23. Lecture 20: Trace defect formula Monday, 20 October, 2008

Reminder. *I am currently examining the algebra* $\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ *to establish, and check various things about, the looping sequence* (20.27) *and hence the construction of the determinant bundle in* (21.12).

Last time I discussed the Riesz regularized integral of classical symbols on any compact manifold with boundary and the residue integral at the boundary. Let us apply this discussion to define a regularized trace functional and a residue trace functional on isotropic pseudodifferential operators

$$
\begin{aligned}
\overline{\mathrm{Tr}} &: \Psi^{m,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p) \longrightarrow \mathbb{C},\\
\mathrm{Tr}_{\mathrm{R}} &: \Psi^{m,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p) \longrightarrow \mathbb{C},\ m\in\mathbb{Z}.
\end{aligned}
\tag{23.1}
$$

The first is supposed to be an extension of the trace functional which is given on smoothing operators by

$$
\mathrm{Tr}: \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}) \ni a \longmapsto (2\pi)^{-k}\int_{\mathbb{R}^{2k}} \mathrm{tr}_{\mathbb{R}^p}(F(t,\tau))dtd\tau.
\tag{23.2}
$$

Here the two parts of the space are treated differently as far as the kernel is concerned, with Weyl coordinates and Fourier transform used in the first part

$$
F(t,\tau,z,z') = \int_{\mathbb{R}^k} e^{-is\cdot\tau} f(t,s,z,z'),\ a(Z,Z',z,z') = f(\frac{Z+Z'}{2}, Z-Z',z,z').
\tag{23.3}
$$

Passing to $\Psi^{k,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ amounts to replacing the Schwartz condition by the partially-Schwartz space $\rho_q^{-m/2}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2k}};\mathcal{S}(\mathbb{R}^{2p}))$. Now, in terms of this quadratic compactification to a ball we know that $\rho_q = (1+|(t,\tau)|^2)^{-1}$ is a boundary defining function – which is to say that $x=R^{-2}$ is also a boundary defining function near the boundary. The symplectic volume form is therefore

$$
\begin{aligned}
|dtd\tau| = R^{2k-1}|dRd\theta| = \frac{1}{2}x^{-k+1}|dxd\theta| \Longrightarrow\\
|dtd\tau| = \rho_q^{-k+1}\nu,\ 0<\nu\in\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2k}};\Omega).
\end{aligned}
\tag{23.4}
$$

Thus the symplectic volume form is the product of a smooth non-vanishing volume form and an element of $\rho_q^{-k+1}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2k}})$. Only in the case $k=1$ is this a smooth volume form. Thus we can use the Riesz regularized index, choosing[4] $\rho_q^{\frac12}$ as the defining function, to define

So we could just take the regularized integral of the 'symbol' (which is the whole operator) and this would give a regularized trace. However, it is better to follow an idea which comes originally from Seeley [11] but was effectively improved by Guillemin [3]. Namely we observe that $\rho_q^{z/2}\in\Psi^{-z}_{\mathrm{qiso}}(\mathbb{R}^k)$ – with no stabilization and remembering the annoying $\frac12$'s. So we can consider the operator product, with A :

$$
\rho_q^{z/2}\circ A \in \Psi^{s+m,-\infty}_{\infty\text{-iso}}(\mathbb{R}^k;\mathbb{R}^p),\ s>-\operatorname{Re} z,
\tag{23.5}
$$

[4] This business about the square-roots and quadratic defining functions is quite irritating; I will have to think of a clearer course of action

where I am using the fact, which I forgot to include earlier, that the stabilized operators are a module over the ustabilized ones (since the unstabilized ones just act 'as a multiple of the identity' in the second variables).

I have just claimed that the composite is an operator with symbol-with-bounds in (23.5). Of course a lot more is true, since we know that the product is given by a bidifferential operator up to any preassigned order[5]

$$\begin{gathered} A\circ B=Q_N(A,B)+Q_{(N)}(A,B),\ Q_N(A,B)=\sum_{|\alpha|+|\beta|\le N}c_{\alpha,\beta}D^\alpha A\cdot D^\beta B,\\ Q_{(N)}:\mathcal{A}^s(\overline{\mathbb{R}^{2k}})\times\mathcal{A}^t(\overline{\mathbb{R}^{2k}})\longrightarrow\mathcal{A}^{s+t+2N}(\overline{\mathbb{R}^{2k}})\ \forall\ s,t\in\mathbb{R}, \end{gathered} \tag{23.6}$$

where we actually know the coefficients. The 'remainder term' is continuous – as is the explict expansion. The same formula applies to the suspended algebra provided we interpret the product as the composition of smoothing operators.

