21. Lecture 18: The determinant bundle Wednesday, 15 October

Even though I have not carefully explained everything that goes into the looping sequence let me proceed to use it to define the determinant bundle – and then see what more we need to do. Here I will proceed in a way that is closely parallel to the content of Section 7. There I used the delooping sequence to (re-)construct the determinant on $G^{-\infty}$. Let me recall that construction now tying in some of the things that have come up in the meantime. The delooping sequence is

$$G^{-\infty}_{\text{sus}}(\mathbb{R}^k) \longrightarrow \tilde{G}^{-\infty}_{\text{sus}}(\mathbb{R}^k) \xrightarrow{R} G^{-\infty}(\mathbb{R}^k). \tag{21.1}$$

The clutching map in (18.16) has the property

$$\mathrm{cl_{eo}}^* \left(\frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{tr}(g^{-1}\frac{dg}{dt})dt \right) = \frac{1}{2}\,\mathrm{tr}(I-\gamma_1) = \mathrm{ind}\,. \tag{21.2}$$

To see this, just compute away:-

$$\begin{aligned} &\mathrm{cl_{eo}}^* \left(\int_{\mathbb{R}} \mathrm{tr}(g^{-1}\frac{dg}{dt})dt \right) \\ &= \int_0^\pi \mathrm{tr}((\cos\Theta - i\sin\Theta I)(-\sin\Theta + i\cos\Theta I) \\ &\qquad - (\cos\Theta - i\sin\Theta\gamma_1)(-\sin\Theta + i\cos\Theta\gamma_1)d\Theta \\ &= i\int_0^\pi \mathrm{tr}(I-\gamma_1)d\Theta = (2\pi i)\,\mathrm{ind} \end{aligned} \tag{21.3}$$

where all the non-trace class terms consisently cancel out. Hence we can say that this is the same 'index functional' on $G^{-\infty}_{\text{sus}}(\mathbb{R}^k)$ as is represented by ind on $\mathcal{H}^{-\infty}(\mathbb{R}^k)$. This will be more fully justified by Fedosov's index theorem a bit later.

So we have defined the first vertical map in

$$\begin{array}{ccccc} G^{-\infty}_{\text{sus}}(\mathbb{R}^k) & \longrightarrow & \tilde{G}^{-\infty}_{\text{sus}}(\mathbb{R}^k) & \xrightarrow{R} & G^{-\infty}(\mathbb{R}^k) \\ \downarrow \mathrm{ind} & & \downarrow \tilde{\eta} & & \downarrow \det \\ \mathbb{Z} & \longrightarrow & \mathbb{C} & \xrightarrow{\exp(2\pi i\cdot)} & \mathbb{C}^* \end{array} \tag{21.4}$$

and – we know that it maps into $\mathbb{Z}$. The second vertical map, $\tilde{\eta}$ we defined by *regularization* – or extension – of the index functional. Namely we just used the 'same' definition but now on the 'one-end-open' loops

$$\tilde{\eta}(\tilde{g}) = \frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{tr}(\tilde{g}^{-1}(t)\frac{d\tilde{g}(t)}{dt})dt. \tag{21.5}$$

This still makes sense since $d\tilde{g}(t)/dt \in \mathcal{S}(\mathbb{R};\Psi^{-\infty}(\mathbb{R}^k))$ because of the flatness condition on these half-open loops. Now, we can no longer see that this extended functional takes integral values – indeed it doesn't – but we checked directly that $\exp(2\pi i\tilde{\eta})$ descends to the quotient and there it satisfies all the properties we want of the determinant, and reduces to it under finite rank approximation. In particular one crucial thing is that $\tilde{\eta}$ is a log-character on the central group

$$\tilde{\eta}(\tilde{g}\tilde{h}) = \tilde{\eta}(\tilde{g}) + \tilde{\eta}(\tilde{h}). \tag{21.6}$$

Exercise 16. (For the enterprising) Show that the sequence obtained as the kernels of the maps in (21.4):
(21.7)

$$\{\mathrm{Id}\} \longrightarrow G^{-\infty}_{\mathrm{sus,ind}=0}(\mathbb{R}^k) \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus},\tilde{\eta}=0}(\mathbb{R}^k) \xrightarrow{R} G^{-\infty}_{\mathrm{iso,det}=1}(\mathbb{R}^k) \longrightarrow \{\mathrm{Id}\}$$

is a reduced classifying sequence for K-theory – meaning it is exact, that the central group is weakly contractible and that the outer groups each just have the bottom homotopy group removed.

