24. Lecture 21: Curvature and Chern class Wednesday, 22 October, 2008

Reminder. *Last time I computed the trace defect for the extension of the trace functional to* $\Psi^{\mathbb{N},-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ *given by Riesz regularization. Today I want to use this to compute the curvature of the determinant line bundle. First I have to make sure of a few technical points, one of these is the proof that the range of the symbol map in* (20.27) *is indeed the index zero component – I will use the trace defect formula to compute the index.*

Recall that $\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ is the space of functions with $\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2k}};\mathcal{S}(\mathbb{R}^{2p})$ with a non-commutative product. Choosing a point on the boundary of the quadratic compactification here – the North Pole – we can consider the subalgebra, $\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$, of functions that vanish in Taylor series there, so identified with

$$\left\{a\in\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2k}};\mathcal{S}(\mathbb{R}^{2p}); a\equiv 0 \text{ at } \{N\}\times\mathbb{R}^{2p},\ N\in\partial^q\overline{\mathbb{R}^{2k}}\right\}. \tag{24.1}$$

I will probably skip the proof of the following result in the lecture. Not that it is unimportant. However, its proof is generally similar to ones we have seen before, although not reducible to them. In particular it makes use of the L^2 boundedness of these operators, which is significant in itself (and one reason the proof is so long).

I should have used this notation before (but only on the insistence of Frédéric Rochon am I introducting it now).

Definition 6. A Fréchet algebra, $\mathcal{A}$, without identity, is said to be a Neumann-Fréchet algebra if the group of invertible elements in $\mathrm{Id}+\mathcal{A}$ is open – it behaves as if the Neumann series converges near the identity. These are often called 'good' Fréchet algebras. Good grief! If it contains the identity then it is Neumann-Fréchet if the elements in a neighbourhood of the identity are invertible.

Proposition 23. *The group* $\dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ *of operators* $A\in\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ *such that* $\mathrm{Id}+A$ *has an inverse of the same form, is open in* $\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$, *i.e. this is a Neumann-Fréchet algebra, and* $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ *is a (relatively) closed normal subgroup.*

Proof. The Fréchet topology is on $\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ comes from the $\mathcal{C}^\infty$ standard toplogy of the supremum norms of derivatives, in this case on $\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2k}})\times\overline{\mathbb{R}^{2p}})$. The rapid vanishing at the boundary in the second factor (of course uniformly in the first factor) to give the Schwartz subspace and at one point in the first factor, give a closed subspace. So we need to show that for a in a small neighbourhood of 0 in this topology, $\mathrm{Id}+b$ has an inverse with $b\in\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$.

To show this we first need to prove the L^2 boundedness of elements this algebra, which acts on $\mathcal{S}(\mathbb{R}^{k+p})$. This can be done using Hörmander's approach, which I will outline at some point. In particular this shows that the L^2 operator norm is continuous on $\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$, from which it follows that $\mathrm{Id}+a$ is invertible on L^2 for a in a neighbourhood of 0. So, it still needs to be seen that this inverse is of the form $\mathrm{Id}+b$, $b\in\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$.

To see this we first show that if $\mathrm{Id}+a$ is L^2 invertible then it must be elliptic, in the sense that $\mathrm{Id}+\sigma_{0,\mathrm{iso}}(a)\in G^{-\infty}_{\mathrm{sus}(2k-1),\mathrm{iso}}(\mathbb{R}^p)$. This can be done by constructive contradiction. That is, non-ellipticity means that $\mathrm{Id}+a(p)$ must be non-invertible as a smoothing perturbation of the identity on $\mathcal{S}(\mathbb{R}^p$ for some $p\in\mathbb{S}^{2k-1}\setminus\{N\}\simeq\mathbb{R}^{2k-1}$

(since at $p = N$ it is the identity). This in turn means there must be an element of the null space. The constructive part is to use this to generate a sequence $u_j \in \mathcal{S}(\mathbb{R}^{k+p})$ which has norm one in L^2 but is such that $(\mathrm{Id}+a)u_j \to 0$ in L^2. The idea here is that the solution should 'concentrated near p' in a sense that can be understood in terms of the symbol of the operator. This violates invertibility. Essentially what is being shown here is that the symbol map extends by continuity to the closure of these operators in the bounded operators on L^2, and so it must be invertible. Again I plan to add a bit about this at some point.

