34. Lecture 31: Iterated fibrations and Multiplicativity Friday, 14 November, 2008

Reminder. *We need to complete the proof of the equality of the topological index, introduced last time, and the semiclassical push-forward map in K-theory.*

First for the new construction for today, although it is not really so new. Namely extending the smoothing algebra and semiclassical and adiabatic constructions to a compact manifold with boundary. A $\mathcal{C}^\infty$ manifold with boundary is a Hausdorff topological space with a covering by open sets on each of which is homeomorphism is given to a (relatively) open subset of $[0,\infty)\times\mathbb{R}^{n-1}$ such that the transition maps, on intersections, are smooth. Note smoothness for a map on $U\subset[0,\infty)\times\mathbb{R}^{n-1}$ means boundedness of all derivatives including up to the boundary.

Given such a manifold Z there are two competing candidates for smooth functions. Namely the 'obvious' $\mathcal{C}^\infty(Z)$ which consists of the functions smooth in local coordinates and $\dot{\mathcal{C}}^\infty(Z)\subset\mathcal{C}^\infty(Z)$ consisting of the smooth functions which also vanish to infinite order at the boundary. The same sorts of definitions make sense on a manifold with corners, but for the moment we only need the case of the product Z^2. Just as in the case of a manifold without boundary, the density bundle Ω is well defined and its sections can be invariantly integrated over compact sets. This means that there are two clases of smoothing operators on Z; those with kernels in $\mathcal{C}^\infty(Z^2;\pi_R^*\Omega)$ and the smaller class with kernels in $\dot{\mathcal{C}}^\infty(Z^2;\pi_R^*\Omega)$. These spaces can be conveniently interpreted as $\mathcal{C}^\infty(Z;\mathcal{C}^\infty(Z;\Omega))$ and $\dot{\mathcal{C}}^\infty(Z;\dot{\mathcal{C}}^\infty(Z;\Omega))$ respectively.

Both spaces are closed under operator composition, essentially by Fubini's theorem with the composition looking the same as in the boundaryless case

$$A\circ B(z,z')=\int_Z A(z,z'')B(z'',z'). \tag{34.1}$$

The two algebras of smoothing operators will be denoted $\Psi^{-\infty}(Z)$ and $\dot{\Psi}^{-\infty}(Z)$, with the 'dot' denoting the infinite vanishing at the boundary.

Similarly there is no difficulty in extending the construction of the semiclassical algebra to this setting, I leave the details to you. However there is one useful thing to note about a compact manifold with boundary. Namely it is always possible to 'double' a compact manifold with boundary Z to a compact manifold without boundary, $2Z$ which as a set is two copies of Z with boundaries identified. In fact $2Z$ is not really well-defined in the sense that there is no natural $\mathcal{C}^\infty$ structure on this double, by there is a choice so that $Z\longrightarrow 2Z$ is a diffeomorphism onto its range, which is one of the copies of Z.

Lemma 38. *Suppose $Z\longrightarrow X$ is an embedding of a compact manifold with boundary (or corners for that matter) as the closure of an open subset of a compact manifold without boundary (which is always possible) then the algebra $\dot{\Psi}^{-\infty}(Z)$ is naturally identified with the subalgebra of $\Psi^{-\infty}(Z)$ corresponding to the kernels with support in $Z\times Z\subset X\times X$.*

Proof. The basic observation is that $\dot{\mathcal{C}}^\infty(Z)$ is identified with

$$\{u\in\mathcal{C}^\infty(X);\operatorname{supp}(u)\subset Z\}. \tag{34.2}$$

Applying this in both factors gives the result, provided densities are taken care of. □

In particular, irrespective of the choice of $\mathcal{C}^\infty$ structure on $2Z$, $\dot\Psi^{-\infty}(Z)$ is the subalgebra of $\Psi^{-\infty}(2Z)$ with kernels supported in $Z\times Z$.

