26. Lecture 23: Iterated periodicity maps
Monday, 27 October, 2008

We have earlier shown that there is a semi-classical relation between involutions on $X \times \mathbb{R}^2$ and X in the form of surjective maps with the homotopically-unique lifting property for compactly-supported families

$$
\begin{tikzcd}
& \mathcal{H}^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^k : \mathbb{R}^p) \arrow[dl, "R"'] \arrow[dr, "\sigma_{\mathrm{ad}}"] & \\
\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}) & & \mathcal{H}^{-\infty}_{\mathrm{sus}(2k),\mathrm{iso}}(\mathbb{R}^p)
\end{tikzcd}
\quad , \ k = 1. \tag{26.1}
$$

This generates the periodicity isomorphism (going from right to left)

$$
[X; \mathcal{H}^{-\infty}_{\mathrm{sus}(2);\mathrm{iso}}(\mathbb{R}^p)] \longrightarrow [X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}]_{\mathrm{c}}. \tag{26.2}
$$

These constructions can be iterated in slightly different sense. We can increase k in (26.1) and iteratively replace X by $X \times \mathbb{R}^2$ in (26.2). However we need to be a little careful to check that these give the same result. In particular we have not shown the surjectivity or lifting property for the map to the left in (26.1) when $k > 1$. Now we will. We have already shown that the adiabatic map, the one to the right, is surjective and has the homotopically-unique lifting property for every k. Thus the diagram (26.1) does generate a map

$$
[X; \mathcal{H}^{-\infty}_{\mathrm{sus}(2k);\mathrm{iso}}(\mathbb{R}^p)] \longrightarrow [X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}]_{\mathrm{c}} \text{ for each } k \geq 1. \tag{26.3}
$$

Proposition 32. *For any $k > 1$ the map* (26.3) *is the isomorphism given by iteration of* (26.2).

Proof. We need to show is that the quantization of involutions (with relatively compact support) on $\mathbb{R}^{2k} \times X$ to involutions on X can be carried out in k steps, quantizing an $\mathbb{R}^2$ each time.

The clearest way to really check that this is possible is to think about the doubly-adiabatic calculus. So, we are interested in operators on $\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^p$ which are separately adiabatic in the two first sets of variables. So the kernels in question are determined by elements

$$
F \in \mathcal{C}^\infty([0,1]_{\epsilon_1} \times [0,1]_{\epsilon_2}; \mathcal{S}(\mathbb{R}^{2k_1+2k_2+2p}) \tag{26.4}
$$

through the adiabatic-Weyl quantization where the actually kernels for $\epsilon < 1 > 0$ and $\epsilon_2 > 0$ are the

$$
\begin{aligned}
&f(\epsilon_1, \epsilon_2, z_1, z_1', z_2, z_2', z, z') \\
&\qquad = \epsilon_1^{-k_1} \epsilon_2^{-k_2} F(\epsilon_1, \epsilon_2, \frac{\epsilon_1(z_1 + z_1')}{2}, \frac{z_1 - z_1'}{\epsilon_1}, \frac{\epsilon_1(z_2 + z_2')}{2}, \frac{z_2 - z_2'}{\epsilon_2}, z, z').
\end{aligned} \tag{26.5}
$$

Since the two adiabatic parameters are in different variables the same argument as before shows that these doubly-adiabatic famililes form an algebra under composition and that function, as in (26.4), determining the composite can be written down quite directly – namely taking the Fourier transform in the second of the adiabatic variables leads to

$$
\hat{H}(t_1, \tau_1, t_2, \tau_2, Z, Z') = \int \hat{F}(t_1, \tau_1, t_2, \tau_2, Z, Z'')\hat{G}(t_1, \tau_1, t_2, \tau_2, Z'', Z'). \tag{26.6}
$$

