22. Lecture 19: Riesz regularization Friday, 17 October

Today, let me go back and fill in some of the gaps, or perhaps just paper over some of the cracks.

First let me say a little more about symbols. I will probably not go through all of this in the lectures but it may help clarify things a bit to separate the symbol spaces from $\mathbb{R}^n$. (Or it may not, depending on your tendencies!) From our point of view symbols are the same things as 'conormal functions at a boundary'. Suppose we have a compact manifold with boundary M; in the case at hand this is a ball – say $\overline{\mathbb{R}^p}$ or ${}^q\overline{\mathbb{R}^p}$. Such a manifold comes equipped with a space of smooth functions $\mathcal{C}^\infty(M)$ with its Fréchet topology of the uniform norm on derivatives over compact subsets of coordinate neighbourhoods. Since it is a manifold with boundary there is a filtration by ideals which vanish to higher and higher order at the boundary. In particular there is always a boundary defining function $0 \le \rho \in \mathcal{C}^\infty(M)$, $\partial M = \{\rho = 0\}$, $d\rho \ne 0$ on ∂M. Then the boundary ideal is

$$\mathcal{J}(\partial M) = \rho\mathcal{C}^\infty(M) = \{u \in \mathcal{C}^\infty(M); u\big|_{\partial M} = 0\}. \tag{22.1}$$

The successive ideals are the powers, in the sense of finite spans of products

$$\begin{gathered}\mathcal{C}^\infty(M) \supset \mathcal{J}(\partial M) \supset \mathcal{J}^2(\partial M) \supset \cdots \supset \mathcal{J}^k(M),\\ \dot{\mathcal{C}}^\infty(M) = \cap_k \mathcal{J}^k(M).\end{gathered} \tag{22.2}$$

We can also add the negative powers, or 'Laurent functions' at the boundary and then think of

$$\rho^{-k}\mathcal{C}^\infty(M) = \{u \in \mathcal{C}^\infty(M \setminus \partial M); \rho^k u \in \mathcal{C}^\infty(M)\} \tag{22.3}$$

as the classical (perhaps more correctly '1-step-classical') symbols of order k, so elements of $\mathcal{J}^k(\partial M)$ are classical symbols of order $-k$, rather perversely.

Now, we can interpolate with the classical symbols of any complex order by taking complex powers of ρ, $\rho^z = \exp(z\log\rho) \in \mathcal{C}^\infty(M \setminus \partial M)$ and then define

$$\rho^z\mathcal{C}^\infty(M) = \{u \in \mathcal{C}^\infty(M \setminus \partial M); \rho^{-z}u \in \mathcal{C}^\infty(M)\}. \tag{22.4}$$

These are the classical symbols of complex order z. Notice that the only inclusions we have are

$$\rho^{z+k}\mathcal{C}^\infty(M) \subset \rho^z\mathcal{C}^\infty(M)\ \forall\ k \in \mathbb{N}_0. \tag{22.5}$$

The topology on $\rho^z\mathcal{C}^\infty(M)$ is the topology of $\mathcal{C}^\infty(M)$ after division by ρ^z. The common subspace of all these spaces is the 'Schwartz space' of smooth functions vanishing to infinte order at the boundary:-

$$\dot{\mathcal{C}}^\infty(M) \subset \rho^z\mathcal{C}^\infty(M) \text{ is a closed subspace.} \tag{22.6}$$

So there is no hope of it being dense!

To arrange density we introduce the 'symbol (or conormal) spaces with bounds'. Let $\mathcal{V}_{\mathrm{b}}(M)$ be the space of smooth vector fields on M which are tangent to the boundary – so in local coordinates $x = \rho$, $y_1, \dots, y_{n-1}$, $\dim M = n+1$, near the boundary, $\mathcal{V}_{\mathrm{b}}(M)$ is spanned over $\mathcal{C}^\infty$ coefficients by $x\partial_x$, ∂_{y_j}. Then set

$$\mathcal{A}^s(M) = \{u \in \mathcal{C}^\infty(M); V_1 \dots V_k\rho^{-s}u \in L^\infty(M)\ \forall\ V_i \in \mathcal{V}_{\mathrm{b}}(M),\ \forall\ k\}. \tag{22.7}$$

This again is a Fréchet space with supremum norms – $\mathcal{V}_{\mathrm{b}}(M)$ is finitely generated as a module over $\mathcal{C}^\infty(M)$ so there really are only countably many conditions here. In fact

$$\times\rho^t : \mathcal{A}^s(M) \longrightarrow \mathcal{A}^{s+t} \text{ is an isomorphism } \forall\ t, s \in \mathbb{R}. \tag{22.8}$$

Note that $\rho^z \in \mathcal{A}^{\operatorname{Re} z}(M)$ so there is only a real, not a complex, order here and

$$\rho^z\mathcal{C}^\infty(M) \subset \mathcal{A}^s(M),\ \ s \le \operatorname{Re}(z). \tag{22.9}$$

