27. Lecture 24: Thom isomorphism Wednesday, 29 October, 2008

Reminder. *Last time I talked a little about symplectic and complex vector bundles and recalled that on a complex vector bundle with hermitian structure one has a well-defined smooth family of harmonic oscillators on the fibres.*

For a symplectic vector bundle over a manifold X we have shown that there is a well-defined non-commuatative product given on $\mathcal{C}^\infty([0,1];\mathcal{S}(W/X))$, the space of smooth functions on $[0,1]\times {}^q\overline{W}$ which are Schwartz on the fibres, see (20.4) for the explicit formula. Thus we have a bundle of algebras where the fibre above $x\in X$ is $\mathcal{C}^\infty([0,1];\mathcal{S}(W))$ and we will denote the space of global sections of this algebra by $\Psi^{-\infty}_{\mathrm{sl,iso}}(W)$. In fact if we can consider two, or even three, symplectic vector bundles, W_1, W_2 and W_3 over X and observe that the algebra defined last time – separately adiabatic in each of the first two variables and just the product at $\epsilon=1$ in the last, is well-defined. We can denote the algebra of global sections of this bundle of algebras as

$$\Psi^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X\; W_2/X\; W_3/X;\mathbb{C}^N) \tag{27.1}$$

where I have thrown in matrix values for good measure. The fibre at some point $x\in W$ is just $\Psi^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}((W_1)_x\;(W_2)_x\;(W_3)_x;\mathbb{C}^N)$ which is essentially the algebra we were looking at last time.

Now there is one important thing to note. Even though this bundle of algebras is 'twisted', because the bundles W_i need not be trivial, its homotopy properties haven't changed much. To see let us note that, given two symplectic vector bundles over X there is a reasonably natural map of algebras:-

$$B:\Psi^{-\infty}_{\mathrm{iso}}(W_1/X)\ni A\longmapsto A\otimes\pi_B(x)\in\Psi^{-\infty}_{\mathrm{iso}}((W_1\times W_2)/X). \tag{27.2}$$

Here π_B is the self-adjoint projection onto the ground state of some chosen smooth family of harmonic oscillators corresponding to a compatible complex structure on W_2. Different choices of compatible complex structure are homotopic and this homotopy lifts to π_B.

We can modify (27.2) to give us a similar map on involutions and invertibles, where I will add a c subscript to indicate that things are compactly supported – appropriately trivial outside a compact set:-

$$\begin{aligned} B_H:\mathcal{H}^{-\infty}_{\mathrm{iso,c}}(W_1/X)\ni I(x)=\gamma_1+a(x)\longmapsto&\\ \gamma_1+a\otimes\pi_B(x)&\in\mathcal{H}^{-\infty}_{\mathrm{iso,c}}((W_1\times W_2)/X)\\ B_H:G^{-\infty}_{\mathrm{iso,c}}(W_1/X)\ni I(x)=\mathrm{Id}+a(x)\longmapsto&\\ \mathrm{Id}+a\otimes\pi_B(x)&\in G^{-\infty}_{\mathrm{iso,c}}((W_1\times W_2)/X).\end{aligned} \tag{27.3}$$

Here the 'stabilization' if different in each case – remember for $\mathcal{H}$ all the objects are in 2×2 matrices over the obvious one. Of course the homotopies implicit in the definition of the components have to have uniformly compact support.

Proposition 33. *The stabilization maps in* (27.3) *induce homotopy equivalences*

$$\begin{aligned}\Pi_0\big(\mathcal{H}^{-\infty}_{\mathrm{iso},c}(W_1/X)\big)&\simeq\Pi_0\big(\mathcal{H}^{-\infty}_{\mathrm{iso},c}((W_1\times W_2)/X)\big)\\ \Pi_0\big(G^{-\infty}_{\mathrm{iso},c}(W_1/X)\big)&\simeq\Pi_0\big(G^{-\infty}_{\mathrm{iso},c}((W_1\times W_2)/X)\big).\end{aligned} \tag{27.4}$$

Note that these objects are the spaces of sections, and we can only map from the base.

