29. Lecture 26: Semiclassical push-forward for fibrations Monday, 3 November, 2008

At this point I want to start the transition to geometric settings and in particular the Atiyah-Singer index theorem. This can be paraphrased in the form: 'The push-forward in K-theory for fibrations is realized by the index of pseudodifferential operators' – although this is slightly misleading since the push-forward is from the fibre-cotangent bundle of the fibration. That is what I want to examine today.

So, let me start with a single compact manifold Z. In fact I will allow it to be a manifold with corners later, but for the moment let us require that it not have a boundary. The basic commutative object is $\mathcal{C}^\infty(Z)$, the space of smooth functions on Z. I will also assume that you know about $\Lambda^k Z$, the bundle of k-forms on Z.

One thing we need to be able to do is to integrate, invariantly. Given the transformation law for integrals under coordinate changes we can only integrate, at least in the usual sense, objects which transform with a factor of the absolute value of the Jacobian unless we assume that the manifold is oriented. The latter works because we then only make coordinate changes with positive Jacobian matrices anyway and volume forms $v \in \mathcal{C}^\infty(Z; \Lambda^n Z)$, $n = \dim Z$, transform with a factor of the Jacobian:-

$$F^*(dz_1 \wedge \cdots \wedge dz_n) = \det\left(\frac{\partial F_i(z)}{\partial z'_j}\right) dz'_1 \wedge \cdots \wedge dz'_n,\ z_j = F_j(z'). \tag{29.1}$$

Here the fibre at a given point $\Lambda^n_z Z$ can be viewed as the space of totally antisymmetic multilinear forms

$$T_z Z \times T_z Z \cdots \times T_z Z \longrightarrow \mathbb{C} \text{ or } \mathbb{R}. \tag{29.2}$$

In general $\Lambda^k_z Z$ is a contraction for $\Lambda^k(T^*_z Z)$ and alternatively on can identify $\Lambda^n_z Z$ as the space of *linear* functions on, i.e. the dual of, $\Lambda^n(T_z Z)$. The latter is a one-dimensional vector space so we can apply the self-proving

Lemma 34. *For any one dimensional real vector space L the space of absolutely homogoneous functions of degree α*

$$f : L \setminus \{0\} \longrightarrow \mathbb{R},\ f(tv) = |t|f(v)\ \forall\ t, \in \mathbb{R} \setminus \{0\},\ v \in V \setminus \{0\} \tag{29.3}$$

is a well-defined one-dimensional vectors space, denoted $\Omega^\alpha V^*$, or ΩV^* when $\alpha = 1$.

It follows easily enough that the fibres $\Omega(\Lambda^n_z Z)$ form a smooth one-dimensional vector bundle ΩZ over Z. This is the space of densities.

Exercise 23. If you have not done this before, check that the integral is well-defined by reference to local coordinates and a partition of unity:

$$\int_Z : \mathcal{C}^\infty(Z; \Omega Z) \longrightarrow \mathbb{R} \text{ or } \mathbb{C}. \tag{29.4}$$

Note that if $v \in \mathcal{C}^\infty(Z; \Lambda^n Z)$ then $|v| \in \mathcal{C}^0(Z; \Omega Z)$ can be integrated and if Z is oriented and $v > 0$ this gives the integral back again (and in that case $|v| \in \mathcal{C}^\infty(Z; \Omega Z)$).

Now consider the product, $Z_1 \times Z_2$, of two compact manifolds. The density bundle on Z_2 can be pulled back to the product, where we can again denote it ΩZ_2 or $\pi_R^*\Omega$ where $\pi_R : Z_1 \times Z_2 \longrightarrow Z_2$ is the projection and we drop, as obvious, the reminder that the bundle comes from the second factor. Fubini's theorem then shows that

$$\int_{Z_2} : \mathcal{C}^\infty(Z_1 \times Z_2; \pi_R\Omega) \longrightarrow \mathcal{C}^\infty(Z_1). \tag{29.5}$$

Exercise 24. Try to write down a clean proof of the existence, and natural properties, of the integration map

$$\int_{Z_2} : \mathcal{C}^\infty(Z_1 \times Z_2; \pi_L^*E \otimes \pi_R\Omega) \longrightarrow \mathcal{C}^\infty(Z_1; E),\ \pi_L : Z_1 \times Z_2 \longrightarrow Z_1 \tag{29.6}$$

being the projection onto the first factor and E being any vector bundle over Z_1.

