infty}_{\mathrm{sus},\mathrm{ind}=0}[\epsilon/\epsilon^2] \\ G^{-\infty}_{\det=1} & \end{tikzcd} \tag{41.22}$$

In fact, as recalled above, we constructed a connection on $\mathcal{L}$ which therefore pulls back to a connection on $\tilde{\mathcal{L}}$.

**So what is a gerbe?** Well, as I said above, there are different points of view on this. In all cases one is supposed to be able to extract a class in $H^3(X,\mathbb{Z})$, where $X$ is the base, from the gerbe. I would distinguish between several different, but closely related objects.