Proof. For a change look at the odd case. The main point is that we certainly know how to do this in case the bundles are trivial, using finite rank approximation, and we have such finite rank approximation available in the general case. Thus the projections, π_N to the span of the eigenfunctions for eigenvalues less than $\operatorname{rank}(W) + 2N + 1$ are all well-defined and smooth. Thus if $a \in \Psi^{-\infty}_{\text{iso}}(W/X)$ then

$$a\pi_N,\ \pi_N a \to a \in \Psi^{-\infty}_{\text{iso}}(W/X). \tag{27.5}$$

The range of π_N is a vector bundle over X which is readily seen to be isomorphic to the Nth symmetric power of the W as a complex bundle. We don't really need to know this, just that it is a vector bundle. It is a standard result that a (compactly supported) vector bundle can be embedded as a subbundle of any vector bundle over the same set with sufficiently large rank – this can be proved by the same sort of crude method as used to embed into a trivial bundle if the rank required is not needed, as it isn't here. Anyway, this means that we can think of the range of $\pi_N(W_1)$, as a vector bundle over $\pi_M(W_2)$ for some M. This allows us to 'identify' the cut-off operator on W_1, $g_N = \pi_N + \pi_N a \pi_N$, which is invertible if N is large enough, as an of invertible family of homorphisms g'_M of π_M on W_2, extending as the identity off the subbundle. This means we can rotate in the usual way between

$$g_N(x) \otimes \pi_B(W_2) \text{ and } \pi_B(W_1) \otimes g'_M(x) \text{ extended as the identity.} \tag{27.6}$$

This is not quite what we want, since we really want to go the other way, but it is easy to extend it a bit so that it is. Namely any element $\operatorname{Id} + a(x) \in G^{-\infty}_{\text{iso,c}}((W_1 \times W_2)/X))$ can be deformed by homotopy to a finite rank perturbation of the identity which acts on the range of $\pi_N(W_1) \otimes \pi_N(W_2)$ for some N. Then $\pi_N(W_2)$ can be complemented to be trivial, and the complement can be embedded in $\pi_M(W_1) - \pi_N(W_1)$ and $\pi_M(W_2) - \pi_N(W_2)$ respectively, for M large enough. Thus we can think of the perturbation as acting on the tensor product of the ranges of two globally trivial families of projections. This actually reduces us to the case right at the beginning (although I did not write down a very elegant proof then). In particular it follows by embedding $\dim \pi_N(W_2)$ copies of the same trivial bundle in W_1 repeatedly in higher 'tranches' $\pi_{M_i}(W_1) - \pi_{M_{i-1}}(W_1)$. Then the argument in the flat case works just as well here to show that such an element can be rotated to act on the range of $\pi_M(W_1) \otimes \pi_1(W_2)$ and hence is in the range of the second map in (27.4).

The part for idempotents can be proved similary. □

Corollary 7. *For any symplectic bundle W (of positive fibre rank of course) over a manifold there are natural idenifications*

$$\begin{aligned} \Pi_0\big(\mathcal{H}^{-\infty}_{\text{iso},c}(W/X)\big) &\simeq \mathrm{K}^0_c(X) \\ \Pi_0\big(G^{-\infty}_{\text{iso},c}(W/X)\big) &\simeq \mathrm{K}^1_c(X). \end{aligned} \tag{27.7}$$

Proof. Apply the preceeding proposition twice in each case to the product of W with a trivial bundle, say with symplectic fibre $\mathbb{R}^2$. □

Now, let us proceed to the Thom isomorphism. This concerns a vector bundle W over a manifold X. As note above, the Thom isomorphism

$$\text{Thom} : \mathrm{K}^0_{\mathrm{c}}(X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X) \tag{27.8}$$

is fixed by the homotopy class of the smoothly varying symplectic form on the fibres of W, so it depends on some orientation information, but otherwise it is well-defined. We will get it, by semiclassical quantization.