Once we know that if $\mathrm{Id}+a$ is an invertible operator on L^2 then it is necessarily elliptic, we can apply usual methods. Namely we can construct a parameterix for $\mathrm{Id}+a$, $\mathrm{Id}+b'$, such that

$$(\mathrm{Id}+b')(\mathrm{Id}+a) - \mathrm{Id} = R_R,\ (\mathrm{Id}+a)(\mathrm{Id}+b') - \mathrm{Id} = R_L \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}). \tag{24.2}$$

Then it follows that the inverse is of the expected form since applying (24.2) on the left and right

$$(\mathrm{Id}+a)^{-1} = \mathrm{Id}+b' - (\mathrm{Id}+a)^{-1}R_L = \mathrm{Id}+b' - R_L - b'R_L + R_R(\mathrm{Id}+a)^{-1}R_L. \tag{24.3}$$

The last term is in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ because of the corner property of these operators.

This shows that $\dot{\Psi}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ (and indeed $\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$) is a Neumann-Fréchet algebra.

The last closure property follows directly from the characterization of the kernels. $\square$

Proposition 24. *The quotient*

$$\dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)/G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}) = G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso}}(\mathbb{R}^p) \tag{24.4}$$

is the group of formal power series in ρ_q, with leading terms forming the component of the identity (i.e. the part on which the index functional vanishes) in $G^{-\infty}_{\mathrm{sus}(2k-1),\mathrm{iso}}(\mathbb{R}^p)$, with arbitrary lower order terms and a $*$-product; for the moment we only prove this for $k = 1$ although it is true in general.

Proof. From Proposition 23 above, first group in (24.4) is an open subset of the smooth functions in (24.1). The subgroup is just the subspace which vanishes in Taylor series on the whole of the product of the boundary of the first factor with the second factor. This is simply because a smoothing perturbation of the identity is invertible if and only if it has an inverse in the larger space, i.e. it is an element of the larger group. Thus the quotient is certainly a subset, and necessarily open, of the space of formaly power series at the boundary of the first factor. The leading term, which is the identity plus a function on the $2k-1$ sphere, valued in smoothing operators, must be invertible, and the perturbation vanishes to infinite order at the point N so it can be identified with an element of $G^{-\infty}_{\mathrm{sus}(2k-1),\mathrm{iso}}(\mathbb{R}^p)$. The induced product on the formal power series is given by the loop product of the leading terms, since this is just the principal symbol, and a $\star$ product in the lower order terms, in the sense that the k the term in the product is give by bidifferential operators on the first k terms of the factors. More about star products below!

If $g \in \dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ then any lower order, formal power series, perturbation of the image in the quotient can be realized as an actual function, $a \in \Psi^{-1,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ with the correct Taylor series. Cutting off near the boundary, this can be seen to have arbitrariy small L^2 norm, and, as discussed in the proof of (23) this implies that $G + a \in \dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$.

Thus it remains to show that the leading part of the quotient in $G^{-\infty}_{\mathrm{sus}(2k-1),\mathrm{iso}}(\mathbb{R}^p)$ is the component of the identity, i.e. on the elements on which the index functional vanishes. It is at this point that we reduce to the case $k=1.$ If $g\in \dot G^{0,-\infty}_{\mathrm{qiso},\mathrm{iso}}(\mathbb{R}^k:\mathbb{R}^p)$ and h is its inverse then certainly

$$\overline{\mathrm{Tr}}([g,h])=0=c\int_{\mathbb{S}}\mathrm{tr}(\frac{\partial h_0}{\partial\theta}g_0(\theta))d\theta=-c\int_{\mathbb{S}}\mathrm{tr}(g_0^{-1}(\theta)\frac{\partial g_0}{\partial\theta}(\theta))d\theta,\ c\neq 0. \tag{24.5}$$

Here the explicit trace defect formula in Proposition 22 has been used. The last integral is the winding number of the determinant of g_0, the leading term of g, i.e. the index functional. Thus the range is contained in the component on which the index vanishes.

