28. Lecture 25: Isotropic families index theorem Friday, 31 October, 2008

This lecture did not go over so well, there were definitely blank stares at what I thought was the punchline! I think one problem was my insistence on working in the generality of a symplectic bundle W over X instead of working on operators on (Schwartz functions on the fibres of) a real vector bundle U over X – which is the special case where $W = U \oplus U'$ with the symplectic form coming from the pairing. It is a bit late now to undo this. In fact I clearly tried to include too much in this lecture, as I discovered when I tried to write it up! Sorry about that, but I will press on to the Atiyah-Singer Theorem. The argument I will use there is essentially the same as this one, so maybe it will become clearer as we go on. This isotropic index theorem is not actually needed in the proof.

Here is the outline I had originally for the lecture, which is pretty much what I did. I will try to write out a different version below in the hopes that it will be more helpful. Of course, part of the problem was that I did not take the time to do things in detail.

Outline:-

(1) Index for $P \in \Psi^k_{\text{iso}}(\mathbb{R}^k)$, elliptic, is an integer.
(2) Index for an elliptic family $P \in \mathcal{C}^\infty(X; \Psi^k_{\text{iso}}(\mathbb{R}^k))$, X compact, is an element of $\mathrm{K}^0(X)$ determined by choosing a parametrix and defining

$$\operatorname{ind}_{\text{iso}}(P) = [I(P,Q)]$$
$$I(P,Q)(x) = \begin{pmatrix} 1 - 2R_L^2 & 2R_L(\mathrm{Id} + R_L)Q \\ 2R_R P & -\mathrm{Id} + 2R_R^2 \end{pmatrix} \in \mathcal{C}^\infty(X; \mathcal{H}^{-\infty}_{\text{iso}}(\mathbb{R}^k),$$
$$R_L = \mathrm{Id} - PQ,\ R_R = \mathrm{Id} - QP, \tag{28.1}$$

from (25.5).

(3) More generally we want to consider a symplectic bundle $W \longrightarrow X$ (I will take the base here to be compact to avoid having to qualify various statements and come back to the non-compact case if necessary – it isn't seriously harder). Then we could take an elliptic family

$$P \in \Psi^k_{\text{iso}}(W/X; \mathbb{C}^N) \tag{28.2}$$

where this stands for the space of sections – so for each $x \in X$ we have an element $P(x) \in \Psi^k_{\text{iso}}(W_x; \mathbb{C}^N)$ which varies smoothly with $x \in X$. In fact, for reasons of generality but also it turns out for topolical reasons that I will mention somewhere, we will consider a pair of complex (smooth of course) bundles over X, $\mathbb{E} = (E_+, E_-)$ which I write as a superbundle for fun and brevity. Then we want an elliptic family

$$P \in \Psi^k_{\text{iso}}(W/X; \hom(\mathbb{E})) \text{ elliptic,} \tag{28.3}$$

which means 'formally mapping sections of E_+ to sections of E_-.' Of course the things we have are not operators so what this means is

$$\begin{gathered} P(x) \in \Psi^k_{\text{iso}}(W_x; \hom(E_+(x), E_-(x)) = \mathcal{C}^\infty({}^q\overline{W_x}; \hom(E_+(x), E_-(x)), \\ P \in \Psi^k_{\text{iso}}(W/X; \hom(E_+, E_-) = \mathcal{C}^\infty({}^q\overline{W}; \pi^* \hom(E_+, E_-) \end{gathered} \tag{28.4}$$

where I have written out what these are as spaces of functions (well, sections of bundles). Thus $\hom(\mathbb{E}) = \hom(E_+, E_-)$ is the bundle of homomorphisms on the fibres. The form algebras when $\mathbb{E} = (E, E)$ is a fixed bundle and more generally form modules and can be composed when the bundles 'in the middle' are the same.

(4) So, what is the index of an elliptic operator (28.3)? It is supposed to be given by the same formula (28.1). Of course we have to remember the bundles. Still, the construction of a smooth parametrix goes through unchanged to give

$$
\begin{gathered}
Q \in \Psi_{\mathrm{iso}}^{-k}(W/X;\hom(\mathbb{E}^-)),\ \mathbb{E}^- = (E_-, E_+), \\
R_L = \mathrm{Id} - PQ \in \Psi_{\mathrm{iso}}^{-\infty}(W/X;\hom(E_-)),\ R_R = \mathrm{Id} - QP \in \Psi_{\mathrm{iso}}^{-\infty}(W/X;\hom(E_+)).
\end{gathered}
\tag{28.5}
$$

So, let us embed $\mathbb{E} \longrightarrow (\mathbb{C}^N, \mathbb{C}^N)$ which is to say, embed both bundles in trivial bundles which can be taken to have the same rank. Let $\pi_\pm(x)$ be the projections onto the ranges of the embeddings of $E_\pm$ in $\mathbb{C}^N$. Now if we look at (28.1) we get the central block in

$$
\tilde{I}(P, Q) = \begin{pmatrix} \mathrm{Id} - \pi_+(x) & 0 & 0 & 0 \\ 0 & 1 - 2R_L^2 & 2R_L(\mathrm{Id} + R_L)Q & 0 \\ 0 & 2R_R P & -\mathrm{Id} + 2R_R^2 & 0 \\ 0 & 0 & 0 & -(\mathrm{Id} - \pi_-(x)) \end{pmatrix}.
\tag{28.6}
$$

So, this has been stabilized into a $\mathbb{C}^2 \otimes \mathbb{C}^N$ and hence can be mapped into $\mathcal{H}_{\mathrm{iso}}^{-\infty}(W \times \mathbb{R}^2/X)$. This then is our index from the point of view of involutions. Later I will do the more conventional (but of course very closely related) stabilization to projections.