I have alread briefly discussed the space

$$\Psi^{-\infty}_{\mathrm{ad,iso}}(W \times \mathbb{R}^{2p}/X; M(N,\mathbb{C})) \subset \mathcal{C}^\infty([0,1] \times \mathcal{S}(W \times \mathbb{R}^{2p}/X; M(N,\mathbb{C})) \tag{27.9}$$

which consists of all the smooth functions on $[0,1]\times W\times\mathbb{R}^{2p}$ with values in $M(N,\mathbb{C})$ and which have entries which are uniformly Schwartz functions on the fibres of $[0,1] \times W \times \mathbb{R}^{2p}$ as a bundle over $X \times [0,1]$. The product is the adiabatic product with respect to the symplectic structure on each fibre W_x. For $N = 2$ we consider the space of adiabatic perturbations of γ_1 and which are involutions

$$\mathcal{H}^{-\infty}_{\mathrm{ad,iso}}(W\times\mathbb{R}^{2p}/X) = \left\{I = \gamma_1 + a,\ a \in \Psi^{-\infty}_{\mathrm{ad,iso}}(W \times \mathbb{R}^{2p}/X; M(N,\mathbb{C})),\ I^2 = \mathrm{Id}\right\}. \tag{27.10}$$

The symbolic properties of this space should by now be fairly clear. Namely the adiabatic sybol map restricts to

$$\sigma_{\mathrm{ad}} : \mathcal{H}^{-\infty}_{\mathrm{ad,iso,c}}(W \times \mathbb{R}^{2p}/X) \longrightarrow \mathcal{C}^\infty_c(X; \mathcal{H}^{-\infty}_{\mathrm{sus}(W),\mathrm{iso}}(\mathbb{R}^p)) = \mathcal{S}_{\mathrm{c}}(W; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p)) \tag{27.11}$$

which is the involutions of the form $\gamma_1 + b$, where $b : W \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p)$ is uniformly Schwartz on the fibres of W and has support in the preimage of a compact set in the base.

Lemma 29. *The symbol map* (27.11) *is surjective and the preimage of each element is connected.*

Since we are now talking about families, necessarily defined on X rather than an arbitrary manifold, this is the analogue of the homotopically-unique lifting property.

Proof. At some point I will write out a general result for these symbolic lifting constructions. This is no different to most others. Namely we use the adiabatic symbol map, the properties if which follow from the case we have been discussing where W is the trivial product $X \times \mathbb{R}^{2k}$ since locally over X there is a trivialization in which the symplectic form reduces to the Darboux form on $\mathbb{R}^{2k}$. Thus we have a short exact and multiplicative symbol sequence

$$\epsilon\Psi^{-\infty}_{\mathrm{ad,iso,c}}(W \times \mathbb{R}^{2p}/X; M(N,\mathbb{C})) \longrightarrow \Psi^{-\infty}_{\mathrm{ad,iso,c}}(W \times \mathbb{R}^{2p}/X; M(N,\mathbb{C})) \xrightarrow{\sigma_{\mathrm{ad}}} \mathcal{S}_{\mathrm{c}}(W; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p)). \tag{27.12}$$

Thus, we can lift the an element in the targen space in (27.11) to $I_0'\gamma_1 + b_0$ where $(I_0')^2 = \mathrm{Id} + \epsilon c$. The same iteration argument used earlier for involutions and Borel's Lemma allows us to improve this to $(I')^2 = \mathrm{Id} + c'$ where c' vanishes to infinite order with ϵ, so is just smooth (and rapidly vanishing) down to $\epsilon = 0$ in the ordinary sense. Then the same integral formula as before allows this to be corrected to an involution in $[0, \epsilon_0]$ for some $\epsilon_0 > 0$ and finally stretching the parameter space we find a lift of the symbol as desired and (27.11) is therefore surjective.

