25. Lecture 22: Isotropic families index ($k = 1$) Friday, 24 October, 2008

Today I am supposed to be proving the weak contractibility of the central group in the looping sequence. With any luck I will get to that as a by-product of the families isotropic index, in K-theory for $k = 1$ (and untwisted). Namely, what I want to do is to define the istropic index map

$$\operatorname{ind}_{\mathrm{iso}} : [X; G^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^p)]_{\mathrm{c}} \longrightarrow [X; \mathcal{H}^{-\infty}_{\mathrm{iso}}]_{\mathrm{c}} \tag{25.1}$$

for any manifold X. Here I have written out the homotopy groups explicitly, since both represent even K-theory, as we already know. The restriction to $k = 1$ shows up in the single suspension on the left (but these arguments do carry over with only relatively minor changes to $2k - 1$ suspension, meaning isotropic operators on $\mathbb{R}^k$ as we will see next week).

For the definition I will use same sort of set up as for Bott periodicity and define some larger spaces. Thus, let

$$\begin{aligned}\dot{\mathrm{Ell}}^{0,-\infty}_{\mathrm{qiso,iso}}&(\mathbb{R}^k; \mathbb{R}^p)\\ &= \{\operatorname{Id} + A \in \dot{\Psi}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k; \mathbb{R}^p); \operatorname{Id} + \sigma_{\mathrm{iso}}(A) \in G^{-\infty}_{\mathrm{sus}(2k-1);\mathrm{iso}}(\mathbb{R}^p)\}.\end{aligned} \tag{25.2}$$

So, this is just the set of elliptic elements perturbations of the identity, the operators with invertible symbols. Then consider a similar space of pairs which are parameterices of each other

$$\begin{aligned}\dot{\mathcal{P}}^{0,-\infty}_{\mathrm{qiso,iso}}&(\mathbb{R}^k; \mathbb{R}^p)\\ &= \{(\operatorname{Id} + A, \operatorname{Id} + B) \in \dot{\Psi}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k; \mathbb{R}^p); (\operatorname{Id} + A)(\operatorname{Id} + B) - \operatorname{Id},\\ &\qquad (\operatorname{Id} + B)(\operatorname{Id} + A) - \operatorname{Id} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}).\end{aligned} \tag{25.3}$$

Proposition 28. *In the diagram*

$$\begin{tikzcd} & \dot{\mathcal{P}}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k; \mathbb{R}^p) \arrow[dl, "p_1"'] \arrow[dr, "\operatorname{ind}_{\mathrm{iso}}"] & \\ \dot{\mathrm{Ell}}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k; \mathbb{R}^p) & & \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}) \end{tikzcd} \tag{25.4}$$

where

$$\begin{gathered}\operatorname{ind}_{\mathrm{iso}}(A, B) = \begin{pmatrix} 1 - 2R_L^2 & 2R_L(\operatorname{Id} + R_L)(\operatorname{Id} + B) \\ 2R_R(\operatorname{Id} + A) & -\operatorname{Id} + 2R_R^2 \end{pmatrix},\\ R_L = \operatorname{Id} - (\operatorname{Id} + A)(\operatorname{Id} + B),\ R_R = \operatorname{Id} - (\operatorname{Id} + B)(\operatorname{Id} + A),\end{gathered} \tag{25.5}$$

the left map is surjective with the lifting property for compactly supported maps and the right map (is well-defined and) has the lifting property up to homotopy, so every compactly supported smooth map into $\mathcal{H}^{-\infty}_{\mathrm{iso}}$ is homotopic to the image of a map into $\dot{\mathcal{P}}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k; \mathbb{R}^p)$.

Proof. Of course the first assertion is that the two maps are well-defined. Certainly the left map is, since it is just projection onto the first factor and this must be elliptic, since the existence of a parameterix implies that the symbol is invertible. I will not dwell on the lifting property for this map, since I discussed the construction earlier. (Where exactly?) A smooth map with compact support into the elliptics can be quantized smoothly, to the identity outside a compact set. Then a smooth family of parametrices can be constructed, also reducing to the identity outside a compact set.

