2. Lecture 2: Finite rank approximation
Friday, 29 August, 2008

From last time recall the definition of the sequential version of the 'smoothing group'

$$
\begin{aligned}
G^{-\infty}(\mathbb{N}) = \big\{a \in \Psi^{-\infty}(\mathbb{N}); \exists\, b \in \Psi^{-\infty}(\mathbb{N}) \text{ satisfying}\\
(\mathrm{Id}+a)(\mathrm{Id}+b) = \mathrm{Id}+a+b+ab = \mathrm{Id} = (\mathrm{Id}+b)(\mathrm{Id}+a)\big\}.
\end{aligned} \tag{2.1}
$$

It is not quite obvious here that the existence of the right inverse, the first identity, implies the existence of a left inverse as in the second identity, and the equality of the two. In fact this is true as we will check later, but for the moment we just require the existence of a two-sided inverse.

The main thing for today is to see that it satisfies variants of the 'obvious' properties of $\mathrm{GL}(N,\mathbb{C})$.

Proposition 2. *The group $G^{-\infty}(\mathbb{N})$ is an open, dense, (path) connected subset of $\Psi^{-\infty}(\mathbb{N})$ in which the product and the map $a \longmapsto b = (\mathrm{Id}+a)^{-1} - \mathrm{Id}$ are continuous.*

To prove these results we will use finite dimensional approximation, so really the same stabilization that was the reason for looking at a group like this in the first place. Let Π_k be the projection on the space $\Psi^{-\infty}(\mathbb{N})$ which 'cuts off the tails' after k terms:

$$
(\Pi_k(a))_{ij} = \begin{cases} a_{ij} & \text{if } 1 \le i, j \le k \\ 0 & \text{if } i > k \text{ or } j > k. \end{cases} \tag{2.2}
$$

Clearly $\Pi_k : \Psi^{-\infty}(\mathbb{N}) \longrightarrow \Psi^{-\infty}(\mathbb{N})$ is linear and continuous – in fact it decreases each of the norms

$$
\|\Pi_k a\|_{(N)} \le \|a\|_{(N)} \tag{2.3}
$$

and $\Pi_k^2 = \Pi_k$.

Proposition 3. *A set $K \subset \Psi^{-\infty}(\mathbb{N})$ is precompact (has compact closure) if and only if each of the norms $\|\bullet\|_{(N)}$ is bounded on K and on such a set*

$$
\|\Pi_k a - a\|_{(N)} \to 0 \text{ uniformly as } k \to \infty \ \forall\, N. \tag{2.4}
$$

Proof. Note that the difference $a - \Pi_k a$ has all entries with $i, j \le k$ vanishing. Thus from the definitions of the norms,

$$
\|a - \Pi_k a\|_{(N)} \le k^{-1}\|a\|_{(N+1)} \tag{2.5}
$$

since at least one of $i, j \ge k$ on all non-zero elements. This shows that (2.4) follows from the assumption that all the norms are bounded on K. This in turn implies sequential precompactness (which is precompactness for a metric space) of a set satisfying these conditions by the usual diagonalization process. That is, given a sequence $a(n)$ in K, $\Pi_k a(n)$ is in a bounded subset of a finite dimensional space, so we can extract successive subsequences such that each $\Pi_k a(n_k)$ converges. Passing to the diagonal subsequence and relabelling it as $a(n)$ it follows that we may assume that $\Pi_k a(n) \longrightarrow \Pi_k a$ for each k and some fixed double sequence a_{ij}. It follows from (2.5) that in fact $a \in \Psi^{-\infty}(\mathbb{N})$ and that $a(n)$ converges to it in $\Psi^{-\infty}(\mathbb{N})$.

The converse is similar, maybe a little easier, and anyway of less interest in what follows, so I leave it as an exercise. □

To prove that $G^{-\infty}(\mathbb{N})$ is open we will also use another property of $\Psi^{-\infty}(\mathbb{N})$.