So, this is just two adiabatic limits going on independently. There are really three different symbol maps. One where $\epsilon_1 = 0$ but $\epsilon_2 > 0$ – but this needs to be understood as an adiabatic family. Another one the other way round. And then the doubly-adiabatic symbol at $\epsilon_1 = \epsilon_2 = 0$. Clearly this latter one is the adiabatic symbol of *each* of the other ones. Said more formally, there are three homomorphisms of algebras:

$$
\begin{gathered}
\Psi^{-\infty}_{\mathrm{ad,ad,iso}}(\mathbb{R}^{k_1} : \mathbb{R}^{k_2} : \mathbb{R}^p) \longrightarrow \Psi^{-\infty}_{\mathrm{sus}(2k_1),\mathrm{ad,iso}}(\mathbb{R}^{k_2} : \mathbb{R}^p),\\
\Psi^{-\infty}_{\mathrm{ad,ad,iso}}(\mathbb{R}^{k_1} : \mathbb{R}^{k_2} : \mathbb{R}^p) \longrightarrow \Psi^{-\infty}_{\mathrm{ad,sus}(2k_2),\mathrm{iso}}(\mathbb{R}^{k_1} : \mathbb{R}^p),\\
\Psi^{-\infty}_{\mathrm{ad,ad,iso}}(\mathbb{R}^{k_1} : \mathbb{R}^{k_2} : \mathbb{R}^p) \longrightarrow \Psi^{-\infty}_{\mathrm{sus}(2(k_1+k_2)),\mathrm{iso}}(\mathbb{R}^p).
\end{gathered}
\tag{26.7}
$$

Here the suspended algebras are just the old algebras depending in a Schwartz manner on the additional parameters.

All three maps are surjective, but they are not jointly surjective. Rather they satisfy precisely the relationships given by the commutative diagram

$$
\begin{tikzcd}
 & \Psi^{-\infty}_{\mathrm{sus}(2k_1),\mathrm{ad,iso}}(\mathbb{R}^{k_2} : \mathbb{R}^p) \arrow[dr] & \\
\Psi^{-\infty}_{\mathrm{ad,ad,iso}}(\mathbb{R}^{k_1} : \mathbb{R}^{k_2} : \mathbb{R}^p) \arrow[ur] \arrow[rr] \arrow[dr] & & \Psi^{-\infty}_{\mathrm{sus}(2(k_1+k_2)),\mathrm{iso}}(\mathbb{R}^p) \\
 & \Psi^{-\infty}_{\mathrm{ad,ad,iso}}(\mathbb{R}^{k_1} : \mathbb{R}^{k_2} : \mathbb{R}^p) \arrow[ur] &
\end{tikzcd}
\tag{26.8}
$$

where all the maps are ‘adiabatic symbols’.

So, here is what we need and a good deal more:

Lemma 27. *For (families of) involutions of the form $\gamma_1 + A$, $A \in \Psi^{-\infty}_{\mathrm{ad,ad,iso}}(\mathbb{R}^{k_1} : \mathbb{R}^{k_2} : \mathbb{R}^p; \mathbb{C}^2)$, the four maps corresponding to ‘restriction’ to $\epsilon_1 = 0 = \epsilon_2 = 0$, $\epsilon_1 = 0, \epsilon_2 = 1$, $\epsilon_1 = 1, \epsilon_2 = 0$ and $\epsilon_1 = \epsilon_2 = 1$ are all surjective with the homotopically-unique lifting property for compactly supported families, and hence induce weak homotopy equivalences between all five spaces of involutions.*

Proof. There is actually nothing really new here, we just have to apply the old procedures thoughtfully. □

In particular of course this completes the proof of Proposition 32. □

Assuming that I have not run out of time, let me now start to discuss the setting of the Thom isomorphism, and closely related isotropic index theorem. This concerns the case of a complex or symplectic vector bundle over a manifold

$$
\begin{tikzcd}
W \arrow[d] \\
X.
\end{tikzcd}
\tag{26.9}
$$

In the symplectic case this is a real vector bundle with a symplectic structure on each fibre, varying smoothly with the base point. Since a symplectic structure on a vector space is just a non-degenerate antisymmetric real bilinear form, the fibres must certainly be even-dimensional.