So, we want to do the 'same thing' but one step up in complexity. Be warned, I am planning to do the next step up too! Now we start with the determinant at the beginning of the looping sequence –

$$(21.8)\qquad \begin{array}{ccccc} \mathbb{C}^* & & \mathbb{C} & & \\ \Big\uparrow{\scriptstyle \det} & & \Big\downarrow & & \\ G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}) & \longrightarrow & \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p) & \xrightarrow{\sigma_{0,\mathrm{iso}\,*}} & G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}(\mathbb{R}^p) \end{array}$$

Here I have added a $\mathbb{C}$ and map to the central group and the final group is supposed to be the image of the 'full' symbol map (which now includes the whole Taylor series at the boundary, hidden in the $\tilde{\sigma}$). The $\mathbb{C}$ is supposed to represent the trivial line bundle, i.e. the top space is really $\mathbb{C} \times \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p)$ but this is a bit messy to write out; $\mathbb{C}$ is of course the fibre so it stands here for the trivial line bundle. The determinant induces a relation on this space

$$(21.9)\quad \mathbb{C} \times \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p) \ni (z,A) \sim_{\det} (z\det(g), Ag) \in \mathbb{C} \times \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p) \quad \text{if } g \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}).$$

This relation is multiplicative, because the determinant is:-
(21.10)

$$(z,A) \sim_{\det} (z\det(g), Ag) \sim_{\det} ((z\det(g))\det(h), (Ag)h) = (z\det(gh), A(gh)).$$

The identifying maps in (21.9) are linear in z so the quotient is a one-dimensional complex vector space for each $\alpha \in G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}(\mathbb{R}^p)$:

$$(21.11)\qquad \mathcal{L}_\alpha = \mathbb{C} \times AG^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p})/\sim_{\det} \ \text{ if } \sigma_{0,\mathrm{iso}\,*}(A) = \alpha \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p})$$

which is independent of the choice of A mapping to α. Note that the exactness of the diagram (21.8) means that the inverse image of any point $\alpha \in G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}$ (accepting for the moment that this *is* the image) is of the form $AG^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p})$ for any particular A in the preimage.

This leads to the full diagram

$$(21.12)\qquad \begin{array}{ccccc} \mathbb{C}^* & & \mathbb{C} & \xrightarrow{\quad\sim_{\det}\quad} & \mathcal{L} \\ \Big\uparrow{\scriptstyle \det} & & \Big\downarrow & & \Big\downarrow \\ G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}) & \longrightarrow & \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p) & \xrightarrow{\sigma_{0,\mathrm{iso}\,*}} & G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}(\mathbb{R}^p) \end{array}$$

We need to be a little careful of the sense in which this is a line bundle, because local triviality isn't so clear. However, except for infinite-dimensional effects this is the construction of the vector bundle associated to a representation (the determinant) of the structure bundle of a principal bundle. To discuss local triviality we need to consider the Fréchet topology on the base, etc. However let me state it as a result before we press on to prove this, and more.

Claim 1. *The determinant function on the structure group in* (21.8) *induces the (locally trivial)* determinant line bundle *over the quotient group.*

I hope the relation of this line bundle to Quillen's original definition from [10] will become abundantly clear as we proceed; for now it may seem rather distant. In fact what we are constructing here is a *universal determinant bundle*. One thing I want to come back to and refine is the following

Claim 2. *The determinant bundle, $\mathcal{L}$, over $G^{-\infty}_{\text{sus}\,*,\text{iso},\text{ind}=0}(\mathbb{R}^p)$ is* universal *for smooth (complex line) bundles over smooth compact manifolds – i.e. any such smooth line bundle is isomorphic to the pull-back of $\mathcal{L}$ under a smooth map into $G^{-\infty}_{\text{sus}\,*,\text{iso},\text{ind}=0}(\mathbb{R}^p)$ (hence defining an even K-class). This much is fairly easy. In fact the same is true for a bundle with connection, it is isomorhic with its connection, to the pull-back of $\mathcal{L}$ with the connection constructed below.*

Note the notion of a *Claim* here is that I believe it to be true but probably do not have a complete proof at hand – so there is always a danger it is not quite right!

Exercise 17. There is a similar construction to this one over the classifying space $\mathcal{H}^{-\infty}(\mathbb{R}^k)$ due I believe to Graeme Segal. This can be found in [9], in the context of trace class operators and groups. In the smooth case we can proceed as follows, and you might like to fill in the details. We have shown that in terms of the action by conjugation on the zero index involutions

$$\mathcal{H}^{-\infty}_{\text{ind}=0}(\mathbb{R}^k) = G^{-\infty}(\mathbb{R}^k;\mathbb{C}^2)/\big(G^{-\infty}(\mathbb{R}^k)\oplus G^{-\infty}(\mathbb{R}^k)\big) \tag{21.13}$$

where the two smaller groups are acting on the first and second components diagonally. So, take the fibre at an involution to be

$$\begin{gathered}\mathcal{L}_I = \big(\mathbb{C}\times\{G\in G^{-\infty}(\mathbb{R}^k;\mathbb{C}^2); I = G^{-1}\gamma_1 G\}\big)\;/\sim_{\det},\\ (z,G)\sim_{\det}(\frac{\det(g_1)}{\det(g_2)}z,(g_1\oplus g_2)G)\;\forall\; g_i\in G^{-\infty}(\mathbb{R}^k),\ i=1,2.\end{gathered} \tag{21.14}$$