So, to the index map. This is in the rather obscure form (25.4) because I have been remiss about filling in the details about the relationship between involutions and vector bundles. It would be more usual (perhaps, it depends a bit on the circles you move in) to express the index map in terms of null bundle and null bundle of parametrix, after stabilization. I did do this quickly earlier too. The explicit map (25.4) has the advantage that it is explicit and defined for any parametrix, without stabilization. Of course, I invite you to do the algebra to show that $\operatorname{ind}_{\mathrm{iso}}(A,B)$ so defined *is an involution*:

$$\operatorname{ind}_{\mathrm{iso}}(A,B)^2=\operatorname{Id}. \tag{25.6}$$

(Which is a strange looking identity.) In doing so it is helpful to note that

$$\begin{aligned} R_LQ=QR_R,\ R_RP=PR_L,\ PQ=\operatorname{Id}-R_R,\\ QP=\operatorname{Id}-R_L \text{ if } Q=(\operatorname{Id}+B),\ P=(\operatorname{Id}+A). \end{aligned} \tag{25.7}$$

At this point it only remains to show the lifting property up to homotopy. Given the normal form for involutions which can be achieved for families, as discussed in Proposition 16 it is really enough to show that the involution corresponding to any pair of families of finite rank, commuting projections can be recovered under the index map. So, as in the periodicity construction we need a 'Bott element'. In this case it is easier – and it is certainly possible I should have done this much earlier. Namely, we know that the annihilation operator, $A=\partial_z+z$, has one dimensional null space but is surjective on Schwartz functions. Of course the order is wrong, but we can just divide by a square root of the harmonic oscillator to make it of order zero with the same property. Then its symbol is not flat at N, but it is equal to 1 at one point. So, deforming it a little more we can find $\operatorname{Id}+A$, $A\in\dot\Psi^0_{\mathrm{qiso}}(\mathbb{R})$ which is surjective, with one dimensional null space. Now to recover a given pair of projections $P_+(x)$, $P_-(x)$ which are smooth families of $N\times N$ matrices, consider

$$\begin{pmatrix} AP_+(x)+\operatorname{Id}(\operatorname{Id}-P_+(x)) & 0\\ 0 & CP_-(x)+\operatorname{Id}(\operatorname{Id}-P_-(x))\end{pmatrix}\in\dot\Psi^0_{\mathrm{qiso}}(\mathbb{R};\mathbb{C}^{2N}). \tag{25.8}$$

Here C is the adjoint of A. It is easy to check (but I have not actually done it ..) that $\operatorname{ind}_{\mathrm{iso}}$ is a family of involutions which 'recovers' these two projections. Of course one should stabilize $\mathbb{C}^{2N}$ into smoothing operators first. □

So, the existence of index map in K-theory, (25.1), follows from the uniqueness up to homotopy of the lifting on the left extended to:

$$\begin{array}{ccc} \mathcal{C}^\infty_c(X;\dot{\mathcal{P}}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)) & \xrightarrow{\operatorname{ind}_{\mathrm{iso}}} & \mathcal{C}^\infty_c(X;\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}))\\ \Big\downarrow{\scriptstyle p_1} & & \\ \mathcal{C}^\infty_c(X;\dot{\mathrm{Ell}}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)) & & \\ \Big\downarrow{\scriptstyle \sigma_{\mathrm{iso}}} & & \\ \mathcal{C}^\infty_c(X;G^{-\infty}_{\mathrm{sus}(2k-1),\mathrm{iso}}(\mathbb{R}^p)). & & \end{array} \tag{25.9}$$

Namely the linear homotopies between quantizations and parameterices shows the uniqueness up to homotopy.

Finally then this isotropic index is rather precise:-

Theorem 7. *This isotropic index map* (25.1) *is an isomorphism with right inverse the clutching map* $\mathrm{cl}_{\mathrm{eo}}$.

Proof. This requires one to check that the symbols constructed in (25.8) are homotopic to that given by the map $\mathrm{cl}_{\mathrm{eo}}$. This implies that $\mathrm{ind}_{\mathrm{iso}}\,\mathrm{cl}_{\mathrm{eo}} = \mathrm{Id}$. Since we know that $\mathrm{cl}_{\mathrm{eo}}$ is an isomorphism at the level of homotopy, i.e. as an inverse to (25.1) it follows that the index is also an isomorphism. □

So, this is the $k = 1$ case of the isotropic families index. We want to generalize it in two ways. First to $k > 1$. In fact the restriction to $k = 1$ only occurs in the construction of the annihilation/creation operators and $\mathrm{cl}_{\mathrm{eo}}$. The more significant extension (which is slightly different in form) is to twist the spaces on which the isotropic algebra acts to be a vector bundle over X, instead of just considering straight families. This leads to the Thom isomorphism. We can think of the group on the left in several ways because of Bott periodicity. The most obvious is to identify it as $\mathrm{K}^1_{\mathrm{c}}(X \times \mathbb{R})$ and hence as $\mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}^2)$. This is how the map is usualy described, as

$$\mathrm{ind}_{\mathrm{iso}} : \mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}^2) \longrightarrow \mathrm{K}^0(X) \tag{25.10}$$

implementing Bott periodicity. In fact it is precisely this relationship I want to exploit in order to define the Thom isomorphism

$$\mathrm{K}^0_{\mathrm{c}}(W) \longrightarrow \mathrm{K}^0(X) \tag{25.11}$$

for any complex, or symplectic, vector bundle over X (so it has even rank as a real bundle).

