33. Lecture 30: Topological index Wednesday, 12 November, 2008

Today I want to go through the definition of the topolical index map, $\mathrm{ind}_{\mathrm{t}}$, for a fibration and then start the proof of the equality

$$\mathrm{ind}_{\mathrm{t}} = (\phi\pi)_!^{(\mathrm{sl})} \tag{33.1}$$

where the notation on the right indicates that this is the (direct) push-forward map in K-theory produced by semiclassical quantization.

The topological index is defined following a construction of Gysin. The basic idea here is to 'trivialize' the topology of a given fibration of compact manifolds

$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & Y \end{tikzcd} \tag{33.2}$$

by embedding into a trivial fibration.

Proposition 48. *For any fibration of compact manifolds,* (33.2)*, there is an embedding as a subfibration of a product* $i : M \longrightarrow M' = \mathbb{R}^N \times Y \xrightarrow{\pi_R} Y$ *giving a commutative diagram*

$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[rr, hook, "i"] \arrow[dr, "\phi"'] & & M' \arrow[r, "\pi_L"] \arrow[dl, "\pi_R"] & \mathbb{R}^N \\ & & Y. & & \end{tikzcd} \tag{33.3}$$

After stabilization, taking the product with some $\mathbb{R}^M$*, any two such embeddings are homotopic.*

Proof. Any compact manifold, such as M, can be embedded in a Euclidean space of sufficiently high dimension – indeed the dimension can be estimated quite well. Here we do not care about the codimension of the embedding. To do this, take a finite covering of M by coordinate neighbourhoods U_i, $i = 1, \dots, k$, on each of which the coordinate map is $F_i : U_i \longrightarrow \mathbb{R}^n$, $n = \dim M$. Then take a partition of unity, χ_i, subordinate to the cover and consider the smooth map

$$i' : M \longrightarrow \mathbb{R}^{nk},\ i(p) = \sum_i e_i \chi_i F_i(p),\ e_i : \mathbb{R}^n \longrightarrow \mathbb{R}^{nk} \text{ being the } i\text{th embedding.} \tag{33.4}$$

This is a globally smooth map which is injective and has everywhere injective differential. It is therefore a global embedding. To get the embedding i giving (33.3) take $i = i' \times \phi : M \longrightarrow \mathbb{R}^N \times Y$ where $N = nk$.

The stable homotopy equivalence we really do not need, but let me indicate how to do it anyway. First, we can always increase the dimension N by adding an extra factor of $\mathbb{R}^p$ to M' and extending the map i by mapping M to 0 in this factor. Given two embeddings, stabilize them to have image spaces of the same dimension, and then stabilize further by adding an extra factor of the stabilized fibre dimension, $\mathbb{R}^N$, to each map, interpreting the first as mapping into the first factor and the second into the second factor. Now simply use the standard rotation between the factors of $\mathbb{R}^N$ to deform one map into the other – checking of course that the conditions persist. □

Having embedded ϕ as a subfibration of a trivial fibration we now use the collar neighbourhood theorem.

Proposition 49. *The image $i(M) \subset \mathbb{R}^N \times Y$ of an embedding* (33.3) *has an open neighbourhood Ω with closure $\overline{\Omega} \subset \mathbb{R}^N \times Y$ a compact manifold with boundary which fibres over $i(M)$ as a radially compactified (real) vector bundle and gives a commutative diagram*

$$
\begin{tikzcd}
\overline{\mathbb{R}^q} \arrow[r, dash] & \overline{\Omega} \arrow[r, hook] \arrow[d, shift right, "\pi"'] & \mathbb{R}^N \times Y \arrow[ddl, "\pi_R"] \\
& i(M) \arrow[u, shift right, "0_\pi"'] \arrow[d, "\phi"] & \\
& Y. &
\end{tikzcd}
\tag{33.5}
$$

Proof. This really is just the collar neighbourhood theorem, perhaps with a little smoothness in parameters. Namely, an embedded submanifold, such as $i(M) \subset \mathbb{R}^N \times Y$ has an open neighbourhood which fibres over the manifold and in such a way that the resulting bundle is diffeomorphic to an open neighbourhood of the zero section of a vector bundle over $i(M)$ with the fibration being the bundle projection. Given this we can easily shrink the neighbourhood a little so that it is the image of a closed ball bundle, and call this $\overline{\Omega}$ and the projection π. The only thing to check is that we can make this bundle structure over $i(M)$ compatible with π_R, meaning that π_R factors through it. This is just the requirement that the vector bundle structure π projects the intersection of $\overline{\varpi}$ with $\mathbb{R}^N \times \{y\}$ to the fibre Z_y above y. To ensure that each of the fibres of $\overline{\Omega}$ over $i(M)$ is contained in one of the fibres $\mathbb{R}^N$ it is enough to recall that one proof of the collar neighbourhood theorem proceeds through the exponential maps of *any* Riemannian metric, in the normal directions to the embedded submanifold. In fact it is enough to use a bundle of directions complementary to the tangent bundle. If the metric is taken to be the product metric on $\mathbb{R}^N \times Y$ and the bundle of initial points for the exponential map to be the normals to Z_y within the fibre then the resulting map π respects the fibres. $\square$

Note that the vector bundle structure defined by π on a neighbourhood is necessarily isomorphic to the normal bundle to $i(M)$ in $\mathbb{R}^N \times Y$ and hence to the bundle of normals to the fibres Z_y in $\mathbb{R}^N$. This is later important when we look at the Chern character of the index, i.e. the Atiyah-Singer index formula.