To see the converse, we need to do a little analysis to reverse the argument above. Namely given the symbol $\mathrm{Id}+a_0$ we can quantize it to a not-necessarily-invertible operator $\mathrm{Id}+A$. However, we know that this has a parametrix $\mathrm{Id}+B$ up to smoothing errors,

$$(\mathrm{Id}+B)(\mathrm{Id}+A),\ (\mathrm{Id}+A)(\mathrm{Id}+B)\in\mathrm{Id}+\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}). \tag{24.6}$$

Indeed, with an error term of order -1 this follows by choosing B with symbol $b=(\mathrm{Id}+a)^{-1}-\mathrm{Id}$. Taking the formal Neumann series for the error term and summing it allows the parametrix to be improved to (24.6). Now, if Π_N is the projector onto the first N eigenfunctions of the harmonic oscillator on $\mathbb{R}^{k+p}$ it follows that

$$(\mathrm{Id}+B)(\mathrm{Id}+A)(\mathrm{Id}-\Pi_N)=(\mathrm{Id}+E(\mathrm{Id}-\Pi_N))(\mathrm{Id}-\Pi_N). \tag{24.7}$$

Since E is a smoothing opertor, $\mathrm{Id}+E(\mathrm{Id}-\Pi_N)\in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ for N large enough. Composing on the left with the inverse gives, with a different operator B',

$$(\mathrm{Id}+B')(\mathrm{Id}+A')=(\mathrm{Id}-\Pi_N),\ A'=A-A\Pi_N-\Pi_N. \tag{24.8}$$

Here A' has the same symbol as A but now must have null space precisely the range of Π_N. Proceeding in the same manner with the adjoint, it follows that $\mathrm{Id}+A'$ must have range of finite codimension. Composing with an element of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+n})$ it can be arranged seen that, a possibly different, $\mathrm{Id}+A'$, but with the same symbol, has null space the range of Π_N and range that of $\mathrm{Id}-\Pi_{N'}$. Then B' can be shifted by a smoothing operator so that (24.8) holds and also

$$(\mathrm{Id}+A')(\mathrm{Id}+B')=\mathrm{Id}-\Pi_N. \tag{24.9}$$

Then we see that the index, in the conventional sense,

$$\begin{aligned}\mathrm{ind}(\mathrm{Id}+A')&=N-N'=\mathrm{Tr}(\Pi_N-\Pi_{N'})\\&=\overline{\mathrm{Tr}}((\mathrm{Id}-\Pi_{N'})-(\mathrm{Id}-\Pi_N))=\overline{\mathrm{Tr}}([\mathrm{Id}+B',\mathrm{Id}+A']).\end{aligned} \tag{24.10}$$

Here we have used the fact that $\overline{\mathrm{Tr}}$ is an extension of the trace functional. Now we can apply the same argument as in (24.5) to see that this is a non-vanishing multiple (-1 I think) of the 'index' in the sense of the winding number of the determinant of g, the symobl of $\mathrm{Id}+A'$. Thus if this vanishes then $N=N'$ and adding Π_N to A' gives an invertible lift of the symbol. □

The use of the trace defect formula in the proof above is very close to the discussion of the curvature below and presages the treatment of 'η forms' later (I hope).

Proposition 25. *The subgroups*

$$\dot{G}^{m,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^{k+p}) \subset \dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p), \tag{24.11}$$

of elements where the perturbation of the identity is of order $m \in -\mathbb{N}$, are normal and the quotient map gives a fibration

$$\begin{tikzcd} \dot{G}^{m,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^{k+p}) \arrow[r] & \dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p) \arrow[d] \\ & G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso}}(\mathbb{R}^p)/\big(\rho_q^{m+1} \equiv 0\big), \end{tikzcd} \tag{24.12}$$

i.e. there is a smooth section in a neighbourhood of each point of the quotient.

Proof. The normality of these subgroups follows directly from the order properties of the product. The handy thing is that the quotients all give fibrations, whereas in the case $m = -\infty$ it is a Serre fibration only – not locally trivial. It suffice to give a section of the projection over a neighbourhood of the identity, since this can be translated to any other point. Since there are only finitely many terms in the power series in the quotient, they can be summed and cut off near the boundary on $\overline{{}^q\mathbb{R}^{2k}}$. Provided the terms are small enough, this gives an invertible perturbation of the identity, following the arguments of Proposition 23, and hence a section – meaning the sequence is a fibration. □

Lemma 26. *The construction in* (21.11), (21.12) *carries over to the sequence* (24.12) *for any $m < -2k$ and constructs a locally trivial line bundle $\mathcal{L}_m$ over the quotient in* (24.12). *Under any of the projection maps for $m' < m$, $m' \in -\mathbb{N}\cup\{-\infty\}$*

$$G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso}}(\mathbb{R}^p)/\big(\rho_q^{-m'+1} \equiv 0\big) \longrightarrow G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso}}(\mathbb{R}^p)/\big(\rho_q^{-m+1} \equiv 0\big), \tag{24.13}$$

$\mathcal{L}_m$ pulls back to be canonically isomorphic to $\mathcal{L}_{m'}$, where $\mathcal{L} = \mathcal{L}_{-\infty}$.