A modification of this argument shows that any two lifts are homotopic as families, i.e. the set of lifts is connected. □

Now, the Thom isomorphism comes from looking at the restriction operator R to $\epsilon = 1$ as before. This gives a diagram (written the other way compared to the perioidicity case) where my notation is a bit out of hand

$$
\begin{tikzcd}[column sep=small]
& \mathcal{H}^{-\infty}_{\mathrm{ad,iso,c}}(W/X:\mathbb{R}^{2p}) \arrow[dl, "\sigma_{\mathrm{ad}}"'] \arrow[dr, "R"] & \\
\mathcal{S}_{\mathrm{c}}(W;\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p)) \arrow[d, "\Pi_o"] & & \mathcal{H}^{-\infty}_{\mathrm{iso,c}}(W\times\mathbb{R}^{2p}/X) \arrow[d, "\Pi_o"] \\
\mathrm{K}^0_{\mathrm{c}}(W) \arrow[rr, "\mathrm{Thom}"] & & \mathrm{K}^0_{\mathrm{c}}(X).
\end{tikzcd} \tag{27.13}
$$

Here the vertical 'Π_0' maps are just the passage to components. Thus we know that both maps on the left side are surjective and that the lift is unique up to homotopy. So the map along the bottom is well-defined

Theorem 8. *[Thom isomorphism] For any symplectic vector budle the map on the bottom in* (27.13) *is an isomorphism and R on the right is surjective.*

So, this is just the same as Bott periodicity in case $W = \mathbb{R}^{2k} \times X$ which we finally discussed properly last time. We already know how to change the dimension 'p' of the isotropic image space, so the Thom map really is well-defined. To prove that it is an isomorphism we will bring out the main tool we have used so far, which is the ability to do two things at once.

As mentioned above, we can consider adiabatic families, as we did last time, with respect to two symplectic bundles W_1 and W_2 over X. The hardes thing here is really the notation! I will let $\mathrm{sus}(W)$, as a subscript, replace $\mathrm{sus}(2p)$ and here mean that we are considering sections which are Schwartz, in an appropriate sense, on the fibres of W. I have already used this without discussing it in (27.11). The doubly adiabatic algebra, which has two parameters, one in the W_1 slots and the other in the W_2 slots, is a non-commutative product on the spaces of sections which have compact support in the base (or arbitrary support, both work but we mostly want the compact case)

$$\{a \in \mathcal{C}^\infty_c([0,1]\times[0,1]\times {}^q\overline{W_1}\times_X {}^q\overline{W_2}\times {}^q\overline{\mathbb{R}^{2p}}); a\equiv 0 \text{ at all boundaries}\}. \tag{27.14}$$

Here the quadratic compactifications can be replaced by the radial ones, since we are considering functions flat at the boundaries anyway. The $\times_X$ means the fibre product, so really we are taking the products of the compactifications of $(W_1)_x$ and $(W_2)_x$ and making them into a bundle over X.

I leave it to you to carefully define the corresponding space of involutions

$$\mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X : W_2/X : \mathbb{R}^p) \tag{27.15}$$

which are doubly-adiabatic-smoothing perturbations of γ_1, so as usual are 2×2 matrices. Now there are a total of seven 'restriction maps' we wish to consider. Six of them correspond to restricting to one of $\epsilon_1=\epsilon_2=0$ (the doubly-adiabatic symbol), to $\epsilon_i=0$, $i=1,2$, the two single adiabatic symbols, $\epsilon_i=1$, $i=1,2$, the two restriction maps and $\epsilon_1=\epsilon_2=1$. The seventh map is the restriction to $\epsilon_1=\epsilon_2$