(5)

Theorem 9. *For an elliptic family* (28.4) *the class of the involution* (28.6) *does not depend on the choices involved and so defines an index class which only depends on the symbol (and of course the bundles):*

$$
\mathrm{ind}_{\mathrm{iso}}(P) = \mathrm{ind}_{\mathrm{iso}}(\sigma(P); \mathbb{E}) \in \mathrm{K}^0(X).
\tag{28.7}
$$

The index map (say from elliptic-parametrix pairs) factors through the semiclassicl index map giving the Thom isomorphism in terms of a map $[\sigma]$ *to be described giving a commutative diagram*

$$
\begin{tikzcd}
\{(P, Q)\} \arrow[rr, "{[\sigma]}"] \arrow[dr, "\mathrm{ind}_{\mathrm{iso}}"'] & & \mathrm{K}_c^0(W) \arrow[dl, "\mathrm{ind}_{\mathrm{sl}} = \mathrm{Thom}"] \\
& \mathrm{K}^0(X). &
\end{tikzcd}
\tag{28.8}
$$

Furthermore the vanishing of $\mathrm{ind}_{\mathrm{iso}}(P)$ *is a necessary and sufficient condition for the existence of a perturbation* $T \in \Psi_{\mathrm{iso}}^{-\infty}(W/X;\mathbb{E})$ *such that* $P + T$ *is invertible with inverse* $(P+T)^{-1} \in \Psi_{\mathrm{iso}}^{-k}(W/X;\mathbb{E}^-)$.

(6) Let me proceed to the idea of the proof without first defining the K-class of the symbol, but that is really what I am working towards. For the moment I will work with projections, but it might be better to do it with involutions. So, what we have done above is embedd the two bundles as

projection-valued sections of $\mathbb{C}^N$ over X. So that is our data:

$$\begin{gathered}\pi_\pm : X \longrightarrow M(N;\mathbb{C}),\ \pi_\pm^2 = \pi_\pm \text{ and } p,\ q \in \mathcal{C}^\infty(\mathbb{S}W; M(N;\mathbb{C})),\\ \pi_-(x)p(w_x) = p(w_x) = p(w_x)\pi_+(x),\ q(w_x)\pi_-(x) = p(w_x) = \pi_+(x)q(w_x),\\ p(w_x)q(w_x) = \pi_-(x),\ q(w_x)p(w_x) = \pi_+(x),\ w_x \in \partial^q\overline{W_x}.\end{gathered}\tag{28.9}$$

Here q is just the inverse of $p = \sigma(P)$ extended as zero outside the bundles. So we need to understand how this gives an element of $\mathrm{K}_{\mathrm{c}}^0(W)$.

(7) To do so, let me generalize the symbolic data in (28.9). Namely we can take exactly the same thing except that we allow the projections to depend on the variables on W but require them to be smooth up to the boudary of the quadratic compactification:-

$$\begin{gathered}\pi_\pm : {}^q\overline{W} \longrightarrow M(N;\mathbb{C}),\ \pi_\pm^2 = \pi_\pm \text{ and } p,\ q \in \mathcal{C}^\infty(\mathbb{S}W; M(N;\mathbb{C})),\\ \pi_- p = p = p\pi_+,\ q\pi_- = p = \pi_+ q,\ pq = \pi_-,\ qp = \pi_+ \text{ on } \partial^q\overline{W}.\end{gathered}\tag{28.10}$$

(8) Now what we want to do is to *quantize* this more general data. We can do this in the following way:-

Proposition 37. *For general data as in* (28.10) *there exist semiclassical families of projections*

$$\begin{gathered}\Pi_\pm \in \Psi^0_{\mathrm{ad}}(W/X;\mathbb{C}^N) \textit{ with } \sigma_{\mathrm{ad}}(\Pi_\pm) = \pi_\pm,\ \sigma_{\mathrm{iso}}(\Pi_\pm) = \pi_\pm \textit{ on } \partial^q\overline{W} \textit{ and}\\ P,\ Q \in \Psi^0(W/X;\mathbb{C}^N) \textit{ s.t. } \sigma_0(P) = p,\ \sigma_0(Q) = q,\\ R(\Pi_-)P = PR(\Pi_+),\ R(\Pi_+)Q = QR(\Pi_-),\\ R_R = R(\Pi_-) - PQ,\ R_L = R(\Pi_+) - QP \in \Psi^{-\infty}_{\mathrm{iso}}(W/X;\mathbb{C}^N).\end{gathered}\tag{28.11}$$

Here R is our usual restriction to $\epsilon = 1$ of the adiabatic family.

Proof. We can quantize the $\pi_\pm$ to semiclassical families, with constant standard symbol (as a function of ϵ.) Such a family is unique up to homotopy through such families. Then we can quantize p and q to operators P' and Q', replace these by $R(\Pi_-)P'R(\Pi_+)$ and $R(\Pi_+)Q'R(\Pi_-)$. Then we need to correct a little to get the remainder terms to be smoothing. □

Really this is just a variant of the elliptic construction.