So, now the density result is easy enough. Take a smooth function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ with $\psi = 1$ in $x < \frac{1}{2}$, $\psi = 0$ in $x > 1$ and consider $\psi_\epsilon = (1-\chi)(\rho/\epsilon) \in \mathcal{C}^\infty(M)$ for $\epsilon > 0$. This vanishes for $\rho < \frac{1}{2}\epsilon$ and is eventually equal to 1 on any compact subset of the interior of M as $\epsilon \downarrow 0$. Then

$$u \in \mathcal{A}^s(M) \Longrightarrow \psi_\epsilon u \to u \text{ in } \mathcal{A}^{s'}(M),\ \ s' < s. \tag{22.10}$$

In particular $\dot{\mathcal{C}}^\infty(M)$ is dense in $\rho^z\mathcal{C}^\infty(M)$ in the topology of $\mathcal{A}^s(M)$ for any $s < \operatorname{Re}(z)$.

The algebra of isotropic pseudodifferential operators discussed above is a non-commutative product on the filtration of Fréchet spaces $\mathcal{A}^s({}^q\overline{\mathbb{R}^{2n}})$ for any n that is, it is a consistent associative product

$$\mathcal{A}^s({}^q\overline{\mathbb{R}^{2n}}) \times \mathcal{A}^t({}^q\overline{\mathbb{R}^{2n}}) \longrightarrow \mathcal{A}^{s+t}({}^q\overline{\mathbb{R}^{2n}})\ \forall\ t, s \in \mathbb{R} \tag{22.11}$$

which restricts to define products

$$\rho^z\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2n}}) \times \rho^{z'}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2n}}) \longrightarrow \rho^{z+z'}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2n}})\ \forall\ z, z' \in \mathbb{C}. \tag{22.12}$$

You might ask: How can one characterize $\rho^z\mathcal{C}^\infty(M)$ inside $\mathcal{A}^s(M)$, which contains it for $s < \operatorname{Re}(z)$?

Proposition 20. *For a compact manifold with boundary there exists a vector field $R \in \mathcal{V}_{\mathrm{b}}(M)$ such that $Rf \equiv f$ modulo $\mathcal{J}^2(M)$ for all $f \in \mathcal{J}(M)$ and then for any $s < \operatorname{Re}(z)$,*

$$\rho^z\mathcal{C}^\infty(M) = \{u \in \mathcal{A}^s(M); (R-z)(R-z-1)(R-z-2)\dots(R-z-k)u \in \mathcal{A}^{s+k}(M)\ \forall\ k \in \mathbb{N}_0\} \tag{22.13}$$

Proof. Is not very hard! □

Following the line of thought related to ρ^z for a boundary defining function ρ, I will next consider Riesz-regularized integrals over M – which is a compact manifold with boundary. Suppose $0 < \nu \in \mathcal{C}^\infty(M; \Omega)$ is a smooth density. In case you don't know about densities I will add some exercises so that you can familiarize yourself with them. For the moment just agree that they are objects which in any local coordinates give a smooth (positive) multiple of the Lebesgue measure in coordinates and that under change of coordinates the factor changes by the absolute value of the Jacobian determinant. Alternatively you can assume that M is oriented (which in our case of the balls it is) and that ν is a smooth volume form which is positive, in the sense that it defines the orientation. Either way, this means that

$$\mathcal{C}^\infty(M) \ni u \longmapsto \int_M u\nu \in \mathbb{C} \tag{22.14}$$

is a continuous linear map on the Fréchet space $\mathcal{C}^\infty(M)$. In fact it extends by continuity, and hence unambiguously, from the subspace $\dot{\mathcal{C}}^\infty(M)$ to

$$\int_M \bullet\nu : \mathcal{A}^s(M) \longrightarrow \mathbb{C},\ \forall\ s > -1. \tag{22.15}$$

The limit at $s = -1$ is just the non-integrability of x^{-1} with respect to dx near 0 on the line.

So, how can we extend this functional? Well, the answer really is that one cannot do it on the spaces $\mathcal{A}^s(M)$ for $s \le -1$. However, one can *extend* the integral to

$$\overline{\int_M} \bullet\ \nu : \rho^{-k}(M) \longrightarrow \mathbb{C},\ \ k \in \mathbb{N}. \tag{22.16}$$

Lemma 24. *If $u \in \rho^{-k}(M)$ and $\nu \in \mathcal{C}^\infty(M;\Omega)$ then*

$$F(z, u\nu, \rho) = \int_M \rho^z u\nu \text{ is holomorphic in } \operatorname{Re} z > k-1 \tag{22.17}$$

and has a meromorphic extension to $\mathbb{C}\setminus\{k-\mathbb{N}\}$ with only simple poles at the points $k-\mathbb{N}$. The residue at $z=0$ (if any) is independent of the choice of ρ.