($=\epsilon$ if you like). These are of the form

(27.16)

$$
\begin{aligned}
\sigma_{\mathrm{ad}(W_1\times W_2)} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X:W_2/X:\mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{sus}(W_1\times W_2),\mathrm{iso}}(\mathbb{R}^p)\\
\sigma_{\mathrm{ad}(W_1)} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X:W_2/X:\mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{sus}(W_1),\mathrm{ad,iso}}(W_2/X:\mathbb{R}^{2p})\\
\sigma_{\mathrm{ad}(W_2)} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X:W_2/X:\mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{ad,sus}(W_2),\mathrm{iso}}(W_1/X:\mathbb{R}^{2p})\\
R_{\epsilon_2=1} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X:W_2/X:\mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{ad,iso}(\omega)}(W_1/X:(W_2\times\mathbb{R}^{2p})/X)\\
R_{\epsilon_1=1} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X:W_2/X:\mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{ad,iso}(\omega)}(W_2/X:(W_1\times\mathbb{R}^{2p})/X)\\
R_{\epsilon_1=\epsilon_2=1} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X:W_2/X:\mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{iso}(\omega)}((W_1\times_X W_2\times\mathbb{R}^{2p})/X)\\
R_{\epsilon_1=\epsilon_2} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X:W_2/X:\mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{ad,iso}}((W_1\times_X W_2)/X:\mathbb{R}^{2p}).
\end{aligned}
$$

Here I just thought of the idea of using $\mathrm{iso}(\omega)$ instead of iso to mean that the space is a symplectic vector space instead of operators on a vector space. The main thing to swallow is that all these maps exist. Here is a diagram in ϵ_1, ϵ_2 space:-

So, there are lots of things we could easily prove about this picture. However, recall that at the level of functions these maps really are restrictions to the sets in question. So they have the obvious consistency properties that I will not write out but will use below.

Now recall what we want to use this set-up for. We want to consider three bundles, namely

$$
\begin{gathered}
W_1\times_X W_2 \longrightarrow X \text{ bundle over X}\\
W_1\times_x W_2 \longrightarrow W_1 \text{ bundle over } W_1\\
W_1 \longrightarrow X \text{ bundle over } X.
\end{gathered}
\tag{27.17}
$$

although for vector bundles the notation $W_1\times W_2$ for the fibre product $W_1\times_X W_2$ is conventional, here I am just trying to emphasize what things really are. In all three case we have the Thom isomorphism and what we will use the doubly-adiabatic set up to show:-

Proposition 34. *For any pair of symplectic vector bundles over a manifold the three Thom maps give a commutative diagram*

(27.18)

$$
\begin{tikzcd}
\mathrm{K}^0_c(W_1\times_X W_2) \arrow[dd, "\mathrm{Thom}"'] \arrow[dr, "\mathrm{Thom}"] & \\
& \mathrm{K}^0_c(W_1) \arrow[dl, "\mathrm{Thom}"] \\
\mathrm{K}^0(X). &
\end{tikzcd}
$$

Proof. The main claim, that I am not for the moment going to write down, is that the *third* map above is surjective. This is the same sort of argument as in Lemma 29 above, with a few extra twists because of the two parameters – but not essentially harder. It follows from this that the first map is also surjective, because this, by consistency of the symbols, is the adiabatic symbol map applied to the range of the third map.

Thus we can start of with an element of $\kappa \in \mathcal{H}^{-\infty}_{\mathrm{sus}(W_2),\mathrm{iso}}(W_1 \times \mathbb{R}^{2p}/X)$ which is in the image, under restriction to $\epsilon_1 = 1$ of an element $K \in \mathcal{H}^{-\infty}_{\mathrm{ad},\mathrm{sus}(W_2),\mathrm{iso}}(W_1/X : \mathbb{R}^{2p})$, meaning its class is in the image of the top sloping Thom map. Using the surjectivity discussed above, this is the image under the third map in (27.17) of an element $\tilde{K}$ in the doubly-adiabatic space. The image of this under the fifth map in (27.17) is therefore a lift of κ which defines the lower Thom map on the right, i.e. restricting this element to $\epsilon_2 = 1$ gives the image of κ in $\mathrm{K}^0(X)$. However the restriction of $\tilde{K}$ under the last map gives the image of K in this space, and by the consistency of these restricition maps these are the same. Thus the Proposition is proved. □