(9) So, we can write down the 'same' involution still defining a K-class:-

$$\tilde{I}(P,Q,\Pi_\pm) = \begin{pmatrix} \mathrm{Id} - \Pi_+(x) & 0 & 0 & 0\\ 0 & 1 - 2R_L^2 & 2R_L(\mathrm{Id} + R_L)Q & 0\\ 0 & 2R_R P & -\mathrm{Id} + 2R_R^2 & 0\\ 0 & 0 & 0 & -(\mathrm{Id} - \Pi_-(x))\end{pmatrix}.\tag{28.12}$$

The uniqueness of the construction up to homotopy shows that this defines an index from the 'general data' in (28.10).

(10) I now need to define the map for the 'generalized data' into $\mathrm{K}_{\mathrm{c}}^0(W)$.

(11) So, how does it improve things to make the problem harder? Well, there is one thing to notice here.

Proposition 38. *The data where $\pi_\pm$ are equal to the same constant projection outside a compact subset of W and $p = q = \mathrm{Id}$ on the range of this projection* *'generates' all generalized data up to operations, stability and homotopy, under which the index is constant.*

So, the proof is really this observation plus the fact that the index in this case is given by semiclassical quantization.

So that was my original outline. Let me approach things from the other end writing out the maps that do exist more carefully. Thank you Paul for pointing out that I had lost a lot qiso's – I was using iso instead. This really does not make any *significant* difference; it just makes a difference. Note that X is taken compact below.

First of all, what exactly is the families isotropic index? Consider the subspace of elliptic operators

$$\mathrm{Ell}^0_{\mathrm{qiso}}(W/X;\mathbb{E}) \subset \Psi^0_{\mathrm{qiso}}(W/X;\mathbb{E}) = \mathcal{C}^\infty({}^q\overline{W};\pi^*\hom(\mathbb{E})). \tag{28.13}$$

As a space of 'functions' the space on the right consists of the smooth sections over ${}^q\overline{W}$, the quadratic compactification of W, of the pull-back of the homomorphism bundle from E_- to E_+ over X. Thus at each point of $w \in {}^q\overline{W}$ one has a linear map from $E_+(x)$ to $E_-(x)$ where $\pi(w) = x$ and this depends smoothly on w. The elliptic elements are those for which the symbol, p, just the restriction to the sphere at infinity, is invertible. In particular the ranks of $E_\pm$ must be equal for there to be any elliptic elements.

Lemma 31. *Any elliptic family $P \in \mathrm{Ell}^0_{\mathrm{qiso}}(W/X;\mathbb{E})$ has a parametrix*

$$Q \in \mathrm{Ell}^0_{\mathrm{qiso}}(W/X;\mathbb{E}^-,)\ \mathbb{E}^- = (E_-,E_+),$$

meaning that

$$R_L = \mathrm{Id} - QP \in \Psi^{-\infty}_{\mathrm{iso}}(W/X;E_+),\ R_R = \mathrm{Id} - PQ \in \Psi^{-\infty}_{\mathrm{iso}}(W/X;E_-) \tag{28.14}$$

and any two parametrices are smoothly homotopic within parametrices.

Proof. Pretty much the same old constructions using the symbol map, iteration and asymptotic summation. □

So we will consider the big set of pairs, of elliptic elements and parametrices as in (28.14), together with smooth embeddings $\pi_\pm$ of the bundles $E_\pm$ as subbundles of $\mathbb{C}^N$ over X and denote this $\mathcal{P}^0_{\mathrm{qiso}}(W/X;\mathbb{E}) = \{P,Q,\pi_\pm\}$.

Definition 7. Let $\mathcal{D}(W)$ be the collection of *elliptic data* for W with elements $(p',\pi_\pm)$ where $\pi_\pm \in \mathcal{C}^\infty(X;M(N,\mathbb{C}))$ are projection-valued and $p \in \mathrm{Iso}(\mathbb{S}W;\pi)$ is a smooth isomorphism between the pull-back of the range of π_+ and the pull-back of the range of π_- over $\mathbb{S}W$.

Each element of $\mathcal{P}^0_{\mathrm{qiso}}(W/X;\mathbb{E})$ defines an involution through

$$I(P,Q,\pi_\pm) = \begin{pmatrix} \mathrm{Id} - \pi_+(x) & 0 & 0 & 0 \\ 0 & 1-2R_L^2 & 2R_L(\mathrm{Id}+R_L)Q & 0 \\ 0 & 2R_RP & -\mathrm{Id}+2R_R^2 & 0 \\ 0 & 0 & 0 & -(\mathrm{Id}-\pi_-(x)) \end{pmatrix} \in \mathcal{H}^{-\infty}(W/X;\mathbb{C}^N) \tag{28.15}$$

where, since we have chosen an embedding of E_+ and E_- into the trivial bundle $\mathbb{C}^N$, so all terms can be regared as 'operators on' $\mathbb{C}^N$. Thus, the 4-fold decomposition of $\mathbb{C}^2 \otimes \mathbb{C}^N = \mathbb{C}^N \oplus \mathbb{C}^N$ in (28.15) is in terms of the ranges of $(\mathrm{Id} - \pi_-(x))$, $\pi_-(x)$, $\pi_+(x)$ and $\mathrm{Id} - \pi_+(x))$. In particular the operators in the central 2×2 block are of the form

$$\begin{pmatrix} \Psi^{-\infty}(W/X; E_-) & \Psi^{-\infty}(W/X; \mathbb{E}) \\ \Psi^{-\infty}(W/X; \mathbb{E}^-) & \Psi^{-\infty}(W/X; E_+) \end{pmatrix}. \tag{28.16}$$

Exercise 22. Make sure that (28.15) is an involution.