Applying this to the product in (23.5) we conclude that as a function

$$\rho_q^{z/2}\circ A=\rho_q^{z/2}P_N(z,\rho_q,A)+Q_{(N)}(z) \tag{23.7}$$

where the leading term is a differential operator applied to A and a polynomial in z while the remainder term is holomorphic as a map

$$\{\operatorname{Re}z>L\}\longrightarrow\Psi^{m-2N-L,-\infty}_{\infty\text{-iso}}(\mathbb{R}^k;\mathbb{R}^p). \tag{23.8}$$

Lemma 25. *For any* $A\in\Psi^{m,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p)$

$$\operatorname{Tr}(\rho_q^{z/2}\circ A) \tag{23.9}$$

is meromorphic in the complex plane with at most simple poles at $z\in m+n-\mathbb{N}$.

Proof. The result follows in $\operatorname{Re}z>L$ for any L by using the splitting (23.7) for large enough N, applying the discussion of Riesz regularization of the integral to the first part and the holomorphy in (23.8) to the second part. □

So, now we can define

$$\begin{gathered} \operatorname{Tr}_{\mathrm{R}}(A)=\lim_{z\to0}z\operatorname{Tr}(\rho_q^{z/2}\circ A),\\ \overline{\operatorname{Tr}}(A)=\lim_{z\to0}\left(\operatorname{Tr}(\rho_q^{z/2}\circ A)-\frac{1}{z}\operatorname{Tr}_{\mathrm{R}}(A)\right) \end{gathered} \tag{23.10}$$

as respectively the residue and the regularized value of the analytic continuation of the trace to $z=0$. The residue trace was defined in the case of the usual pseudodifferential algebra on a compact manifold by Wodzicki [12].

Proposition 21. *If the order of* A *is less than* $-2k$ *then* $\overline{\operatorname{Tr}}(A)=\operatorname{Tr}(A)$. *The residue trace is a trace functional,* $\operatorname{Tr}_{\mathrm{R}}([A,B])=0$, *it vanishes on operators of order less than* $-2k$, *is given explicitly by the residue integral*

$$\operatorname{Tr}_{\mathrm{R}}(A)=(2\pi)^{-k}\,{}^{R}\!\!\int A\omega, \tag{23.11}$$

and the regularized trace satisfies the trace defect formula

$$\overline{\operatorname{Tr}}([A,B])=\frac{1}{2}\operatorname{Tr}_{\mathrm{R}}\left([B,\log\rho_q]A\right),\ \forall\ A,B\in\Psi^{\mathbb{N},-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p). \tag{23.12}$$

[5]Again I should have included this earlier, I will!

Proof. When the order of A is less than $-2k$, the trace of $\rho_q^{z/2} \circ A$ is holomorphic in a neighbourhood of $z=0$. Evaluating there, $\rho_q z/2 = 1$ is the identity in the (unstabilized) algebra so indeed $\overline{\mathrm{Tr}}(A) = \mathrm{Tr}(A)$. Thus $\overline{\mathrm{Tr}}$ is an extension of the trace functional.

To compute $\mathrm{Tr}_{\mathrm{R}}([A,B])$ observe that this is, by definition, the residue at $z=0$ of the analytic continuation of

$$\mathrm{Tr}(\rho_q^{z/2} \circ ([A,B]) = \mathrm{Tr}([B,\rho_q^{z/2}]A) - = \mathrm{Tr}([A,\rho_q^{z/2}]B) \tag{23.13}$$

where we have used the trace identity for $\mathrm{Re}\, z >> 0$ and the uniqueness of analytic continuation. Using the decomposition of the product in (23.6) the commutator here can be written as a sum

$$[B,\rho_q^{z/2}] = Q_N(B,\rho_q^{z/2}) - Q_N(\rho_q^{z/2},B) + Q_{(N)}(z) \tag{23.14}$$

where if N is large enough the second term is holomorphic and uniformly of order less than $-2k$ up to $z=0$ after composition with A. Thus, the trace of this term is regular at 0 so does not contribute to the residue; only the first two terms in (23.14) contribute for N large enough. The leading, commutative product, term cancels in the commutator so in every remaining term, $\rho_q^{z/2}$ is differentiated at least once. This produces a factor of z with the trace of the coefficient having at most a simple pole at $z=0$ by the discussion above. Thus there is no pole at $z=0$ and $\mathrm{Tr}_{\mathrm{R}}([A,B]) = 0$ always.