We will make extensive use of smoothing operators. Let me set these up first for any pair of compact manifolds Z_i, $i = 1, 2$ with complex vector bundles E_i over them. Namely a smoothing operator is a continuous linear map

$$A : \mathcal{C}^\infty(Z_2; E_2) \longrightarrow \mathcal{C}^\infty(Z_1; E_1) \tag{29.7}$$

which is given by the generalization of (29.6). Namely there must exist a Schwartz kernel $A \in \mathcal{C}^\infty(Z_1 \times Z_2; \mathrm{Hom}(E_2, E_1) \otimes \pi_R^*\Omega)$ such that

$$Au(z_1) = \int_{Z_2} A(z_1, z_2)u(z_1) \tag{29.8}$$

Here $\mathrm{Hom}(E_2, E_1)$ is a bundle over $Z_1 \times Z_2$ which has fibre at a point (z_1, z_2) the linear space of linear maps $T : (E_2)_{z_2} \longrightarrow (E_1)_{z_1}$ – it is unfortunate about the reversals here. Standard linear algebra gives a natural isomorphism

$$\mathrm{Hom}(E_2, E_1) = E_1 \boxtimes (E_2)^*. \tag{29.9}$$

Then (29.8) reduces to (29.6) since it means we 'contract' in E_2 – or apply the homomorphism – to get

$$A(z_1, z_2)u(z_2) \in (E_1)_{z_1} \otimes \Omega(Z_2)_{z_2} \tag{29.10}$$

and then we can integrate.

Exercise 25. I have not discussed the Fréchet topology on $\mathcal{C}^\infty(Z; E)$ for a vector bundle over Z – it is basically the same as the earlier spaces such as $\mathcal{S}(\mathbb{R}^n)$. In fact it is isomorphic to this space! You might wish to go through the topology carefully (and think about the isomorphism which will appear a little later).

For the moment we are most interested in the case $Z_1 = Z_2 = Z$ and $E_1 = E_2 = E$, then $\mathrm{Hom}(E) = \mathrm{Hom}(E, E)$ is a bundle over Z^2. The 'usual' homomorphism bundle, $\hom(E) = \hom(E, E)$ is a bundle over Z which is the restriction of $\mathrm{Hom}(E)$ to the diagonal.

Lemma 35. *The space $\mathcal{C}^\infty(Z^2; \mathrm{Hom}(E) \otimes \pi_R^*\Omega)$ is an associative, non-commutative, Neumann-Fréchet algebra, denoted $\Psi^{-\infty}(Z; E)$ under the operator product*

$$AB(z, z') = \int_Z A(z, z'')B(z'', z'). \tag{29.11}$$

Proof. I leave it to you to check that similar arguments as in the isotropic case show that the product is continuous and that it has the corner property. For the moment this means that with seminorms (based on continuous derivatives in local coordinates and trivializations)

$$\|A_1A_2A_3\|_k \le C\|A_1\|_k\|A_2\|_0\|A_3\|_k. \tag{29.12}$$

This again is Fubini's theorem. It follows that for A in a neighbourhood of the zero in $\Psi^{-\infty}(Z;E)$ the operator $\operatorname{Id}+A$ is invertible with inverse $\operatorname{Id}+B$, $B\in\Psi^{-\infty}(Z;E)$. $\square$

Exercise 26. Let $G^{-\infty}(Z;E)$ be the corresponding Fréchet group – it is in fact an open dense subset of $\Psi^{-\infty}(Z;E)$ with the 'drop the Id' identification. I invite you to check that many things we have done previously hold for this group. It has a determinant function, admits finite rank approximation and it is a classifying space for K^1. I will write down the Chern forms and so on later.

This is our replacement in the geometric setting for $\Psi^{-\infty}(\mathbb{R}^k;\mathbb{C}^N)$. Moreover we can generalize at least some of the things we have done before. First we can introduce the corresponding semiclassical algebra. The scaling here is slightly different to the isotropic case, but this does not in the end make very much difference. Of course we immediately know what smooth dependence on a parameter, even one in a manifold, means.