Proposition 39. *Corollary 7 shows that the involution* (28.15) *defines an element of* $\mathrm{K}^0(X)$ *and this induces a commutative diagram*

$$\begin{tikzcd} & \mathcal{P}^0_{\mathrm{qiso}}(W/X; \mathbb{E}) \arrow[dl, "\sigma"'] \arrow[dr, "{[I(P,Q,\pi_\pm)]}"] & \\ \mathcal{D}(W) \arrow[rr, "\mathrm{ind}_{\mathrm{iso}}"'] & & \mathrm{K}^0(X). \end{tikzcd} \tag{28.17}$$

Proof. We have to check that the class $[I(P, Q, \pi_\pm)]$ defined by applying Corollary 7 to (28.15) is independent of the choices of parametrix, Q, of quantization P and of embedding of $E_\pm$ in $\mathbb{C}^N$, including the indepence of N. In fact the last of these is the simplest since increasing N just corresponds to stabilization which is already part of the definition of the map in $\mathrm{K}^0(X)$. By definition the class $[I(P, Q, \pi_\pm)]$ is homotopy invariant – notice that the notation really is inadequate since it depends on the *identification* of $E_\pm$ with the ranges of $\pi_\pm$. Fixing everything else, independence of the choice of P and Q follows from the fact that the linear family $(1-t)P_0 + tP_1$ between any two quantization (operators with symbol p) consists of quantizations and the construction of parametrices can be carried out uniformly in an additional parameter (which can be hidden in X). Thus, it suffices to suppose that P is fixed. Then the linear homotopy $(1-t)Q_0 + tQ_1$ consists of parametrices. Thus it suffices to consider P and Q fixed and change the embeddings. Changing the embedding of $E_\pm$ with $\pi_\pm$ fixed means conjugating by isomorphisms on the ranges of π_+ and π_- in the middle block (but not the outer block) in (28.5). These can be rotated away after stabilising a bit. On the other hand, if $\pi'_\pm$ is another family of projections with range bundle isomorphic to $E_\pm$, for the same N then there are necessarily elements $F_\pm \in \mathcal{C}^\infty(X; \mathrm{GL}(N, \mathbb{C}))$ conjugating $\pi_\pm(x)$ to $\pi'_\pm(x)$ for each x and homotopic to the identity, see Proposition 40 – these can be deformed away. $\square$

Proposition 40. *If* $F_i : E \longrightarrow \mathbb{C}^N$ *are two embeddings of a complex vector bundle into a trivial bundle then, after stabilizing by further embedding as* $\tilde{F}_i = F_i \oplus 0 : E \longrightarrow \mathbb{C}^N \oplus \mathbb{C}^N$ *there is an element* $A \in \mathcal{C}^\infty(X; \mathrm{GL}(N+M, \mathbb{C}))$ *which is homotopic to the identity and conjugates the range of* F_1 *to the range of* F_2.

Proof. Let $\pi_i \in \mathcal{C}^\infty(X; M(N, \mathbb{C})$ be the orthogonal projections onto the ranges of the F_i. We can consider the joint embedding $F_1 \oplus F_2 : E \oplus E \longrightarrow \mathbb{C}^{2N}$ which has range $\pi_1(x) \oplus \pi_2(x)$ at each point. Consider the 'rotation' on $\mathbb{C}^{2N}$ obtained by decompositing into the four pieces

$$G_t : (v, w) \longmapsto \pi(x)v + (\mathrm{Id} - \pi_1(x))w + \pi_2(x)w + (\mathrm{Id} - \pi_2(x))w$$
$$\longmapsto (\cos t)\pi(x)v + (\sin t)F_1(F_2)^{-1}(\pi_2(x)v) + (\mathrm{Id} - \pi_1(x))w$$
$$- (\sin t)F_2(x)F_1^{-1}\pi(x)v + (\cos t)\pi_2(x)w + (\mathrm{Id} - \pi_2(x))w. \tag{28.18}$$

Clearly, $G_0 = \mathrm{Id}$ and $G_{\pi/2}$ conjugates the range of $\pi_1 \oplus 0$ to the rang of $0 \oplus \pi_2$. Following this by a 1-parameter family of rotations between the two factors, starting at the identity and finishing at a map which exchanges the factors (and reverses one sign) finally gives a bundle isomorphism of $\mathbb{C}^{2N}$ which intertwines the two projections and is connected to the identity – where it is easy to make the family smooth:

$$g_0 = \mathrm{Id},\ g_1^{-1}(\pi_1 \oplus 0)g_1 = \pi_2 \oplus 0. \tag{28.19}$$