Check that this is a line bundle – linearity and local triviality. Note that if we took the product of the determinants instead of the quotient it we would produce a trivial line bundle (since the determinants are consistent with that on the big group). You could even try to see that the pull-back of the determinant line bundle over the index zero component of $G^{-\infty}_{\text{sus},\text{iso}}(\mathbb{R}^k;\mathbb{C}^2)$ under cl_{eo} in (18.16) is isomorphic to the bundle from (21.14) – it is. Report back any success here, since I haven't done this explicitly myself myself (yet). Note that there is a subtlety, since the line bundle above is defined on the $*$ extension of the image group, so you need to do something about that first for this last part. You can see what to do about this, at least in part, from the discussion below.

So, apart from proving the claim above, I want to do a little more – and this is where the $*$ part of the quotient group starts to come into its own. The analogy between the treatment of the determinant via the delooping sequence and the determinant bundle might seem somewhat forced. To make it more apparent that they really are closely related, consider the problem of construction a connection on $\mathcal{L}$ in (21.12). That is, we want to know how to differentiate sections. One can construct a connection using local trivializations, but here we can do it directly because of the quotient construction of the bundle itself. Namely all we need is a connection on the trivial bundle over the 'big group' which descends to a connection on the quotient. Since the trivial bundle is, ahem, trivial, any connection on it is the sum of the trivial connection, d, and a 1-form:

$$\nabla = d - \gamma,\ \gamma \in \mathcal{C}^\infty(\dot{G}^{0,-\infty}_{\text{qiso}}(\mathbb{R};\mathbb{R}^p);\Lambda^1). \tag{21.15}$$

So what do we need for this connection to 'descend' to a connection on $\mathcal{L}$ over $G^{-\infty}_{\text{sus}\,*,\text{iso},\text{ind}=0}(\mathbb{R}^p)$? There are outstanding issues of local triviality etc., but basically we need to know that when applied to the lift of a local section of $\mathcal{L}$ to the trivial bundle over $\dot{G}^{0,-\infty}_{\text{qiso}}(\mathbb{R};\mathbb{R}^p)$ we get the lift of a section. A lifted section, which is just a complex-valued function, must have the transformation law along a fibre of $\dot{G}^{0,-\infty}_{\text{qiso}}(\mathbb{R};\mathbb{R}^p)$ given by

$$e(Ag) = \det(g)e(A)\ \forall\ g \in G^{-\infty}_{\text{iso}}(\mathbb{R}^{1+p}) \tag{21.16}$$

so what we need is that

$$\det(g)^{-1} d\det(g) - \gamma(Ag) = 0 \text{ on } AG^{-\infty}_{\text{iso}}(\mathbb{R}^{1+p}) \tag{21.17}$$

for all A. So, how do we construct such a γ? Basically, (21.17) just says that restricted to a fibre,

$$\gamma = d\log\det = \operatorname{tr}(g^{-1}dg). \tag{21.18}$$

So the 'obvious' thing to do is proceed as we did for $\tilde{\eta}$, somehow regularized the formula on the right to get a 1-form on the central group. This we will do as follows:-

Proposition 19. *The trace functional on $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^p \times \mathbb{R}^k)$ has an extension to a continuous linear functional*

$$\overline{\operatorname{Tr}} : \Psi^{0,-\infty}_{\text{iso}}(\mathbb{R}^p : \mathbb{R}^k) \longrightarrow \mathbb{C} \tag{21.19}$$

by Hadamard regularization of the integral and

$$\gamma = -\overline{\operatorname{Tr}}(\tilde{g}^{-1}d\tilde{g}) \in \mathcal{C}^\infty(G^{0,-\infty}_{\text{qiso}}(\mathbb{R};\mathbb{R}^p);\Lambda^1)) \tag{21.20}$$

gives a connection on the trivial bundle through (21.15) *which descends to $\mathcal{L}$; the curvature of this line bundle is the 2-form part of the Chern character.*

Question 3. What does the line bundle $\mathcal{L}$ represent – why did Quillen call it the determinant line bundle?

Answer 3. The determinant line at any point consists of all the *possible*, or perhaps one should say *reasonable*, values of the determinant for the operator (or object) in question. If the determinant bundle were trivial then it would be possible to give a global definition of the determinant; if not (which is the case on the whole space) then not. One thing I hope to do in the sequel is to find a big subgroup on which the determinant bundle *is* trivial – although at this stage I am still not quite convinced that it exists in a useful form.