Lemma 1. *The algebra $\Psi^{-\infty}(\mathbb{N})$ has the 'corner property' that for any a, b, $c \in \Psi^{-\infty}(\mathbb{N})$ and any N,*

$$\|abc\|_{(N)} \le C\|a\|_{(N)}\|b\|_{(1)}\|c\|_{(N)},\ N \ge 1. \tag{2.6}$$

Here C is actually independent of N but that is not really the point. As you will see when we get to the geometric realizations of this setup, (2.6) corresponds to the 'smoothing property' of these operators

Proof. This is just the same sort of estimate as before:

$$i^N j^n|(abc)_{ij}| \le \sum_{l,m} i^N|a_{il}||b_{lm}|j^N|c_{mj}| \le (\sum_{l,m} l^{-2}m^{-2})\|a\|_{(N)}\|b\|_{(1)}\|c\|_{(N)}. \tag{2.7}$$

□

Proof of Proposition 2. So, we want to show that for each point $a \in G^{-\infty}(\mathbb{N})$ there is an open ball centred at a with respect to one of the norms which is contained in $G^{-\infty}$. We will use a Neumann series argument. Clearly the group product is continuous, since it is $(a,b) \longmapsto a+b+ab$. Thus, if $\mathrm{Id}+a \in G^{-\infty}$ and U is a neighourhood of $\mathrm{Id} \in G^{-\infty}$ then $(\mathrm{Id}+a)U$ is a neighbourhood of $\mathrm{Id}+a$. Thus it suffices to show that

$$\{a \in \Psi^{-\infty}(\mathbb{N}); \|a\|_{(1)} < 1\} \subset G^{-\infty}(\mathbb{N}). \tag{2.8}$$

To see this consider the Neumann series for the inverse

$$(\mathrm{Id}+a)^{-1} = \mathrm{Id} + \sum_{j=1}^{\infty}(-1)^j a^j. \tag{2.9}$$

This is Cauchy with respect to the norm $\| \bullet$ $_{(1)}$ provided $\|a\|_{(1)} < 1$. Of course, to get (2.8) we need to show that it is Cauchy with respect to all the norms, since that implies that it is Cauchy with respect to the distance. This is where Lemma 1 comes in, since if $\|a\|_{(1)} = c < 1$ then from (2.6)

$$\|a^{j+2}\|_{(N)} \le C\|a\|^2_{(N)}c^j \tag{2.10}$$

which implies that the sequence is Cauchy with respect to each $\| \bullet \|_{(N)}$. Thus the sequence in (2.9) does indeed converge. The limit is a two-sided inverse to $\mathrm{Id}+a$.

The continuity of the inverse map follows from this argument and the continuity of the product is clear.

Next we want to show that $G^{-\infty}(\mathbb{N})$ is connected. Here we can use the finite dimensional approximation to good effect. Since we know that $\Pi_k a \to a$ as $k \to \infty$ and now that $G^{-\infty}(\mathbb{N})$ is open, it follows that $ta + (1-t)\Pi_k a \in G^{-\infty}(\mathbb{N})$ for $t \in [0,1]$ if k is large enough. Thus $\mathrm{Id}+\Pi_k a \in G^{-\infty}(\mathbb{N})$ is connected to a. From the uniqueness of the inverse in a group, $\mathrm{Id}_{k\times k}+\Pi_k a \in \mathrm{GL}(k,\mathbb{C})$ when thought of as a finite dimensional matrix. Here we are using the fact that we can embed $\mathrm{GL}(k,\mathbb{C})$ in $G^{-\infty}$ by subtracting the identity in $\mathrm{GL}(k,\mathbb{C})$ from it, extending the resulting matrix as zero for $i,j > k$ and then adding the formal identity to it afterwards.