□

Theorem 10. *(Essentially Theorem 9 above). The relation $\sim$ on $\mathcal{D}(W)$ generated by stability, bundle isomorphisms and homotopy on p, gives a natural isomorphism $\mathcal{D}(W)/\sim = \mathrm{K}^0_c(W)$ which leads to a commutative diagram*

$$\begin{tikzcd}
& \mathcal{P}^0_{\mathrm{qiso}}(W/X;\bullet) \arrow[dl, "\sigma"'] \arrow[dr, "{[I(P,Q,\pi_\pm)]}"] & \\
\mathcal{D}(W) \arrow[rr, "\mathrm{ind}_{\mathrm{iso}}"'] \arrow[dr, "\sim"'] & & \mathrm{K}^0(X) \\
& \mathrm{K}^0_c(W) \arrow[ur, "\mathrm{Thom}=p_{\mathrm{sl}}"'] &
\end{tikzcd} \tag{28.20}$$

under which the isotropic index map factors through the semiclassical realization of the Thom isomorphism. The vanishing of $\mathrm{ind}(\sigma(P)$ *in* $\mathrm{K}^0(X)$ *is a necessary and sufficient condition for the existence of a perturbation $T \in \Psi^{-\infty}_{\mathrm{iso}}(W/X;\mathbb{E})$ such that $P+T$ is invertible with inverse in $\Psi^0_{\mathrm{iso}}(W/X;\mathbb{E}^-)$ and is also equivalent to the exists of an homotopy, through elliptic elements starting from a stabilization of P, to the identity.*

So this is the families isotropic index theorem.

Now the main aim is to prove (28.20) identifying the isotropic index map with the Thom isomorphism. It would be logical to discuss the relation $\sim$ on the symbol data, however as in the outline above, I prefer to launch into a discussion of 'generalized symbol data'. The key ingredient in the proof of (28.20) is then that the index map can be extended to this more general data.

Definition 8. The space $\widetilde{\mathcal{D}}^0(W)$ of *generalized elliptic data* for W consists of the elements $(p, \pi_\pm)$ where $\pi_\pm \in \mathcal{C}^\infty({}^q\overline{W}; M(N,\mathbb{C})$ are projection-valued and $p \in \mathrm{Iso}(\mathbb{S}W;\pi)$ is a smooth isomorphism between the range of π_+ and the range of π_- over $\mathbb{S}W$.

So the only sense in which this is generalized compared to Definition 7 is that the projections are smooth on the whole of the quadratic compactification of W – rather than being pulled-back from X and so constant on the fibres. Of course

$$\mathcal{D}(W) \subset \widetilde{\mathcal{D}}^0(W). \tag{28.21}$$

The main thing we need, and at this stage it may seem just like a strange generalization, is to define the index map on the whole of this generalized data. So let me consider a big version of $\mathcal{P}^0_{\text{qiso}}(W/X;\bullet)$ discussed above.

Definition 9. Let $\widetilde{\mathcal{P}}^0(W/X)$ consist of all elements $(P,Q,\Pi_\pm)$ where for some N,

$$\begin{gathered}\Pi_\pm\in\Psi^0_{\text{ad,qiso}}(W/X;\mathbb{C}^N),\ \Pi^2_\pm=\Pi_\pm,\\ \sigma_{\text{iso}}(\Pi_\pm)\text{ is independent of }\epsilon,\\ P,\ Q\in\Psi^0_{\text{qiso}}(W/X;\mathbb{C}^N)\text{ satisfy}\\ P=R(\Pi_-)P=PR(\Pi_+),\ Q=R(\Pi_+)Q=QR(\Pi_-),\\ R_L=R(\Pi_+)-QP,\ R_R=R(\Pi_-)-PQ\in\Psi^{-\infty}_{\text{iso}}(M/X;\mathbb{C}^N)\end{gathered}\tag{28.22}$$

where R is the restriction of the adiabatic family to $\epsilon=1$.

Again we certainly have

$$\mathcal{P}^0(W)\subset\widetilde{\mathcal{P}}^0(W/X)\tag{28.23}$$

where an element $(P,Q,\pi_\pm)$ can be regarded as an element of $\widetilde{\mathcal{P}}^0(W/X)$ since $\pi_\pm$ are just smooth matrices over X so can be thought of as adiabatic families, just constant matrices on each fibre, which are then completely independent of ϵ. In particular in this case $\Pi_\pm=\pi_\pm$ with the isotropic symbol reducing to $\pi_\pm$ again and this is constant in ϵ.

Lemma 32. *The adiabatic and istropic symbol maps induce a surjective map*

$$\widetilde{\mathcal{P}}^0(W/X)\ni(P,Q,\Pi_\pm)\longmapsto(p,\pi_\pm=\sigma_{\text{sl}}(\Pi_\pm))\in\widetilde{\mathcal{D}}^0(W).\tag{28.24}$$

Proof. The existence of the map (28.24) is just a matter of checking the consistency conditions. Namely, the adiabatic symbols of $\Pi_\pm$ are projections $\pi_\pm\in\mathcal{C}^\infty({}^q\overline{W};M(N,\mathbb{C})$. The compatibility between adiabatic and istropic symbols means that the istropic symbol restricted to $\epsilon=0$ is $\pi_\pm$ restricted to the boundary, $\mathbb{S}W$. Then the insistence in (28.22) that the isotropic symbol be constant in ϵ means this holds everywhere. Then it follows from the other conditions in (28.22) that $p=\sigma_{\text{iso}}(P)$ satisfies

$$\pi_-p=p\pi_+=p\tag{28.25}$$

and the existence and properties of Q then it means it is an isomorphism from the range of π_+ to the range of π_- at each point of $\mathbb{S}W$.