Proof of Theorem 8 – Thom isomorphism. Consider any symplectic vector bundle W. We know that this can be given a complex structure compatible with its symplectic structure (and determined up to homotopy). As a complex vector bundle, W can be embedded as a subbundle of $\mathbb{C}^N$ for some N and hence complemented to this trivial bundle with another complex bundle $\tilde{W}$. Conversely $\tilde{W}$ has a symplectic structure so we can arrange, after a homotopy of the symplectic structure which does not affect the Thom map, that

$$W \times_X \tilde{W} = W \oplus \tilde{W} = \mathbb{R}^{2N} \times X \tag{27.19}$$

with consistent symplectic structures. Now Proposition 34 gives the commutative diagram

$$\begin{tikzcd}
\mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2N} \times X) \arrow[dd, "\mathrm{Thom}=p_{\mathrm{sl}}"'] \arrow[dr, "\mathrm{Thom}"] & \\
& \mathrm{K}^0_{\mathrm{c}}(W) \arrow[dl, "\mathrm{Thom}"] \\
\mathrm{K}^0(X). &
\end{tikzcd} \tag{27.20}$$

Thus the vertical map is Bott periodicity, so is an isomorphism. It follows that the lower map on the right, the Thom map for W as a bundle over X is surjective. It also follows that the upper map on the right, which is the Thom map for $W \times_X \tilde{W} = \mathbb{R}^{2N} \times X$ as a symplectic bundle over W is injective.

We have shown that the Thom map is *universally* surjective. However the upper map on the right, which we know to be injective, is an example of such a map. So it must also be surjective, hence an isomorphism. Hence the general Thom map on the lower right is also always an isomorphism. □

Here is some material that seems to have been orphaned; I will work out where to put it some time!

Let me recall, and extend, some of the basic results about the space $\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p)$; especially since the treatment I gave was rather brief, to say the least.

Proposition 35. *Two compactly supported smooth maps $I_i : X \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p)$, $i = 0, 1$, (through such maps) if and only if they are conjugate under a smooth compactly supported map $g : X \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p; \mathbb{C}^2)$ which is homotopic through such* *maps to the identity:*

$$I_1(x) = g^{-1}(x) I_0(x) g(x) \ \forall \ x \in X. \tag{27.21}$$

Proof. Certainly if $g = g_1$ for a compactly supported homotopy $g_\cdot : [0,1] \times X \longrightarrow G^{-\infty}_{\text{iso}}(\mathbb{R}^p; \mathbb{C}^2)$ with $g_0 = \text{Id}$ then $I_t = g_t^{-1} I_0 g_t(x)$ is an homotopy from I_0 to I_1.

To see the converse, we will solve a differential equation. Recall that for an homotopy of involutions

$$\dot{I}_t I_t + I_t \dot{I}_t = 0 \Longrightarrow \dot{I}_t = I_t^+ \dot{I}_t I_t^- + I_t^- \dot{I}_t I_t^+, \tag{27.22}$$

meaning that the derivative must be off-diagonal with respect to the involution. Now, if we want to solve $I_t = g_t^{-1} I_0 g_t$, for the moment for fixed x, we can differentiate and try to solve

$$\dot{I}_t = \gamma(t) = -(g_t^{-1} \dot{g}_t) I_t + I_t (g_t^{-1} \dot{g}_t). \tag{27.23}$$

Now, it follows from (27.22) that this identity is satisfied if we take arrange that

$$g_t^{-1} \dot{g}_t = \frac{1}{2}\big(- I_t^+ \dot{I}_t I_t^- + I_t^- \dot{I}_t I_t^+ \big). \tag{27.24}$$