This is important for our proof and also allows us to *define* the adiabatic algebra for $Z\times\mathbb{R}^n$ for instance as the subalgebra

$$\text{(34.3)}\quad \dot\Psi^{-\infty}_{\text{ad,iso}}(Z;\mathbb{R}^n)=\big\{A\in\Psi^{-\infty}_{\text{ad,iso}}(2Z;\mathbb{R}^n);\\ \text{the kernel has }\operatorname{supp}(A)\subset(0,1)\times Z\times Z\times\mathbb{R}^n\times\mathbb{R}^n\big\}.$$

This saves quite a bit of work and allows everything to be extended to fibrations etc although there are still some things to check. Let me just restate the basic result we have used in the compact boundaryless case in this context.

Proposition 50. *For a fibration of compact manifolds where the total space M has boundary, but the base Y does not,*

$$\text{(34.4)}\quad \dot{\mathcal{H}}^{-\infty}_{\text{ad,iso}}(M/Y;\mathbb{R}^n)=\{I=\gamma_1+a;\ a\in\dot\Psi^{-\infty}_{\text{ad,iso}}(M/Y;\mathbb{R}^n)\otimes M(2,\mathbb{C}),\ I^2=\mathrm{Id}\}$$

has a semiclassical symbol map which induces an 'homotopy equivalence' (identity on components)

$$\text{(34.5)}\qquad \dot{\mathcal{H}}^{-\infty}_{\text{ad,iso}}(M/Y;\mathbb{R}^n)\xrightarrow{\sigma_{\text{ad}}}\mathcal{S}(T^*(M/Y);\mathcal{H}^{-\infty}_{\text{iso}}(\mathbb{R}^n))$$

which via restriciton to $\epsilon=1$ induces the push-forward map

$$\text{(34.6)}\quad \mathrm{K}^0_c(T^*((M\setminus\partial M)/Y))\simeq\Pi_0(\mathcal{S}(T^*(M/Y);\mathcal{H}^{-\infty}_{\text{iso}}(\mathbb{R}^n))\xrightarrow{R_{\epsilon=1}}\\ \dot{\mathcal{H}}^{-\infty}_{\text{iso}}(M/Y;\mathbb{R}^n)\simeq\mathrm{K}^0(Y).$$

Proof. Everything here is pretty much as before but I should really go through it step by step. In particular the last part, which is the fact that the homotopy classes of sections of the bundle over Y of involutions which are fibre-smoothing perturbations of γ_1 reduces to the K-theory of Y – again this uses the existence of finite-rank exhausting families of projections. ☐

Now, having extended the semiclassical quantization, or push-forward, map to fibrations where the fibres are compact manifolds with boundary it is important to note that this is related to the isotropic case.

Proposition 51. *Under the compactification map $\mathbb{R}^n\hookrightarrow\overline{\mathbb{R}^n}$ the algebras $\dot\Psi^{-\infty}(\overline{\mathbb{R}^n})$ and $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ are identified.*

Proof. This is basically the identification of $\mathcal{S}(\mathbb{R}^n)$ with $\dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^n})$. ☐

Now the same thing is almost true of the adiabatic versions of these algebras. The only difference is the (by some accounts weird) scaling in the isotropic case. Indeed the kernel in the isotropic case can be written

$$\text{(34.7)}\quad \epsilon^{-n}F(\epsilon,\frac{\epsilon(z+z')}{2},\frac{z-z'}{\epsilon},Z,Z')=T_\epsilon\epsilon^{-2n}F(\epsilon,\frac{z+z'}{2},\frac{z-z'}{\epsilon^2},Z,Z')T_\epsilon^{-1}$$

where T_ϵ is the coordinate change $z\longmapsto z/\epsilon$.

Proposition 52. *The parameter-dependent coordinate transformation, T_ϵ, reduces an isotropic-adiabatic family of operators on $\mathbb{R}^n$ to an adiabatic family on the manifold with boundary $\overline{\mathbb{R}^n}$ with parameter ϵ^2.*

This is enough to take care of almost all of the commutativity results we need, except for the most important one. Namely we need to show the commutativity of the top triangle in (33.11).