Consider what (33.5) shows for the fibre cotangent bundle $T^*(M/Y)$ for the original fibration ϕ. In the construction above, $\overline{\Omega}$ has been identified smoothly with the total space of a radially compactified vector bundle $U \longrightarrow M$, if we use i to identify M with $i(M)$. This means that the fibre cotangent bundle of $\overline{\Omega}$, as a fibration over $i(M)$ is identified with

$$T^*(\Omega/Y) \simeq T^*(M/Y) \oplus (U \oplus U^*) = T^*(M/Y) \oplus W \tag{33.6}$$

as vector bundles over M. Here $W = U \oplus U^*$ has a natural symplectic structure given in terms of the pairing of U and its dual U^*. Thus, by the Thom isomorphism

$$\mathrm{K}^0_{\mathrm{c}}(T^*(M/Y)) \simeq \mathrm{K}^0_{\mathrm{c}}(T^*(\overline{\Omega}/Y)) \tag{33.7}$$

where we regard W as a symplectic vector bundle over $T^*(M/Y)$.

Now, $\Omega \hookrightarrow \mathbb{R}^N \times Y$ is an open subset, with consistent fibration. Thus the fibre cotangent bundle also embeds as an open subset

$$T^*(\Omega/Y) \hookrightarrow T^*((\mathbb{R}^N \times Y)/Y) = T^*(\mathbb{R}^N) \times Y = \mathbb{R}^{2N} \times Y. \tag{33.8}$$

Thus compactly supported K-theory on the open subset is mapped into compactly supported K-theory of the larger open set

$$\iota : \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/Y)) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2N} \times Y). \tag{33.9}$$

Finally we can apply Bott periodicity to get a composite map which can be written out in steps:

$$\begin{tikzcd}
\mathrm{K}^0_{\mathrm{c}}(T^*(M/Y)) \arrow[r, "\mathrm{Thom}"] \arrow[d, "\mathrm{ind}_{\mathrm{t}}"'] & \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/Y)) \arrow[d, "\iota"] \\
\mathrm{K}^0(Y) & \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2N} \times Y) \arrow[l, "\mathrm{Bott}"]
\end{tikzcd} \tag{33.10}$$

but in principle might depend on the embedding.

Exercise 32. Show that this topological index map does not change under stabilization by additional Euclidean factors in the embedding and also under homotopies of the embedding. Hence conclude that it is in fact well-defined.

I do not feel the need to show the independence of the choice of embedding in the definition of $\mathrm{ind}_{\mathrm{t}}$ because we can show (33.1) without using this. Since the map on the right, given by semiclassical quantization, knows nothing of the embedding this will show the naturality of the topological index as well.

Theorem 13. *The identity* (33.1) *between semiclassical push-forward and topological index maps holds for any embedding of M as a subfibration of a trivial fibration as in* (33.3).

Proof. The strategy is to follow semiclassical quantization around the diagram (33.10) where we have to be a bit careful of some of the identifications that have been made. So we need to check that all the maps in the following diagram are well-defined and all the triangles commute:

$$\begin{tikzcd}
\mathrm{K}^0_{\mathrm{c}}(T^*(M/Y)) \arrow[dd, "q_{\mathrm{sl}}"'] & \mathrm{K}^0_{\mathrm{c}}(\tilde{W}/T^*(M/Y)) \arrow[l, "q_{\mathrm{sl}}"'] \arrow[d, "i"] \arrow[ddl, "q_{\mathrm{sl}}"'] \\
 & \mathrm{K}^0_{\mathrm{c}}(T^*(\overline{\Omega}/Y)) \arrow[d, "\iota"] \arrow[dl, "q_{\mathrm{sl}}"] \\
\mathrm{K}^0(Y) & \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2N} \times Y) \arrow[l, "q_{\mathrm{sl}}"']
\end{tikzcd} \tag{33.11}$$

Perhaps unhelpfully there are five maps labelled q_{sl} and I have added an extra step compared to (33.10) corresponding to the identification of the normal neighbourhood of $i(M)$ with the normal bundle. Thus the map on the left is semiclassical quantization (of involutions) on the fibres of a fibration. The top map is isotropic quantization (in the same general sense) for a symplectic vector bundle over a base – in this case the base is $T^*(M/Y)$. This we know gives the Thom isomorphism so gives the top arrow in (33.10) after reversal. The top sloping map is the combination of these –

isotropic on the fibres of a fibration over a fibre-bundle of manifolds. We need to check that this is well-defined and gives a commutative first triangle. As you can imagine at this point, the commutativity is some double-adiabatic argument but slightly different to what we did before since one part is a compact manifold and the other is a symplectic vector bundle – before they were both bundles. There is a more significant difference in that this is not the fibre product of two fibrations but a double fibration, one is above the other. This means the double-adiabatic algebra needs to be a little different. The second sloping arrow is somewhat new. I mentioned this at some point, but this is the 'same' as the left arrow except that now we have a fibration where the fibres are compact manifolds with boundary. Once this quantization map is defined we need to show commutativity of the triangle above it, meaning that the 'isotropic' quantization can be replaced by the 'manifold' quantization (hence coordinate invariant) in this case. Again I mentioned earlier that this was pretty obvious, but it does need to be done. Finally the bottom q_{sl} is again adiabatic quantization. So once again the commutativity here is the key with the Bott periodicity map isotropic but the one above it not defined precisely this way. □