In particular $\mathcal{L}$ as constructed originally is indeed locally trivial, so Claim 1 is vindicated.

Moreover the same is true of the connections. Namely, for $m < -k$ each of the determinant line bundles $\mathcal{L}_m$ has a connection given as the quotient of the connection

$$d - \gamma,\ \gamma = \frac{1}{2}\overline{\mathrm{Tr}}\big(g^{-1}dg + (dg)g^{-1}\big) \tag{24.14}$$

where the regularized trace is defined above.

Proposition 26. *The 1-form γ on $\dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ defined in* (24.14) *induces a connection on each $\mathcal{L}_m$, $m < -2k$. In case $k = 1$, the curvature is*

$$c\int_{\mathbb{R}}(\mathrm{tr}\,\big(g^{-1}\frac{dg}{dt}(g^{-1}dg)^2\big)dt \tag{24.15}$$

Proof. Let me concentrate on the case $m = -\infty$. The case for finte m corresponds to the extension of the determinant to the groups $G^{-m}_{\mathrm{iso}}(\mathbb{R}^{k+p}$ by continuity – where it continues to be multiplicative.

So, even though the projection in the delooping sequence may not be a fibration, the determinant bundle itself is locally trivial. Namely over any small open set in one of the finite order quotients there is a section and this lifts to the preimage of the open set in $G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso}}(\mathbb{R}^p)$ and hence to a section of the trivial bundle over the preimage in $\dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p)$ with the transformation law

$$f(Ag)=f(A)\det(g),\ \forall\ g\in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}). \tag{24.16}$$

Conversely, a local section of the trivial bundle on the preimage of a set descends to be a section of $\mathcal{L}$ over that set if and only if (24.16) holds. Now, the connection on the trivial bundle is

$$\nabla(f)(A)=df-\gamma(A)f \tag{24.17}$$

so

$$\begin{aligned}\nabla(f)(Gg)&=d(f\det(g))-\gamma(Ag)f(Gg)=\det(g)(df-\gamma(A)f)\\&+\operatorname{tr}(g^{-1}dg)f-\frac{1}{2}\overline{\operatorname{Tr}}(g^{-1}A^{-1}d(Ag)+d(Ag)g^{-1}A^{-1}-A^{-1}dA-(dA)A^{-1}).\end{aligned} \tag{24.18}$$

The combination of the last terms vanishes, since commutation by g preserves $\overline{\operatorname{Tr}}$ (since the smoothing term permits approximation by smoothing terms) so

$$\begin{aligned}\overline{\operatorname{Tr}}(g^{-1}A^{-1}d(Ag))&=\overline{\operatorname{Tr}}(A^{-1}dA)+\operatorname{Tr}(g^{-1}dg),\\ \overline{\operatorname{Tr}}(d(Ag)g^{-1}A^{-1})&=\overline{\operatorname{Tr}}((dA)A^{-1})+\operatorname{Tr}(g^{-1}dg).\end{aligned} \tag{24.19}$$

Thus the connection descends to $\mathcal{L}$. The curvature can be compute on the central group – of course it must also descend to the quotient. It will be exact on the large group (which is actually weakly contractible as we shall see) but not on the quotient – reconcile yourself with this as necessary!

The curvature is $-d\gamma$, or $-d\gamma/2\pi i$ according to taste. Anyway, it is enough to compute:-

$$\begin{aligned}d\gamma&=-\frac{1}{2}\overline{\operatorname{Tr}}(A^{-1}(dA)A^{-1}dA-(dA)A^{-1}(dA)A^{-1})\\&=\frac{1}{2}\overline{\operatorname{Tr}}\big([(dA)A^{-1}dA,A^{-1}]\\&=-\operatorname{Tr}_{\mathrm{R}}\big(A^{-1}[A,\log\rho](A^{-1}(dA))^2\big)\\&=-c\int_{\mathbb{R}}\operatorname{tr}(a^{-1}\frac{da}{dt}(a^{-1}da)^2)dt,\ a=\sigma_{0,\mathrm{iso}}(A),\end{aligned} \tag{24.20}$$

using the trace defect formula and in the last line the explicit formula in case $k=1$. Here, you may recognize the 2-form part of the Chern character on the suspended group (of course I have lost track of the constants at the moment, but it is actually equal to it – at least up to sign). □

It is very unlikely I will get to this until some time later!