So, the surjectivity is just the converse, that every such triple in $\widetilde{\mathcal{D}}^0(W)$ arises this way. This is our usual constructive task – to find $\Pi_\pm$ and P, Q as in (28.24) given p and $\pi_\pm$. First think about the $\pi_\pm$. The joint surjectivity of semiclassical and isotropic symbols means that we can choose $\Pi'_\pm\in\Psi^0_{\text{ad,iso}}(W/X;\mathbb{C}^N)$ with the symbolic conditions in (28.24) – since the compatibility condition is evidently satisfied. Now, I leave it to you to go back and see that $\Pi'_\pm$, which are necessary projections up to leading order, in both the semiclassical and the isotropic sense, can be deformed to be actual projections with the same symbols, i.e. by adding terms which are lower order in both senses. In brief this comes from the same iterative arguments with the symbols as before. Do it first at the semiclassical face, checking that the resulting correction (after summing the Taylor series) is of order at most -1 in the isotropic sense. Then do the same iterative correction using the isotropic symbol and note that all terms, and hence the asymptotic sum, can be chosen to vanish to infinite order at $\epsilon=0$ (so they aren't really semiclassical, just smooth in ϵ). This corrects $\Pi'_\pm$ to be projections modulo and error $(\Pi'_\pm)^2 - \Pi'_\pm$ which is of order $-\infty$ and vanishes to infinite order at $\epsilon = 0$. Now, the integral argument allows this to be corrected to a family of projections in $\epsilon < \epsilon_0$ for some $\epsilon_0 > 0$. Since the isotropic symbol is constant anyway, reparameterizing allows the family to be 'extended' all the way to $\epsilon = 1$.

Now, having constructed $\Pi_\pm$ we need to construct P and Q to satisfy the remaining conditions. We can certainly choose $P' \in \Psi^0_{\mathrm{qiso}}(W/X;\mathbb{C}^N)$ with $\sigma_{\mathrm{iso}}(P') = p$. Replacing it by $P = R(\Pi_-)P'R(\Pi_+)$ does not change the symbol, given the properties of p and the $\pi_\pm$ and of course implies that $P = R(\Pi_-)P = PR(\Pi_+)$. So, it remains to construct Q satisfying the remaining properties. By assumption p has a generalized inverse $q \in \mathcal{C}^\infty(\mathbb{S}W; M(N,\mathbb{C}))$ such that $pq = \pi_-$, $qp = \pi_+$ on $\mathbb{S}W$. First take $Q' \in \Psi^0_{\mathrm{qiso}}(W/X;\mathbb{C}^N)$ with $\sigma_{\mathrm{iso}}(Q') = q$ and set $Q_0 = R(\Pi_+)Q'\Pi_-$. We have everything but the last line in (28.22) and we have this to first order, because of the properties of p and q – namley

$$R'_L = R(\Pi_+) - Q_0P \in \Psi^0_{\mathrm{qiso}}(W/X;\mathbb{C}^N) \text{ has } \sigma_{\mathrm{iso}}(R'_L) = \pi_+ - qp = 0 \Longrightarrow R'_L \in \Psi^{-1}_{\mathrm{qiso}}(W/X;\mathbb{C}^N). \tag{28.26}$$

Thus we wish to successively add lower order terms to make successive symbols vanish. We can in fact add any term of order -1 or the form $R(\Pi_+)Q_1R(\Pi_-)$ to Q_0 and this has arbitrary symbol of order -1 of the form $\pi_+q_1\pi_-$. Moreover it follows by composing the identity in (28.26) on the right with Π_- and the left with Π_+ that the symbol of R'_L of order -1 is of this form. So, iterating this argument and asymptotically summing we can arrange that Q satisfies the first identity on the last line in (28.22) with everything else still holding. So, it remains to ensure the last condition – which can certainly be done by the obvious variant of the preceeding argument, but we need both to hold at once! So, go back to the previous construction and proceed by induction. The extra step is that at stage p we have arranged that R'_L is of order $-k-1$ and R'_R is of order $-k$ and we want to correct the second without destroying the first. The term we add to Q' is $\Pi_+Q'_k\Pi_-$ where $\sigma_{-k}(Q'_k) = q_k = -\sigma_{-k}R'_R$. However, from the definition of $R'_R = \Pi_+ - PQ'$, $R'_RP = PR'_L$ has vanishing symbol of order $-k$, so $q'_kp = 0$ from which it follows that both conditions then hold at order $-k$ and the induction can continue.

Thus indeed the operators in (28.22) can be constructed and surjectivity follows, proving the Lemma. □

Proposition 41. *The analogue of Proposition 39 holds for the 'generalized' parametrix sets and elliptic data, so inducing a commutative diagram which restricts to (28.17):-*

$$\begin{tikzcd} & \widetilde{\mathcal{P}^0}_{\mathrm{qiso}}(W/X;\mathbb{E}) \arrow[dl, "\sigma"'] \arrow[dr, "{[I(P,Q,\Pi_\pm)]}"] & \\ \widetilde{\mathcal{D}}(W) \arrow[rr, "\mathrm{ind}_{\mathrm{iso}}"'] & & \mathrm{K}^0(X). \end{tikzcd} \tag{28.27}$$