Thus, we can simply (try to) solve

$$\begin{gathered} \dot{g}_t = \frac{1}{2} g_t \gamma(t), \ g_0 = \text{Id} \\ \Longleftrightarrow g_t = \text{Id} + a(t), \ a(t) = \int_0^t \big(\gamma(s) + a(s)\gamma(s)\big) ds. \end{gathered} \tag{27.25}$$

Now, the integral equation has a unique solution by standard contraction arguments, and it follows from this uniqueness that the solution is smooth in the parameters. Moreover it follows that $g_t(x) = \text{Id} + a(t,x)$ is always invertible, and is equal to the identity outside a compact set in X. For instance the invertibility follows by following the determinant since

$$\frac{d}{dt} \log \det(g_t(x)) = \operatorname{tr}(\gamma(t,x)). \tag{27.26}$$

Exercise 21. Do it – check that it works in each seminorm and from uniqueness the solution to (27.25) is Schwartz.

Now, going backwards it follows that g_t implements the conjugation we want. $\square$

Lemma 30. *For a symplectic vector bundle W over X, two elements*

$$I_i \in \mathcal{H}^{-\infty}_{\text{iso},c}(W/X; \mathbb{C}^N) \ i = 0, 1,$$

are in the same component if and only if there exists $g \in G^{-\infty}_{\text{iso}}(W/X; \mathbb{C}^N \otimes \mathbb{C}^2)$ in the component of the identity such that $I_1 = g^{-1} I_0 g$.

Proof. The uniqueness of the method used in the previous proof means that it works in the same way for sections of these bundles over X and then this is simply a restatement of the conclusion. $\square$

I did show earlier that any smooth map of compact support $I : X \longrightarrow \mathcal{H}^{-\infty}_{\text{iso}}(\mathbb{R}^p)$ are homotopic to simple sections of the form

$$
\begin{gathered}
\tilde{I} = \gamma_1 - 2E_+ \otimes P_-(x) + 2E_+ \otimes P_+(x),\ P_\pm(x)^2 = P_\pm(x),\\
\Pi_N P_+(x) = P_+(x)\Pi_N = P_+(x),\\
(\Pi_M - \Pi_N)P_-(x) = P_+(x)(\Pi_M - \Pi_N) = P_+(x)\ \forall\ x \in X,\\
P_\pm(x) = A_\pm P_\mp(x) B_\pm \text{ are constant for } x \in X \setminus K,\ K \Subset X.
\end{gathered}
\tag{27.27}
$$

Here M and N are integers and $A_\pm$ and $B_\pm$ are matrices acting on the range of π_M.

To extend this to the case of sections of a symplectic bundle W is straightforward except that we cannot demand the constancy outside a compact set unless we demand that the bundle W itself is trivial, and has constant symplectic structure outside such a set – I have been a bit cavalier about this. Fortunatly it is not really a problem. Instead we just demand the conjugation equivalence in the complement of a compact set (and I will change things retrospectively at some point).

Proposition 36. *Any family of involutions, $I \in \mathcal{H}^{-\infty}_{\text{iso}}(W/X)$, is homotopic over any open subset $\Omega \subset X$ with compact closure to one of the form*

$$
\begin{gathered}
\tilde{I} = \gamma_1 - 2E_+ \otimes P_-(x) + 2E_+ \otimes P_+(x),\ P_\pm(x)^2 = P_\pm(x),\\
\Pi_N(x) P_+(x) = P_+(x)\Pi_N(x) = P_+(x),\\
(\Pi_M(x) - \Pi_N(x))P_-(x) = P_+(x)(\Pi_M(x) - \Pi_N(x)) = P_+(x)\ \forall\ x \in X,\\
P_\pm(x) = A_\pm(x) P_\mp(x) B_\pm(x) \textit{ in } \Omega' \setminus \Omega,\ \Omega' \textit{ open},\ \overline{\Omega'} \Subset X,\ \overline{\Omega} \subset \Omega'.
\end{gathered}
\tag{27.28}
$$