The residue at zero is the 'boundary integral' or 'residue integral' and will be denoted

$$ {}^{\mathrm{R}}\!\!\int u\nu = \lim_{z\to 0} zF(z, u\nu, \rho). \tag{22.18}$$

The regularized value at $z = 0$ is the regularized integral

$$\overline{\int_M} u\nu = \lim_{z\to 0}\big(F(z, u\nu, \rho) - \frac{1}{z}\, {}^{\mathrm{R}}\!\!\int u\nu\big). \tag{22.19}$$

In contrast to the residue integral, this functional does depend on the choice of ρ if $k \ge 1$. Note that these are both functionals on $u\nu \in \rho^{-k}\mathcal{C}^\infty(M;\Omega)$ which are consistent on restriction from $\rho^{-k}\mathcal{C}^\infty(M;\Omega)$ to $\rho^{-k+1}\mathcal{C}^\infty(M;\Omega)$.

Proof. For any k holomorphy of $F(z, u\nu, \rho)$ in $\operatorname{Re} z > k-1$ follows from the absolute convergence of the integral defining it in (22.17), the fact that any one deriviative with respect to $\operatorname{Re} z$ or $\operatorname{Im} z$ is also absolutely convergent, since it only introduces another $|\log\rho|$ of growth, and the holomorphy of the integrand. Now, we can split the integral into a part near the boundary and a part away from the boundary using ρ:-

$$\begin{gathered} F(z, u\nu, \rho) = F'(z, u\nu, \rho) + F''(z, u\nu, \rho),\\ F'(z, u\nu, \rho) = \int_{\rho<\delta} \rho^z u\nu,\ \ F''(z, u\nu, \rho) = \int_{\rho>\delta}\rho^z u\nu \end{gathered} \tag{22.20}$$

where $\delta > 0$ is fixed and is chosen so small that there is a smooth product decomposition, of M, in $\rho \le \delta : \{\rho \le \delta\} \simeq [0,\delta]_\rho \times \partial M$. The term F'' is entire in z, by the same reasoning as above. For the second term we can write the product

$$u\nu = (\sum_{j=0}^{L} \rho^{j-k}u'_j + \rho^{-k+L+1}v_L)d\rho d\nu_{\partial M},\ v_k \in \mathcal{C}^\infty([0,\delta]\times\partial M),\ u_j \in \mathcal{C}^\infty(\partial M) \tag{22.21}$$

for any $L \ge k$. This comes from the product decomposition of the measure $u\nu$, into the product of $d\rho$, a smooth measure, $\nu_{\partial M}$, on ∂M and function $\rho^{-k}u'$ with

$u' \in \mathcal{C}^\infty([0,\delta] \times \partial M)$. Then (22.21) is just the Taylor series expansion up to order L.

The remainder term here makes a contribution to F' of the form

(22.22) $F'_L(z) = \int_0^\delta \rho^{z+1+L-k} v'_k d\rho d\nu_{\partial M}$ which is holomorphic in $\operatorname{Re} z > k-L-2$

by the same reasoning. Thus in this half-space, which of course increases with L,

$$F'(z) = F'_L(z) + \sum_{j=0}^{L} (z+j-k+1)^{-1} \delta^{z+j-k+1} \int_{\partial M} u'_j d\nu_{\partial M}. \tag{22.23}$$

Since δ^z is entire, this shows that F' only has simple poles at the points $z \in k - \mathbb{N}$ in the half-plane $\operatorname{Re} z > k - L$. Thus indeed $F(z, u\nu, \rho)$ is meromorphic as claimed.

So, it remains to show that the residue at $z = 0$ – which of course can only be non-zero if $k \in \mathbb{N}$ – is independent of the choice of ρ. The other residue-functionals are not independent in this way. Any two boundary defining fucntions are positive smooth multiples of each other so a second can be connected to the first by a smooth curve

$$\rho_s = ((1-s)1 + sa)\rho = A(s)\rho,\ 0 < a \in \mathcal{C}^\infty(M),\ 0 < A \in \mathcal{C}^\infty([0,1]_s \times M). \tag{22.24}$$

Inserting this into the definition of F we get

$$F(z, u\nu, \rho_s) = \int_M \rho^z A(s)^z u\nu \tag{22.25}$$

in the half-plane of holomorphy.

The arguments above now apply uniformly in $s \in [0,1]$, with the extra, entire, factor of $A(s)^z$. It follows by differentiation that all residues and the analytic continuation are smooth in s. Now

$$\frac{d}{ds} F(z, uv, \rho_s) = z \int_M \rho^z \frac{dA(s)}{ds} A(s)^{z-1} uv. \tag{22.26}$$

The same argument regarding meromorphy can now be applied to the integral on the right, so it can only have a simple pole at $z = 0$. The extra factor of z ensures that there is no such pole, so the residue of $F(z, uv, \rho_s)$ at $z = 0$ is constant in s and hence indeed independent of the choice of ρ. $\square$