Proof. The main thing to notice is that modifying (28.15) to

$$I(P,Q,\Pi_\pm)= \begin{pmatrix} \operatorname{Id}-\Pi_+(x) & 0 & 0 & 0\\ 0 & 1-2R_L^2 & 2R_L(\operatorname{Id}+R_L)Q & 0\\ 0 & 2R_RP & -\operatorname{Id}+2R_R^2 & 0\\ 0 & 0 & 0 & -(\operatorname{Id}-\Pi_-(x)) \end{pmatrix} \in \mathcal{H}^{-\infty}(W/X;\mathbb{C}^N) \tag{28.28}$$

gives a family of involutions with essentially the same homotopy properties as in the proof of Proposition 39. I will write a little more, especially about the conjugation which shows up when we change things – here it is a bit more general but the same arguments work. □

So, if you believe all that, observe that something rather pleasant happens in that we have another 'extreme' subset of $\widetilde{\mathcal{D}}^0(W)$ (the other one being $\mathcal{D}^0(W)$. Namely consider

$$\mathcal{D}^{-\infty}(W)=\{(\pi_\infty,\pi_\pm);\\ \pi_+=\pi_\infty+a_+,\ a_+\in\mathcal{S}(W/X;M(N,\mathbb{C}),\ \pi_-=\pi_\infty\in M(N,\mathbb{C})\}. \tag{28.29}$$

Thus in this subset, π_- is constant, π_+ is a Schwartz perturbation of $\pi_-=\pi_\infty$ and p is the identity map on the range of π_∞. Thus the $-\infty$ is denotes, that the elements are smoothing perturbations of constant objects (we could allow π_- to have a Schwartz term too).

Lemma 33. *The equivalence relation $\sim_{-\infty}$ on $\mathcal{D}^{-\infty}(W)$ in which elements can be stabilized, by the addition of the identity or of zero on a complementary subspace (so increasing N), or subject to homotopies within $\mathcal{D}^{-\infty}(W)$, so preserving the constancy of π_- on ${}^q\overline{W}$ etc, but allowing π_∞ to vary in $M(N,\mathbb{C})$ with the parameter, gives a natural isomorphism*

$$\mathcal{D}^{-\infty}/\sim_{-\infty}\xrightarrow{\ \equiv\ }\mathrm{K}^0_c(W). \tag{28.30}$$

Proof. We define the map (28.30) directly. For an element (π_+,π_∞) in $\mathcal{D}^{-\infty}(W)$ (so $\pi_-=\pi_\infty$ is a projection in $M(N,\mathbb{C})$ and π_+ is a family of projections on W which is a Schwartz perturbation of π_∞) let $M=\operatorname{rank}(\pi_\infty$. The cases $M=N$ and $M=0$ are trivially globally constant. So, we can add either an identity block of size $N-2M$ if this is positive or a zero block of size $2M-N$ in the opposite case, to arrange that $N=2N$ keeping equivalence under $\sim_{-\infty}$. Now, all projections in $M(N,\mathbb{C})$ of given rank are are homotopy where the curve is obtained by conjugation with a curve in $\mathrm{GL}(N,\mathbb{C})$. So after an admissible homotopy under $\sim_{-\infty}$ we can arrange that $N=2p$ is even and under a decomposition $\mathbb{C}^{2p}=\mathbb{C}^2\otimes\mathbb{C}^p$

$$\pi_\infty=E_+\otimes\operatorname{Id}_{p\times p}\Longrightarrow I(w)=\pi_+(w)-(\operatorname{Id}-\pi_+(w))\longrightarrow\mathcal{S}(W;\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R})) \tag{28.31}$$

where as usual we are further stabilizing by using the harmonic oscillator basis of $\mathcal{S}(\mathbb{R})$ to map into $\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R})$.

It remains to check that passing to homotopy classes in (28.31) projects to a map (28.30) into $\mathrm{K}^0_{\mathrm{c}}(W)$ and that this map is an isomorphism. I omit the details but surjectivity is clear enough by finite rank approximation and after stabilization homotopy on the left in (28.31) exhausts the freedom on the right. □

Proof of Theorem 10. The crucial observation is that we know already how to quantize the data in $\mathcal{D}^{-\infty}(W)$ in (28.29). So we prove the main result by looking at the expanded and rearranged version of (28.17):-

(28.32)

$$\begin{tikzcd}
\mathcal{P}^0_{\mathrm{qiso}}(W/X;\bullet) \arrow[r, hook] \arrow[d, "\sigma"'] & \widetilde{\mathcal{P}}^0_{\mathrm{qiso}}(W/X;\bullet) \arrow[d, "\sigma"'] & \mathcal{P}^{-\infty}_{\mathrm{iso}}(W/X;\bullet) \arrow[l, hook'] \arrow[d, "\sigma"'] & \\
\mathcal{D}(W) \arrow[r, hook] \arrow[d, "{[\sigma]}"'] & \widetilde{\mathcal{D}}(W) \arrow[d, "{[\sigma]}"'] & \mathcal{D}^{-\infty}(W) \arrow[l, hook'] \arrow[d, "{[\sigma]}"'] & \\
\mathcal{D}(W)/\sim \arrow[r, equal] \arrow[dr, "\mathrm{ind}_{\mathrm{iso}}"'] & \widetilde{\mathcal{D}}(W)/\sim \arrow[r, equal] \arrow[d, "\mathrm{ind}_{\mathrm{iso}}"'] & \mathcal{D}^{-\infty}(W)/\sim_{-\infty} \arrow[r, equal] \arrow[dl, "\mathrm{ind}_{\mathrm{iso}}"'] & \mathrm{K}^0_{\mathrm{c}}(W) \arrow[dll, "\mathrm{Thom}"] \\
& \mathrm{K}^0(X). & &
\end{tikzcd}$$

So, we proceed to check that this diagram commutes.