So, the connectedness of $G^{-\infty}(\mathbb{N})$ follows from the connectedness of each of the $\mathrm{GL}(k,\mathbb{C})$ (well, we only need this for k large enough). This of course is well known. One way to see it is to use a little spectral theory. If $a \in \mathrm{GL}(k,\mathbb{C})$ then aa^* is positive definite, in particular is selfadjoint with positive eigenvalues, so has a positive square root and defining u by $a = (aa^*)^{\frac{1}{2}}u$ makes u unitary. Moreover the curve $t(a^*a)^{\frac{1}{2}} + (1-t)\,\mathrm{Id}_{k\times k}$ connects the positive definite factor to the identity through positive, hence invertible, elements. Thus it is enough to show that $\mathrm{U}(k)$, the group of unitary matrices is connected. The spectral decomposition of u gives an orthonormal basis of eigenvectors on each of which u acts as $e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Rotating this to 1 on each eigenspace connects u to the identity. Thus each $\mathrm{U}(k)$ and hence $G^{-\infty}(\mathbb{N})$ is connected. □

As for $k \times k$ matrices, it is nice to know that invertibility is determined by the non-vanishing of a 'character', which is to say a multiplicative map defined on $\Psi^{-\infty}(\mathbb{N})$ in the sense that

$$\det\left((\mathrm{Id} + a)(\mathrm{Id}_b)\right) = \det(\mathrm{Id} + a)\det(\mathrm{Id} + b). \tag{2.11}$$

This is often called the 'Fredholm determinant'.

Proposition 4. *There is an entire analytic function*

$$\Psi^{-\infty}(\mathbb{N}) \ni a \longmapsto \det_{Fr}(a) = \det(\mathrm{Id} + a) \in \mathbb{C} \tag{2.12}$$

such that $G^{-\infty}$ is the complement of its null space and if $a = \Pi_k a$ then

$$\det_{Fr}(a) = \det(\mathrm{Id} + a) = \det(\mathrm{Id}_{k\times k} + \Pi_k a) \tag{2.13}$$

reduces to the usual determinant.

So the proof I have in mind is the first use I will make of differential forms on $G^{-\infty}$. There are other, possibly simpler, proofs but this one has the virtue of linking up with the Chern classes later on – in fact that is what we are discussing here, the first odd chern class.

I will therefore launch into a brief discussion of analysis on $G^{-\infty}(\mathbb{N})$. This is fairly straightforward since $G^{-\infty}(\mathbb{N})$ is open in $\Psi^{-\infty}(\mathbb{N})$; it is therefore a complete metric space, so we certainly know what continuity means. For differentiability I will take a strong definition – there are lots of possibilities on Fréchet manifolds but many of them coincide here. So, first note that as an open set of a linear space, the tangent space at any point can be identified with $\Psi^{-\infty}(\mathbb{N})$ itself. For a function $f : U \longrightarrow \mathbb{C}$ where $U \subset \Psi^{-\infty}(\mathbb{N})$ is open, to be differentiable at a we will require the existence of a continuous linear map $Df(a) : \Psi^{-\infty}(\mathbb{N}) \longrightarrow \mathbb{C}$ such that

$$\begin{gathered} f(a+b) - f(a) - Df(a)\cdot b = o(\|b\|_{(N)}) \text{ for } N \text{ sufficiently large} \\ \Longleftrightarrow \\ \exists\ N \text{ such that } \forall\ \delta > 0\ \exists\ \epsilon > 0 \text{ for which} \\ \|b\|_{(N)} < \epsilon \Longrightarrow \|f(a+b) - f(a) - Df(a)\cdot b\|_{(N)} \le \delta\|b\|_{(N)}. \end{gathered} \tag{2.14}$$

Note that if N is large enough and $\epsilon > 0$ is small enough then $a+b \in U$ if $\|b\|_{(N)} < \epsilon$.

The special properties of $G^{-\infty}(\mathbb{N})$ allow us to require as part of the definition of once continuous differentiability, which or course requires differentiability at each point, that (2.14) hold everywhere with the same N and that the derivative

$$Df : U \times \Psi^{-\infty}(\mathbb{N}) \longrightarrow \mathbb{C} \text{ be continuous with respect to } \|\bullet\|_{(N)} \tag{2.15}$$

on both factors. This does not make much sense unless U contains open $\|\bullet\|_{(N)}$-balls around each of its points – which of course is the case for $G^{-\infty}(\mathbb{N})$.