For the semiclassical calculus we want to consider the appropriate subspace of kernels

$$A_\bullet\in\mathcal{C}^\infty((0,1];\Psi^{-\infty}(Z;E)=\mathcal{C}^\infty((0,1]\times Z^2;\operatorname{Hom}(E)\otimes\pi_R^*\Omega)$$

where I am even too lazy to write the extra pull-backs from Z^2 to $(0,1]\times Z^2$. So, of course the crucial thing is to specify exactly what happens as $\epsilon\downarrow 0$. We demand two things of the kernels A_ϵ. First assume $E=\mathbb{C}$:

$$\begin{gathered}\text{If } K\subset Z^2 \text{ is closed and } K\cap\operatorname{Diag}=\emptyset \text{ then}\\ A_\epsilon\to 0 \text{ rapidly with all derivatives as } \epsilon\downarrow 0 \text{ on } K.\end{gathered}\tag{29.13}$$

$$\begin{gathered}\text{If } U\subset Z \text{ is a coordinate chart then } \exists\ F_U\in\mathcal{C}^\infty([0,1]\times U\times\mathbb{R}^n) \text{ s.t.}\\ A_\epsilon(z,z')=\epsilon^{-n}F_U(\epsilon,z,\frac{z-z'}{\epsilon})|dz'| \text{ on } (0,\epsilon_0(K))\times K\times K,\ \epsilon_0(K)>0\ \forall\ K\Subset U.\end{gathered}\tag{29.14}$$

So, there are two main changes compared to the Euclidean case. First we need to specify the rapid vanishing away from the diagonal – this is true in the Euclidean case anyway – since we do not have global coordinates. Secondly, the scaling is different in (29.14) – it simply does not make sense to scale the base variable since they lie in U. I have also not made the 'Weyl' change from z to $(z+z')/2$ but this is only because I would have to put a double covering – since $(z+z')/2$ is not in U in general.

Exercise 27. Work out the wording for (29.14) in terms of Weyl coordinates.

This is a seriously overspecified definition. Even so, this would not make much sense if it wasn't really local:-

Exercise 28. Check that (29.14) for *all* coordinate charts is equivalent to the same definition for a covering by coordinate charts, given (29.13). Check at the same time that the bundle E can be put back in where in (29.14) U should be such that E is trivial over it and then f should take values in $M(N,\mathbb{C})$ where N is the rank of E.

Proposition 43. *The semiclassical families form an algebra under operator composition, denoted $\Psi_{\mathrm{sl}}^{-\infty}(Z;E)$ with a well-defined symbol map giving a multiplicative short* *exact sequence*

$$\epsilon\Psi_{\mathrm{sl}}^{-\infty}(Z;E) \longrightarrow \Psi_{\mathrm{sl}}^{-\infty}(Z;E) \xrightarrow{\sigma_{\mathrm{sl}}} \mathcal{S}(T^*Z;\hom(E)), \tag{29.15}$$

where in any local coordinate system

$$\sigma_{\mathrm{sl}}(z,\zeta) = \int_{\mathbb{R}^n} e^{-iZ\cdot\zeta} F_U(0,z,Z). \tag{29.16}$$

Proof. Use the preceeding exercise to reduce the problem to local coordinates and then check directly by changing variable as we did before. Most importantly, check that the leading part of F_U, $F_U(0,z,Z)|dZ|$ is actually a well-defined density on T_zZ for each $z \in Z$, so (29.16) makes sense and gives a well-defined function on the cotangent bundle – the density is absorbed by the Fourier transform. □

Digression 1. I am pretty unhappy having to do local coordinate proofs like the one above – that I have not done. So, I now resort to global definititons of things like the semiclassical calculus described above, where the composition and symbolic properties become geoemtrically complelling. In this case it is convenient to use the notion of real blow up. Since I do not have the time to discuss this in the course I have not used it, although I have come close. You could look at the notes from my introductory lectures at MSRI this year but it can also be found in lots of other places. So, let me just assume you know what blow up is. The manifold we want to consider, a manifold with corners, is by definition

$$Z^2_{\mathrm{sl}} = [[0,1]\times Z^2; \{0\}\times \mathrm{Diag}],\ \beta_{\mathrm{sl}} : Z^2_{\mathrm{sl}} \longrightarrow [0,1]\times Z^2. \tag{29.17}$$

That is, it is the kernel and parameter space, blown up at the diagonal at $\epsilon = 0$. This is a manifold with corners with a 'front face' corresponding to the blow up – it is a bundle over $\{0\}\times\mathrm{Diag}$ which is naturally isomorphic to $\overline{TZ}$, the radial compactification of the tangent bundle of Z. The other, or 'old', boundary hypersurface is the closure of the preimage of $\{0\}\times(Z^2\setminus\mathrm{Diag})$. It is naturally diffeomorphic to $[Z^2;\mathrm{Diag}]$, the product with the diagonal blown up. The intersection of these two faces, the corner, is naturally the sphere bundle, the boundary of the radial compactification of the tangent bundle of Z.