Going down the left side is repeating the discussion above, that we know how to define the isotropic families index by looking at the family of involutions $I(P, Q, \pi_\pm)$. The image, in $\mathrm{K}^0(X)$ of this isotropic index map factors through the symbol data, into $\mathcal{D}^0(W)$ and further under the equivalence relation $\sim$ to the quotient. So the map from top left down the left side and to $\mathrm{K}^0(X)$ is the isotropic index in the sense of $[I(P, Q, \pi_\pm)]$. The same is true down the middle column, except that the problem has been 'aggrandized' by includion of semiclassical Toeplitz objects – the $\Pi_\pm$. We also know the top two maps from the left column, given by inclusion, give commutative squares. Similarly for the next map down we know there is an inclusion-induced map from left to right where I have equality, and this gives a commutative left side. Everything is the same on the right side, again with a map now from right to left third down; the equality to $\mathrm{K}^0_{\mathrm{c}}(W)$ is Lemma 33.

So the only things left to show are that the maps in the third row are isomorphism, and given their naturality as inclusion maps can then be regarded as equalities plus the proof of the commutativity of the lower right triangle.

Let's do the last part first. Going way back to the discussion of the Bott element. Modulo checking the details it is clear enough. The 'isotropic index' in this case is obtained by semiclassical quantization of the one projection π_+ – the isotropic part of the quantization is trivial since there we just have π_∞ itself. So, this map and the Thom isomorphism are given by semiclassical quantization. Unfortunately on one side it is given by quantization of a projection and the other side by an involution. Of course the projection is supposed to be the positive part of the involution, so the exact correspondence needs to be checked.

By Lemma 33 it is enough to consider the case where $\pi_\infty = E_+$ on $\mathbb{C}^2 \otimes \mathbb{C}^N$ where E_+ is the projection onto the first element of $\mathbb{C}^2$, tensored with the identity of course. The semiclassical quantization of $\pi_+ \in \mathcal{C}^\infty({}^q\overline{W}; M(N, \mathbb{C}))$ to a family of projections Π_+ also gives a semiclassical quantization of the involution $\pi_+ - (\mathrm{Id} - \pi_+) = \gamma_1 + a$ with a Schwartz. So, it is only necessary to check that the isotropic index, which is $[I(P, Q, \Pi_\pm)]$ for this very special data, is the same as the class of semiclassical quantization of the involution at $\epsilon = 1$ which defines the Thom isomorphism.

A bit more detail needed here.

Proving the horizontal maps are equalities is showing that every class in $\widetilde{\mathcal{P}}/\sim$ can be represented uniquely by an element either in $\mathcal{P}/\sim$ or $\mathcal{P}^{-\infty}/\sim_{-\infty}$. It is straightforward.

There is still more to Theorem 10 apart from (28.20) – which certainly follows from (28.32). Namely, what happens if the index vanishes. Going through the proof of the equalities of the quotients in (28.32) and of course using the fact that the Thom map is an isomorphism, one concludes that the vanishing of the isotropic index implies that $(P, Q, \pi_\pm) \sim 0$. Looking at the equivalence relation, the implication is that the symbol p, between the original bundles, is, after stabilization by the identity on some additional bundle, homotopic to a bundle isomorphism – can be deformed to be constant on the fibres of $\mathbb{S}W$ over X. This is one of the two claims.

The other one is more interesting analytically so I will extract it for later reference. □

Proposition 42. *If $P \in \Psi^0_{\mathrm{qiso}}(W/X;\mathbb{E})$ is elliptic then there is a perturbation $T \in \Psi^{-\infty}_{\mathrm{qiso}}(W/X;\mathbb{E})$ such that $P+T$ has a generalized inverse $Q \in \Psi^{-\infty}_{\mathrm{qiso}}(W/X;\mathbb{E}^-)$ meaning*

$$\begin{cases} \varpi_L = \mathrm{Id}_+ - Q(P+T) & \in \Psi^{-\infty}_{\mathrm{iso}}(W/X;E_+) \\ \varpi_R = \mathrm{Id}_- - (P+T)Q & \in \Psi^{-\infty}_{\mathrm{iso}}(W/X;E_-) \end{cases} \text{ are projections} \tag{28.33}$$

and the index is represented by the K-class $\mathrm{ind}_{\mathrm{iso}}(P) = \mathrm{Ran}(\varpi_L) \ominus \mathrm{Ran}(\varpi_R)$. If this K-class vanishes then T can be chosen so that $P+T$ is invertible.

So the isotropic index is the precise obstruction to perturbative invetibility without any need to stabilize. This is basically because there is 'enough room' in the smoothing terms.

Proof. Replace P by $P(\mathrm{Id} - \Pi_N) = P + T$ where Π_N is the sequence of harmonic oscillator projections and check that for large N this has 'null space the range of Π_N' (these aren't actual operators) and (28.33) can be arranged. If the index K-class vanishes this means that after increasing N enough (effectively stabilizing) the bundles defined by $\Pi_N = \Pi_L$ and Π_R are isomorphic. This there is a further perturbation T which maps precisely from the 'null space' of $P(\mathrm{Id} - \Pi_N)$ to the range of Π_R and hence $P(\mathrm{Id} - \Pi_N) + T$ is invertible with the invserse as claimed. □