Now, what does this tell us about $\dot G^{0;-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$? Basically half of what we want, I would say the hard half given where we are. Namely

Corollary 4. *If* $f : X \longrightarrow G^{-\infty}_{\mathrm{sus;iso}}(\mathbb{R}^k)$ *(is compactly supported) then* $f = \sigma_{\mathrm{iso}}(F)$ *for a compactly supported family* $F : X \longrightarrow \dot G^{0;-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ *implies that* f *is homotopic to the identity.*

Proof. The index vanishes, so $[f] \in [X; G^{-\infty}_{\mathrm{sus;iso}}(\mathbb{R}^k)]_{\mathrm{c}}$ must vanish. □

In fact the converse is also true but this involves another argument which we can subsume into what is the last part of the looping sequence.

Proposition 29. *The group* $\dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R};\mathbb{R}^p)$ *is weakly contractible; any compactly supported map into it is homotopic to the constant map to the identity.*

Proof. Given such a map, F, it follows from the corollary that $\sigma_{\mathrm{iso}}(f)$ is homotopic to the identity. Let f_t be such an homotopy, with $f_0 = \mathrm{Id}$, $f_1 = f$. Since it is an elliptic family, and has an invetible quantization at $t = 0$, this can be lifted to an homotopy $F_t : X \times [0,1] \longrightarrow \dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R};\mathbb{R}^p)$ with symbol family f_t and $F_0 = \mathrm{Id}$. This might seem to solve the problem, but not so fast! It follows that F_1 is *a* lift of f to be a family of invertibles, but it is not clear that it is the one we started with. Since we can deform lower order terms in the symbol away, we can arrange that

$$F_1^{-1}F \in \mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{p+k})) \tag{25.12}$$

since the two quantizations then differ by smoothing terms. So the remaining problem is to show that the image of a family of smoothing perturbations can be deformed away in the bigger group $\dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R};\mathbb{R}^p)$. In fact with the use of a bit more homotopy theory we already have enough to show this. However, I think it is worth doing directly. So the proof of this Proposition is completed by the next. □

Proposition 30. *If $g : X \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{p+k})$ is a compactly supported smooth map then there is an homotopy $G_t \in \mathcal{C}^\infty_c(\dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R};\mathbb{R}^p))$ with $G_1 = g$ and $G_0 = \mathrm{Id}$.*

Proof. This can be done quite explicitly using the creation and annihilation operators. Here is the idea but I have not checked the details at all. The crucial point is that we have found an operator of index 1.

First retract g to be of finite rank, on some $\mathbb{C}^N$. Then take two copies and consider the creation and annihilation operators acting on $\mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^2$ as a 2×2 matrix with matrix-operator values

$$\begin{pmatrix} \pi_1 & C \\ A & 0 \end{pmatrix} \tag{25.13}$$

This is invertible, since the null space of A and the lack of range of C are compensated for by the off-diagonal π_1, projecting onto the null space. Now, tensor with matrices on $\mathbb{C}^N$ and consider

(25.14)

$$\begin{pmatrix} \cos(\theta) & g(x)\sin(\theta) \\ -g^{-1}(x)\sin(\theta) & \cos(\theta) \end{pmatrix}^{-1} \begin{pmatrix} \pi_1 \otimes g(x) + \cos(\theta)(\mathrm{Id} - \pi_1) & g(x)\sin(\theta)C \\ -g^{-1}(x)\sin(\theta)A & \cos(\theta). \end{pmatrix}$$

These are invertible matrices, the first normalizes the symbol to be the identity at N, the point where A and C are both flat to 1.

Then, consider the reverse homotopy through invertibles

$$\begin{pmatrix} g(x)\cos(\theta) & g(x)\sin(\theta) \\ -g^{-1}(x)\sin(\theta) & g^{-1}(x)\cos(\theta) \end{pmatrix}^{-1} \times \begin{pmatrix} g(x)\pi_1 + g(x)\cos(\theta)(\mathrm{Id} - \pi_1) & g(x)\sin(\theta)C \\ -g^{-1}(x)\sin(\theta)A & g^{-1}(x)\cos(\theta). \end{pmatrix} \tag{25.15}$$

This starts at the same matrix, at $\theta = \pi/2$ and deforms back to the identity. As I say, I haven't checked this. □

There is another approach to proving Proposition 30 which is a bit more machine-heavy but has other advantages, as I hope we will see. To do this we need the adiabatic limit for operators of order 0.

