39. Topic 8: More on the determinant bundle
In place of lecture for Wednesday, 27 November, 2008

These are as yet very crude notes.

Consider the 3×3 commutative block in which the groups are only roughly identified:-

(39.1)

$$
\begin{matrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{matrix}
\quad = \quad
\begin{tikzcd}
G^{-\infty}_{\mathrm{sus}} \arrow[r] \arrow[d] & \dot G^0_{\mathrm{sus}} \arrow[r, "\sigma"] \arrow[d] & G^{-\infty}_{\mathrm{sus}(2)} \arrow[d] \\
\tilde G^{-\infty}_{\mathrm{sus}} \arrow[r] \arrow[d, "R"] & \dot G^0_{\mathrm{sus\,pt}\,\infty} \arrow[r, "\sigma"] \arrow[d, "R"] & \tilde G_{\mathrm{sus}(2)} \arrow[d, "R"] \\
G^{-\infty} \arrow[r] & \dot G^0 \arrow[r, "\sigma"] & G^{-\infty}_{\mathrm{sus,ind}=0}
\end{tikzcd}
$$

In more detail:-

- G_{11} : This is the classifying group for even K-theory $G^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^2)$ consisting of the elements $a \in \mathcal{S}(\mathbb{R}^5)$ where the first variable is a parameter, so the product is pointwise in this variable and in the last four variables is as smoothing operators on $\mathcal{S}(\mathbb{R}^2)$ and $\mathrm{Id} + a(t)$ is required to be invertible for all t.
- G_{21} : This is the contractible, half-free version of the preceeding group - it consists of smooth loops in $G^{-\infty}(\mathbb{R}^2)$ which have Schwartz derivative and tend to Id as $t \to -\infty$.
- G_{31} : This is the classifying group for odd K-theory $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^2)$.
- G_{*1} : Is therefore the (flat) delooping sequence for $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^2)$.
- G_{12} : This is the symbolically suspended group of invertible isotropic pseudodifferential operators on $\mathbb{R}$ with values in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R})$ and normalization condition. As functions the kernels can be identified with functions on $\overline{\mathbb{R}^3} \times \mathbb{R}^2$ which are $\mathcal{C}^\infty$ and Schwartz in the last two variables. The first variable, t, is a parameter and the functions are required to vanish to infinite order at C which is a great half circle in the t direction. The are quantized to operators by Weyl quantization in the second two variables and then we require $\mathrm{Id} + a(t)$ to be invertible for all t. This group is contractible.
- G_{22} : This group is supposed to be similar to the previous one except it is now of product type. As functions the elements are smooth on $[\overline{\mathbb{R}^3}, \{t = \infty\}] \times \mathbb{R}^2$ and vanish to infinite order at the lift of C to the blow up – which means the closure of the complement of $t = \infty$. The product extends to these more general functions and we look at the group of invertible perturbations as before. This is also a contractible group.
- G_{32} : This is really a $*$-extended version of the usual group $\dot G^0_{\mathrm{iso}}(\mathbb{R}; \mathbb{R})$. The latter consists of the smooth functions on $\overline{\mathbb{R}^2} \times \mathbb{R}^2$ which are Scwartz in the last two variables, flat at a point C' at the on the bounding sphere and such that $\mathrm{Id} + a$ is invertible. The $*$-extension adds arbitrary lower order terms in $\dot\Psi^0_{\mathrm{iso}}(\mathbb{R}; \mathbb{R})$ which do not affect invertibility.
- G_{*2} : This is an exact sequence of contractible groups!

G_{13} : This is the image of the full symbol map from G_{12}. It consists of a $*$-algebra where, after some reorganization, all terms are Schwartz maps from $\mathbb{R}^2$ into Schwartz operators on $\mathbb{R}$ and the leading term is such that $\mathrm{Id}+b$ is invertible. This is a classifying space for odd K-theory.

G_{23} : This is a half-open version of the preceeding group. That is the individual terms are not Schwartz but are (I think after rearrangement) Schwartz in one variable with values in the half-open flat loops in the other; it has a $*$-product. It is again a contractible group.

G_{33} : This is a $*$-extension of $G^{-\infty}_{\mathrm{sus,ind}=0}(\mathbb{R})$.

G_{i*} : For each i this is quantization sequence.

Thus the operators in the top left block of four groups all correspond to certain functions on $\mathbb{R}^5$. The top two of the right column and the left two on the bottom row correspond to functions on $\mathbb{R}^4$ and the bottom right group to functions on $\mathbb{R}^3$. In all cases the last two variables are Schwartz. So we can really imagine the functions as being on $\mathbb{R}^3$, $\mathbb{R}^2$ and $\mathbb{R}$ respectively.

Log-multiplicative functionals:

(1) $\mathrm{ind}: G_{11} \longrightarrow \mathbb{Z}$, $\mathrm{ind}(g) = \frac{1}{2\pi i}\int_{\mathbb{R}} \mathrm{tr}(g^{-1}\dot{g}(t))dt$.
(2) $\eta : G_{12} \longrightarrow \mathbb{C}$, $\eta(g) = \overline{Tr}(g^{-1}\dot{g})$ where $\overline{Tr}$ is the regularized trace-integral which is a trace on the algebra.
(3) $\tilde{\eta} : G_{21} \longrightarrow \mathbb{C}$, $\tilde{\eta}(g) = \frac{1}{2\pi i}\int_{\mathbb{R}} \mathrm{tr}(g^{-1}\dot{g}(t))dt$ which makes sense because of the flatness of the loops.
(4) $\tilde{\eta} : G_{22} \longrightarrow \mathbb{C}$, $\tilde{\eta}(g) = \overline{Tr}(g^{-1}\dot{g})$ where $\overline{Tr}$ is the regularized trace-integral which is a trace on the algebra, since the parameter is the 'good' variable in product suspension.

These four maps are consistent under inclusion – i.e. they are all restrictions of the last map. Thus, restricting to the null spaces of these maps we get a commutative square in the top left corner. The exponential, $\exp(2\pi i\tilde{\eta})$ on G_{21} descends to G_{31} where it is the multiplicative Fredholm determinant. The exponential $\exp(2\pi i\eta)$ on G_{12} again descends to 'our' multiplicative determinant on the doubly suspended group. Again we can restrict to the subgroup where $\det = 1$ in these two cases and get short exact sequences on the top row and left column.

Exercise 33. Extend this commutative diagram to the whole 3×3 square. In particular show (I believe Frédéric Rochon has already done this) that the image groups under R and σ respectively in G_{32} and G_{23} are the full groups as before – the same as without the $\tilde{\eta} = 0$ restriction. This shows how we can kill the determinant line bundle since the resulting group in the 33 slot is the central extension of $G^{-\infty}_{\mathrm{sus,ind}=0}$ by the determinant bundle.

Proposition 57. *The fact that the determinant bundle is 'primitive' as in* (24.21) *is equivalent to the fact that the non-zero elements give a* $\mathbb{C}^*$ *central extension:*

$$\mathbb{C}^* \longrightarrow \mathcal{L}^* \longrightarrow G^{-\infty}_{\mathrm{sus,ind}=0}. \tag{39.2}$$

Exercise 34. Check it. Also, while you are at it, define an Hermitian inner product on the determinant line bundle which reduces this to a U(1) extension. In the geometric case this was done by Bismut and Freed.

I will use this central extension to define and discuss the (reduced) K-theory 2-gerbe later.

Remark 2. The right hand column, in the unreduced picture, constructs the determinant bundle via the $*$-extended, suspended delooping sequence.