Again if A is of order less than $-2k$ then so is $\rho_q^{z/2} \circ A$ near $z=0$ where it is holomorphic. The reside, and hence the residue trace of A, therefore vanishes. Similarly for general A the difference between $\mathrm{Tr}(\rho \circ A)$ and the integral of the commutative product $\rho_q^{z/2} A$ involves all the terms in $Q_N(\rho_q^{z/2}, A)$ after the constant term and the remainder. For N large enough, the latter can not contribute to the residue at $z=0$. All the other terms involve at least one derivative falling on $\rho_q^{z/2}$ so again cannot contribute to the residue trace of A. Thus (23.11) follows by the definition of the Riesz regularization of the integral.

It remains to prove the trace defect formula (23.12). Following the discussion above, especially (23.13), $\overline{\mathrm{Tr}}([A,B])$ is the regularized value of the analytic continuation of the trace of the product of (23.14) and A. For large N the second term is holomorphic near $z=0$ and of low order so

$$\mathrm{Tr}(Q_{(N)}(z)A) = \mathrm{Tr}(Q_{(N)}(0)A) \text{ is regular near } z=0. \tag{23.15}$$

However, $Q_{(N)}(z)$ is the 'low order part of the comutator $[B,\rho_q^{z/2}]$. At $z=0$ $\rho_q^{z/2} = 1$ is the identity operator so all the leading terms vanish (since they involve differentiation of 1 so this low order part also vanishes, since the whole commutator vanishes. It follows that the right side of (23.15) vanishes at $z=0$. Thus $\overline{\mathrm{Tr}}([A,B])$ is the regularized value at $z=0$ of the analytic continuation of

$$\mathrm{Tr}\left(\left(Q_N(B,\rho_q^{z/2}) - Q_N(\rho_q^{z/2},B) \right) \circ A \right). \tag{23.16}$$

Again, $\rho_b^{z/2}$ is differentiated at least once, producing a factor of z. Thus the analytic continuation is regular at $z=0$. Writing

$$d\rho_q^{z/2} = \left(\frac{z}{2}\frac{d\rho_q}{\rho_q}\right)\rho_q^{z/2} \tag{23.17}$$

it follows that if any subsequent derivative falls on the last factor then this produces an overall factor of z^2 and hence does not contribute to the regularized trace. Thus the effect is the same as if all derivatives acting on the appropriate factor in Q_N in (23.16) fall on $\log \rho_q$. That is, the regularized trace is the same as the regularized value of

$$\frac{z}{2}\operatorname{Tr}\left(([B, \log \rho_q] \cdot \rho_q^{z/2}) \circ A\right). \tag{23.18}$$

Again expanding out the product with A, the low order term is holomorphic – so does not contribute – and any differentiation of $\rho_q^{z/2}$ produces another factor of z so also does not contribute. Thus the value of the integral at $z = 0$ reduces to the residue trace and (23.12) follows. □

The case $k = 1$ is particularly simple, since then we can easily compute the right side of (23.12). The difficulty of this computation is greater when $k > 1$ since the residue trace occurs higher and higher in the Taylor series expansion of the symbol of an element of $\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ as k increases.

Proposition 22. *If* $k = 1$ *the trace defect formula* (23.12) *involves only the* principal symbols *of* A *and* B *:*

$$\overline{\operatorname{Tr}}([A, B]) = c\int_{\mathbb{S}} \operatorname{tr}(\frac{\partial b(\theta)}{\partial \theta} a(\theta))d\theta = -c\int_{\mathbb{S}} \operatorname{tr}(\frac{\partial a(\theta)}{\partial \theta} b(\theta))d\theta, \\ a = \sigma_0(A),\ b = \sigma_0(b),\ A,\ B \in \Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}; \mathbb{R}^p). \tag{23.19}$$