When talking about the isotropic product, leading to the algebra $\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ that is underlying the looping sequence, I carried along, at least for a while, the adiabatic version of it. Namely the product in (20.3) is actually smooth down to $\epsilon = 0$, just as in the case of smoothing operators discussed earlier. This means we can set up an algebra of adiabatic operators of order 0 which is just the space of functions

$$\mathcal{C}^\infty([0,1]_\epsilon \times {}^q\overline{\mathbb{R}^{2k}} \times ; \mathcal{S}(\mathbb{R}^{2p})) \tag{25.16}$$

with the product given by (20.3). Without going into details this will have both an adiabatic symbol, at $\epsilon = 0$, extending the one in the smoothing case, and now also a 'regular' symbol at the boundary in the second variable (or a 'full symbol if we take Taylor series at this boundary). This second symbol depends on $\epsilon \in [0,1]$, but just as a parameter since the leading term is always just the product (including the operator product in the last variables of course). The lower order terms, the star product, does depend on ϵ. We also have the restriction to $\epsilon = 1$ and if you recall this is how I finally talked about Bott periodicity. I will denote this adiabatic algebra

$$\Psi^{0,-\infty}_{\mathrm{ad\,qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p) \tag{25.17}$$

Now to this we can add the 'dot' conditon of having the functions vanish to infiniite order at the fixed point N on the boundary of the quadratic compactification of $\mathbb{R}^{2k}$. This leads to the subalgebra

$$\dot{\Psi}^{0,-\infty}_{\mathrm{ad\,qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p). \tag{25.18}$$

Proposition 31. *The adiabatic algebras of order* 0 *in* (25.17) *and* (25.18) *(which are non-unital) are Neumann-Fréchet algebras.*

Proof. This requires a combination of the proofs of the earlier cases. □

So now I have at least described the corresponding group of invertible perturbations of the identity by elements of (25.18)

$$\dot{G}^{0,-\infty}_{\mathrm{ad\,qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p). \tag{25.19}$$

What good is it? Well, it has the same symbol maps as the algebra but now valued in the groups. This gives us a diagram where the bottom line is the looping sequence, the middle line is the corresponding sequence with this new group in the middle and the top line is in principle the adiabatic part – what we get by taking the adiabatic symbol (i.e. restricting to $\epsilon = 0$ to the extent that it makes sense).

$$\begin{tikzcd}
G^{-\infty}_{\mathrm{sus(2),iso}}(\mathbb{R}^p) \arrow[r] & \tilde{G}^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R};\mathbb{R}^p) \arrow[r] & G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso},\mathrm{ind}=0}(\mathbb{R}^p) \\
G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}:\mathbb{R}^p) \arrow[r] \arrow[u, "\sigma_{\mathrm{ad}}"] \arrow[d, "R"] & \dot{G}^{0,-\infty}_{\mathrm{ad,iso}}(\mathbb{R};\mathbb{R}^p) \arrow[r, "\sigma_{0,\mathrm{iso}}"] \arrow[u, "\sigma_{\mathrm{ad}}"] \arrow[d, "R"] & \mathcal{C}^\infty([0,1];G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso},\mathrm{ind}=0}(\mathbb{R}^p)) \arrow[u, "\epsilon=0"] \arrow[d, "R"] \\
G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}) \arrow[r] & \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p) \arrow[r, "\sigma_{0,\mathrm{iso}}"] & G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso},\mathrm{ind}=0}(\mathbb{R}^p)
\end{tikzcd} \tag{25.20}$$

Now, I need to discuss what is happening here carefully, but the – maybe somewhat surprising – point to grasp is that the top row is in fact the delooping sequence for the (terminal) group which is the component of the identity in the suspended group we are by now getting familiar with. So what this diagram is supposed to show conceptually is that the delooping sequence is 'just' the partly quantized delooping sequence for the loop group. Now, before I go into a term by term discussion of (25.20) let me show why it might help us.

Claim 3. *All the vertical arrows, up and down, in* (25.20) *are surjective weak homotopy equivalences with the lifting property for compact families, all three rows are exact and the central column consists of weakly contractible groups.*

So this is just making the same claim but more so! In fact the very central group here is easily seen to be contractible by hand. The surjectivity of the middle R

would imply directly the weak contractibility of the middle group for the looping sequence, but this is where I have failed to find a direct proof.

However, just look at the bottom left rectangle and see that we can use it, if we know a bit – particularly the commutativity – to give a proof of Proposition 30. What we are given is a compactly supported map into the bottom left group. By the lifting property property this comes from a map into the 'adiabatic' group on the left of the middle row. This can then be sent into the really central group. As I said above, it is easy to see geometrically that this group is weakly contractible. Thus it can be deformed away here. Mapping the homotopy forward to the central group on the bottom row gives, by commutativity, an homotopy trivializing the image of the original map.