Proposition 53. *Let $\pi_U : U \longrightarrow M$ be a real vector bundle over the total space of a fibration* (33.2) *then the semiclassical push-forward maps give a commutative diagram*

$$\begin{tikzcd}
\mathrm{K}^0_c(T^*(M/Y) \oplus (U \oplus U')) \arrow[d, "q_{\mathrm{sl}}"'] \arrow[dr, "q_{\mathrm{sl}}"] & \\
\mathrm{K}^0_c(T^*(M/Y)) \arrow[r, "(\phi\pi)_!"'] & \mathrm{K}^0(Y)
\end{tikzcd} \tag{34.8}$$

where the sloping map is given by semiclassical quantization on the fibres, which compact are manifolds with boundary, of $\phi\overline{\pi_U} : \overline{U} \longrightarrow Y$, the vertical map is given by isotropic quantization on the fibres of U and the horizontal map is given by semiclassical quantization on the fibres of ϕ.

Proof. The proof is very close to the similar commutation result for the direct sum of two symplectic bundles. There are two differences, first of course one of the fibrations has fibres which are compact manifolds and the second difference is that U is a bundle over M, not over Y, so this is not a fibre product of bundles over Y. In particular there is only one form of (34.8) – it does not make sense to try to quantize the ϕ fibration *before* the quantization on the fibres of U since the fibres vary along the fibres of M. Still, pretty much the approach works.

Thus, we wish to construct and use a double-adiabatic algebra of smoothing operators. Consider what the kernels should be. There are two parameters, ϵ and δ and in terms of local coordinates y in the base, z on the fibres of ϕ and u linear coordinates on U_m, locally trivialized, the kernels should be of the form

$$\epsilon^{-n}\delta^{-p}F(\epsilon, \delta, y, z, \frac{z-z'}{\epsilon}, \frac{\epsilon^{\frac{1}{2}}\delta(u+u')}{2}, \frac{u-u'}{\epsilon^{\frac{1}{2}}\delta}) \tag{34.9}$$

where F is smooth in all variables and Schwartz in the last three collections of variables. Note the difference with the double isotropic case, the ϵ semiclassical parameter (de-) quantizes in both variables, whereas the δ parameter does so only in the fibre variables.

So the kernels are specified locally near the *fibre diagonal* which is $z = z'$ by (34.9) and away from $z = z'$ the kernels are supposed to be smooth in the z and z' variables (the difference does not make sense since they are generally in different coordinate patches) and rapidly vanishing with all derivatives as $\epsilon \downarrow 0$. The behaviour in u and u' is already specified globally on the fibres of U since they are linear.

Of course the main thing to show is that these operators form an algebra. However this is not significantly different from the earlier discussions. Certainly for $\epsilon >$ this is just an adiabatic family in the isotropic smoothing operators on the fibres of U so it is only necessary to check what happesn as $\epsilon \downarrow 0$. The rapid vanishing in the off-diagonal part in the z, z' variables quelches all other behaviour as is easily seen. Thus it suffices to look at the composition of two kernels of the form (34.9) with compact support in the one coordinate patch Ω in the local fibres

Z and with $U = \mathbb{R}^p$ locally. The composite is then

$$(34.10)\quad \epsilon^{-2n}\delta^{-2p}\int_\Omega\int_{\mathbb{R}^p} F(\epsilon,\delta,y,z,\frac{z-z''}{\epsilon},\frac{\epsilon^{\frac12}\delta(u+u'')}{2},\frac{u-u''}{\epsilon^{\frac12}\delta})$$
$$G(\epsilon,\delta,y,z'',\frac{z''-z'}{\epsilon},\frac{\epsilon^{\frac12}\delta(u''+u')}{2},\frac{u''-u'}{\epsilon^{\frac12}\delta})$$
$$=\epsilon^{-n}\delta^{-p}H(\epsilon,\delta,y,z,\frac{z-z'}{\epsilon},\frac{\epsilon^{\frac12}\delta(u+u')}{2},\frac{u-u'}{\epsilon^{\frac12}\delta})$$

where

$$(34.11)\quad H(\epsilon,\delta,y,z,Z,t,s),\ =\int_\Omega\int_{\mathbb{R}^p} F(\epsilon,\delta,y,z,Z',t+\frac{\epsilon\delta^2(r+s)}{4},\frac{s-r}{2})$$
$$G(\epsilon,\delta,y,z-\epsilon Z',Z-Z',t+\frac{\epsilon\delta^2(r-s)}{4},\frac{s+r}{2}).$$

Secondly we need to understand the symbolic properties in the two, or in some sense three, adiabatic limits. These follow directly from (34.11). □