There is another property of this construction of the determinant line bundle which is a consequence of a wider conjugation-invariance property of the determinant.

Proposition 27. *The determinant line bundle is* primitive *in the sense that for any two elements $\alpha,\beta\in G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso,ind}=0}(\mathbb{R}^p)$ there is a natural isomorphism*

$$\mathcal{L}_\alpha\otimes\mathcal{L}_\beta\simeq\mathcal{L}_{\alpha\beta} \tag{24.21}$$

which makes the complement of the zero section $\mathcal{L}^* \subset \mathcal{L}$ into a group which is a $\mathbb{C}^*$ extension of the base, so

$$\mathbb{C}^* \longrightarrow \mathcal{L}^* \longrightarrow \dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p) \tag{24.22}$$

is a short exact sequence of groups.

Proof. The additional conjugation invariance referred to above, is that

$$\det(A^{-1}gA) = \det(g) \ \forall\ A \in G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p),\ g \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}). \tag{24.23}$$

Given this the statement of the Proposition follows readily. Namely if A, $B \in \dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ project to α, $\beta \in G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso}}(\mathbb{R}^p)$ then for $g \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ the map

$$(z\det(g), Ag)(z'\det(g'), Bg') \longmapsto (zz'\det(gg'), AB(B^{-1}gB)g') \tag{24.24}$$

descends to an identification of $\mathcal{L}_\alpha \otimes \mathcal{L}_\beta$ with $\mathcal{L}_{\alpha\beta}$. This can clearly be interpreted as a group product

$$(l,\alpha)\cdot(l',\beta) \longmapsto (l\otimes l', \alpha\beta) \tag{24.25}$$

on $\mathcal{L}^*$ which reduces to $\mathbb{C}^*$ on the trivial fibre above Id. This gives the short exact sequence (24.22).

So, it remains to prove (24.23). There is a smooth curve, $g(t)$, connecting g to the identity in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$. Certainly

$$A^{-1}gA = A^{-1}(\mathrm{Id} + a)A = \mathrm{Id} + A^{-1}aA \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}) \tag{24.26}$$

so consider

$$\begin{aligned} &\frac{d}{dt}\log\big(\det(g(t))^{-1}\det(A^{-1}g(t)A)\big) \\ &= \frac{d}{dt}\log\big(\det(g^{-1}(t)A^{-1}g(t)A)\big) = \mathrm{Tr}\,\big(A^{-1}g^{-1}(t)Ag(t)\frac{d}{dt}(g^{-1}(t)A^{-1}g(t)A)\big) \\ &= \mathrm{Tr}\,\big(A^{-1}g^{-1}(t)g'(t)A - A^{-1}g^{-1}(t)Ag'(t)g^{-1}A^{-1}g(t)A\big) = 0. \end{aligned} \tag{24.27}$$

In the last step the commutation of factors of A is justifed by the fact that there is one factor, g', which is smoothing so A can be approximated by smoothing operators, in the topology of symbols-with-bounds, of marginally positive order, in such a way that the product converges in smoothing operators. Since the determinants are equal when $g = \mathrm{Id}$ they are equal everywhere. $\square$

Exercise 18. Did I explain (24.27) clearly enough? If not, go through the following. If $A \in \Psi^{s,-\infty}_{\infty\text{-iso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ and $a \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ then $Aa \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ depends continuously on A, as does aA, in the topology of $\Psi^{s,-\infty}_{\infty\text{-iso}}(\mathbb{R}^k : \mathbb{R}^p)$ for any $s' \geq s$. Since $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ is dense in $\Psi^{s,-\infty}_{\infty\text{-iso}}(\mathbb{R}^k : \mathbb{R}^p)$ in this weaker topology, for any $s' > s$, and on this dense subspace $\mathrm{Tr}(Aa) = \mathrm{Tr}(aA)$ it follows in general.

Exercise 19. Write out in some reasonably explicit form the product on the 'star-extended' loop group $G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}(\mathbb{R}^p)$ and show that it can be extended to the whole of the group $G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso}}(\mathbb{R}^p)$ – which is defined to have the 'same' product but arbitrary invertible leading term (rather than index zero) and arbitrary lower order terms as before. Show that the determinant line bundle can be transferred from the component of index zero to the other components by choosing a base point in each. Is it possible to extend the group property to the whole thing? Assuming that you agree with me that this does not seem possible, can you explain why?