The blow-down map can be composed with the projections to get, for instance

$$\tilde\pi_R = \pi_R\circ\beta : Z^2_{\mathrm{sl}} \longrightarrow Z \text{ and } \tilde\beta = \pi_{Z^2}\circ\beta : Z^2_{\mathrm{sl}} \longrightarrow Z^2 \tag{29.18}$$

which are also smooth.

Proposition 44. *The kernels of semiclassical operators, forming* $\Psi_{\mathrm{sl}}^{-\infty}(Z;E)$ *can be identified naturally (by continuity from* $\epsilon > 0$*) with*

$$\big\{A \in (\beta^*(\epsilon))^{-n}\mathcal{C}^\infty(Z^2_{\mathrm{sl}};\tilde\beta^*\operatorname{Hom}(E)\otimes\tilde\pi_R^*\Omega);\\ (\tilde\beta^*\epsilon)^n A \equiv 0 \ \mathit{at}\ \beta^{-1}(\{0\}\times(Z^2\setminus\mathrm{Diag})\big\}. \tag{29.19}$$

Thus, except for the power of ϵ (which can be hidden in the density if one prefers) the kernels are smooth on Z^2_{sl}. The semiclassical symbol then comers from the restriction of the kernel to the front face. If the ϵ factor is absorbed into the density, this is naturally a Schwartz function on TZ with values in the fibre density. The Fourier transform along the fibres then gives the function $\sigma_{\mathrm{sl}}(A)$. The exactness of (29.15) is then just the fact that vanishing at the front face produces a similar kernel with an extra factor of ϵ – since the kernel vanishes rapidly at the 'old' face by assumption.

The product itself can be usefully viewed in this picture too. I may put in a description here if I have an idle moment!

So, all this is setting up the semiclassical calculus of smoothing operators on a compact manifold. Naturally we want to go further and the main immediate extension is to operators on the fibres of a fibration. This is the setting of the Atiyah-Singer theorem.

Recall that a smooth map $\phi : M \longrightarrow Y$ between manifolds (compact or not) is a fibration if there exists another manifold Z such that each point $\bar{y} \in Y$ has an open neighbourhood $U \subset Y$ corresponding to which there is a diffeomorphism F_U giving a commutative diagram

$$\begin{tikzcd} \phi^{-1}(U) \arrow[rr, "F_U"] \arrow[dr, "\phi"'] & & U \times Z \arrow[dl, "\pi_L"] \\ & U. & \end{tikzcd} \tag{29.20}$$

Exercise 29. Recall that the implicit function theorem shows that if M and Y are compact and connected then the condition that ϕ be a submersion, that its differential be surjective at each point of M, implies that it is a fibration. The connectedness conditon can be dropped with minor consequences. Dropping compactness is more serious.

If Y is connected, or by *fiat* in the definition above, the manifold Z is fixed, up to diffeomorphism. I will use the notation

$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & Y \end{tikzcd} \tag{29.21}$$

for a fibration and denote the fibre above $y \in Z$ by $Z_y = \phi^{-1}(y)$. There is no specific map from Z to a given fibre, but such a diffeomorphism does exist by hypothesis from (29.20).

Now, if I am to get this far I will have to be quick. Let me just say, the coordinate invariance of the smoothing algebra and the semiclassical algebra on a fixed manifold, Z, means that it can be transferred to the fibres of ϕ in such a way that we know what

$$\Psi^{-\infty}(M/Y; E) \text{ and } \Psi^{-\infty}_{\mathrm{sl}}(M/Y; E) \tag{29.22}$$

are, where E is a bundle over M (not necessarily coming from a bundle over Y). They are the spaces of smooth sections of bundles of (families of) operators. In the first case we get for each $y \in Y$ an element of $\Psi^{-\infty}(Z_y; E_y)$ where $Z_y = \phi^{-1}(y)$ and E_y is the restriction of E to this submanifold (which of course is diffeomoprhic to Z). In the second case we get a semiclassical family, an element of $\Psi^{-\infty}_{\mathrm{sl}}(Z_y; E_y)$. Enough said, well not quite. We need a little of the geometry of fibrations.

