36. Topic 5: Atiyah-Hirzebruch theorem In place of lecture for Wednesay, 20 November, 2008

Theorem 14. *For any smooth manifold, X, the total Chern character gives an isomorphsim*

$$\mathrm{Ch}^* : \mathrm{K}_c^*(X) \otimes \mathbb{C} \xrightarrow{\simeq} \mathrm{H}_c^*(X;\mathbb{C}). \tag{36.1}$$

This was originally proved by Atiyah and Hirzebruch using a spectral sequence argument coming from a filtration of K-theory and cohomology (which indeed works for any generalized cohomology theory) based on the 'skeleton' of the manifold as a CW complex. I will outline here a more pedestrian argument, which is essentially sheaf theory, corresponding to the Mayer-Vietoris complex. In fact it is really a Čech-theoretic version of the argument of Atiyah and Hirzebruch.

First recall the long exact sequence for K-theory for a manifold relative to its boundary – Proposition 47. Although I did not go through the proof in detail, any reasonable proof extends to the non-compact case to give the analogous sequence with compact supports:-

$$\begin{tikzcd} \mathrm{K}_c^0(X,\partial X) \arrow[r] & \mathrm{K}_c^0(X) \arrow[r] & \mathrm{K}_c^0(\partial X) \arrow[d, "\mathrm{cl}_{\mathrm{eo}}"] \\ \mathrm{K}_c^1(\partial X) \arrow[u, "\mathrm{cl}_{\mathrm{oe}}"] & \mathrm{K}_c^1(X) \arrow[l] & \mathrm{K}_c^1(X,\partial X) \arrow[l] \end{tikzcd} \tag{36.2}$$

From this we can pass to the Mayer-Vietoris sequence for a decomposition into manifolds with boundary in the following sense. Let X be a (generally non-compact) manifold. Let $\rho_i \in \mathcal{C}^\infty(X)$ be two real functions such that $H_i = \{\rho_i\}$ are smooth disjoint hypersurfaces on which $d\rho_i \neq 0$ and in addition

$$H_1 \subset X_2 = \{\rho_2 \geq 0\},\ H_2 \subset X_1 = \{\rho_1 \geq 0\},\ X = X_1 \cup X_2. \tag{36.3}$$

Since they do not intersect, these hypersurfaces lie in the interior of the 'other' manifold with boundary. Thus $Y = X_1 \cap X_2$ is also a manifold with boundary.

Picture.

Proposition 54. *There is a long exact (Mayer-Vietoris) complex in K-theory*

$$\begin{tikzcd} \mathrm{K}_c^0(Y\setminus\partial Y) \arrow[r] & \mathrm{K}_c^0(X_1\setminus H_1)\oplus \mathrm{K}_c^0(X_2\setminus H_2) \arrow[r] & \mathrm{K}_c^0(X) \arrow[d] \\ \mathrm{K}_c^1(X) \arrow[u] & \mathrm{K}_c^1(X_1\setminus\partial X_1)\oplus \mathrm{K}_c^1(X_2\setminus\partial X_2) \arrow[l] & \mathrm{K}_c^1(Y\setminus\partial Y). \arrow[l] \end{tikzcd} \tag{36.4}$$

Here the top right and bottom left horizontal maps are the sums of 'inclusions' given by extending maps trivial to the boundary to be trivial across it. The other two horizontal maps are also given by the two inclusion maps, with appropriately chosen signs. The vertical, connecting, homomorphisms are the sums, with orientations, of the restrictions to two H_i composed with $\mathrm{cl}_{\mathrm{oe}}$ or $\mathrm{cl}_{\mathrm{eo}}$ and then embedded in the interior of Y.

Proof. I leave the proof that this is a complex and exactness – which is clear except on the sides – as an extended exercise, at least for the moment. □

Proposition 55. *If the sequence* (36.4)*, and the corresponding sequence for cohomology with compact supports, are wrapped up then the combined Chern characters give a commutative diagram (i.e a natural transformation)*

$$
\begin{tikzcd}
 & & \mathrm{H}^*_c(X) \arrow[dddd, bend left] \\
 & & \mathrm{K}^*_c(X) \arrow[u, "\mathrm{Ch}^*"'] \arrow[dd] \\
\mathrm{H}^*_c(\mathring{X}_1) \oplus \mathrm{H}^*_c(\mathring{X}_2) \arrow[uurr] & \mathrm{K}^*_c(\mathring{X}_1) \oplus \mathrm{K}^*_c(\mathring{X}_2) \arrow[l, "\mathrm{Ch}^*"'] \arrow[ur] & \\
 & & \mathrm{K}^*_c(Y \setminus \partial Y) \arrow[ul] \arrow[d, "\mathrm{Ch}^*"'] \\
 & & \mathrm{H}^*_c(Y \setminus \partial Y) \arrow[uull]
\end{tikzcd}
\tag{36.5}
$$

Proof. This mainly involves the earlier discussion of the behaviour of the Chern character under $\mathrm{cl}_{\mathrm{oe}}$ and $\mathrm{cl}_{\mathrm{eo}}$. □

Any compact manifold can be reconstructed from combinations of this type:-

Proposition 56. *For any compact manifold X there are two sequences of open submanifolds $X'_j \subset X$, with $X'_0 = \emptyset$ and $X'_N = X$, and $B'_j \subset X$ such that $X'_j = X'_{j-1} \cup B_j$, the closures $X_{j-1}\overline{X'_{j-1}}$ and $B_j = \overline{B'_j}$ in X'_j are smooth manifolds with boundary with X'_j decomposed in terms of them as in* (36.3) *for each j and such that the intersection $B'_j \cap X'_{j-1}$ is a finite union of disjoint balls.*

Proof. This can be accomplished by covering X with finitely many sufficiently small balls with respect to some Riemann metric and then slightly adjusting the radii to avoid unpleasant intersections. In particular this gives a good open cover. □

Discuss tensor products of abelian groups briefly and that

$$\mathrm{Ch}^* : \mathrm{K}^*_c(X) \otimes \mathbb{C} \longrightarrow \mathrm{H}^*_c(X). \tag{36.6}$$

Proof of Theorem 14. First check that for an open ball the combined Chern character does indeed give an isomorphism

$$\mathrm{Ch}^* : \mathrm{K}^*_c(B) \otimes \mathbb{C} \longrightarrow \mathrm{H}^*_c(B; \mathbb{C}). \tag{36.7}$$

It follows that this is equally true for a finite union of disjoint open balls.

Now, proceeding inductively we may assume that the same is true for X'_j for $j < k$. Then in the (36.5), after tensoring with $\mathbb{C}$, for X'_j relative to X_{j-1} and B_j, two if the Ch^* arrows are known to be isomorphisms. It follows, from diagram chasing often called the 'Five Lemma' that the third is also an isomorphism, proving the desired result. □