The relationship between the symplectic and complex structures can be seen geometrically by defining a metric which is compatible with the symplectic structure. Take any real fibre metric h and look at the duality map it defines relative to ω, the symplectic structure:

$$\omega_x(v,w) = g_x(v, J'_x w),\ J'_x : W_x \longrightarrow W_x. \tag{26.10}$$

This uses the non-degeneracy of each of the forms, the one symmetric the other antisymmetric, from which it follows that J'_x is a smooth isomorphism. In fact it is necessary skew-adjoint (it is real) with respect to g since

$$g_x(v, J'_x w) = \omega_x(v,w) = -\omega_x(w,v) = -g_x(w, J'_x v) = -g(J'_x v, w) \tag{26.11}$$

for all v, w. Thus it follows that the eigenvalues of J'_x are pure imaginary and non-zero, since J'_x is invertible from the non-degeneracy of ω and g. Now, applying the same procedure as we did earlier in turning near projections to projections, we can 'compress' J'_x to J_x, also a smooth family of isomorphism which have eigenvalues $\pm i$. Now, J_x is a complex structure on W_x and the metric

$$h(v,w) = -\omega(v, J_x w) \tag{26.12}$$

is the real part of an hermitian parametrix with imaginary part ω.

Exercise 20. Show conversely that on a complex vector space, a choice of positive-definite hermitian inner product generates a symplecti structure, as the imaginary part of this inner product, on the underlying real vector space such that the given complex structure on that even-dimensioanl vector space over the reals is the one constructed above.

Now, we have constructed a complex structure on the vector space which is consistent with the symplectic structure we can introduce the higher dimensional analogue of the annhilation and creation operators and in particular the harmonic oscillator. We already know these in local coordinates, i.e. on $\mathbb{R}^{2n}$. The annihilation operators, $A_j = \partial_{x_j} + x_j = x_j + iD_{x_j}$, where $D_x = \frac{1}{i}\partial/\partial x$, can be assembled into the creation complex:

$$\begin{gathered} A : \mathcal{S}(\mathbb{R}^{2n}; \Lambda^{k,0}) \longrightarrow \mathcal{S}(\mathbb{R}^{2n}; \Lambda^{k+1,0}), \\ u = \sum_\alpha u_\alpha dz^\alpha \longmapsto \sum_\alpha \sum_j A_j u_\zeta dz_j \wedge dz^\alpha. \end{gathered} \tag{26.13}$$

Lemma 28. *With $\Lambda^k = \Lambda^k V$ for an hermitian vector space V, the annihilation complex* (26.13) *is well-defined, i.e. is independent of the choice of complex orthonormal basis used to define it.*

The adjoint complex is the 'creation complex'. Check that $AC + CA = \Box$ acts on complex forms in each degree and reduces to the harmonic oscillator on zero forms.

Corollary 5. *The choice of an hermitian structure on a complex vector space, or of a compatible metric on a symplectic vector space, fixes the associated harmonic oscillator which reduces to the standard harmonic oscillator in local coordinates in which the structures are reduced to the standard ones on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$.*

What we most want from this is uniform finite-rank approximability of smoothing operators. Let $\Psi^{-\infty}_{\text{iso}}(W/X) = \mathcal{S}({}^q\overline{W})$ denote the smooth functions on the quadratic compactification of (the fibres of) a symplectic vector space, made into the space of smooth sections of the bundle of Schwartz-smoothing algebras on the fibres.

Corollary 6. *On the fibres of a complex, or a real symplectic, vector bundle W over a manifold X there is a sequence of Schwartz-smoothing projections, $\Pi_N \in \Psi^{-\infty}_{\mathrm{iso}}(W/X)$, on the fibres, smooth over the base, such that for any $A \in \Psi^{-\infty}_{\mathrm{iso}}(W/X)$, $\chi\Pi_N A$, $\chi A\Pi_N \to A$ in $\Psi^{-\infty}_{\mathrm{iso}}(W/X)$ for any $\chi \in \mathcal{C}^\infty_c(X)$.*