The pull-back of the cotangent bundle of the base $\phi^* T^* Y \longrightarrow T^* M$ is a subbundle and the quotient is denoted

$$T^*(M/Y) = T^*M/\phi^*T^*Y, \ \pi : T^*(M/Y) \longrightarrow M. \tag{29.23}$$

Its fibre can be thought of as the space of fibre-differentials at that point of M.

We will let $G^{-\infty}(M/Y;E)$ be the group of invertibles in $\operatorname{Id}+\Psi^{-\infty}(M/Y;E)$ and $\mathcal{H}^{-\infty}(M/Y;E)$ the space of involutions of the form γ_1+a, $a\in\Psi^{-\infty}(M/Y;\mathbb{C}^2\otimes E)$ where γ_1 is the usual 2×2 matrix.

One of the more serious generalization of the isotropic picture that we need is

Proposition 45. *If Y is compact and* (29.21) *is a fibration with compact fibres (of positive dimension) then for any bundle E over M there are natural identifications*

$$\begin{aligned}\Pi_0(G^{-\infty}(M/Y;E)) &= \mathrm{K}^1(Y),\\ \Pi_0(\mathcal{H}^{-\infty}(M/Y;E)) &= \mathrm{K}^0(Y).\end{aligned}\tag{29.24}$$

To prove this I will rely on a construction that I will not give in the lectures. Not that it is hard, just that it is not so amusing.

Proposition 46. *For any fibration,* (29.21)*, with compact total space and any complex vector bundle E there is a sequence of elements $\Pi_j\in\Psi^{-\infty}(M/Y;E)$ which are projections, $\Pi_j^2=\Pi_j$ and are such that $A\Pi_j\to A$ and $\Pi_jA\to A$, as $j\to\infty$, for each $A\in\Psi^{-\infty}(M/Y;E)$.*

Proof of Proposition 45. We can retract onto operators acting on sections of the range of the bundle Π_j, for j large enough, and then stabilize to get elements of $\mathrm{K}^1(Y)$ or $\mathrm{K}^0(Y)$ and conversely. I should do this properly, but it is similar to the corresponding proof for a symplectic bundle, once we have the Π_j's. □

This again means that the twisting by the fibration does not matter and extends the claims made above that $G^{-\infty}(Z;E)$ is classifying for odd K-theory.

Finally then after wading through all this stuff, we get a theorem which should be almost self-proving at this stage. Let $\mathrm{GL}(E)$ be the bundle of invertible linear maps on the fibres of E and $\mathcal{H}(E)$ be the bundle with fibre the involutions on the fibres of $\mathbb{C}^2\otimes E$.

Theorem 11. *In case the base and fibre of the fibration* (29.21) *is compact, the semiclassical symbol restricts to give surjective maps with connected fibres*

$$\begin{aligned}G^{-\infty}_{\mathrm{sl}}(M/Y;E) &\longrightarrow \mathcal{S}(T^*(M/Y);\pi^*\,\mathrm{GL}(E))\\ \mathcal{H}^{-\infty}_{\mathrm{sl}}(M/Y;E) &\longrightarrow \mathcal{S}(T^*(M/Y);\pi^*\mathcal{H}(E)).\end{aligned}\tag{29.25}$$

Complementing E to a trivial bundle and using the standard stabilizations maps gives

$$\begin{aligned}\Pi_0(\mathcal{S}(T^*(M/Y);\pi^*\,\mathrm{GL}(E))) &\longrightarrow \mathrm{K}^1_c(T^*(M/Y))\ \textit{and}\\ \Pi_0(\mathcal{S}(T^*(M/Y);\pi^*\mathcal{H}(E))) &\longrightarrow \mathrm{K}^0_c(T^*(M/Y))\end{aligned}\tag{29.26}$$

which cover the images as E varies and then, using (29.24)*, define push-forward maps*

$$\begin{aligned}p_{\mathrm{sl}}:\mathrm{K}^1_c(T^*(M/Y))\ni[\sigma_{\mathrm{sl}}(A)] &\longmapsto [R_{\epsilon=1}(A)]\in \mathrm{K}^1(Y),\\ p_{\mathrm{sl}}:\mathrm{K}^0_c(T^*(M/Y))\ni[\sigma_{\mathrm{sl}}(A)] &\longmapsto [R_{\epsilon=1}(A)]\in \mathrm{K}^0(Y).\end{aligned}\tag{29.27}$$