20. Lecture 17: Isotropic calculus and looping sequence Monday, 13 October

I am including here more detail (and may add even more later) than I will give in the lecture where I will assume some familiarity with pseudodifferential operators. In fact, in the lecture, I started at (20.27) and tried to explain the nature of the space in the middle that we need to construct – and then talked a little about the isotropic calculus (the case $\epsilon = 1$ of what follows) and the corresponding group.

Earlier, I worked out the product formula for semiclassical families of smoothing operators, in terms of their 'renormalized' kernels, $a_\epsilon(t,t') = \epsilon^{-n}F(\epsilon, \frac{1}{2}\epsilon(t+t'), (t-t')/\epsilon)$ where F is Schwartz on $\mathbb{R}^{2n}$ and smooth in ϵ. From (10.5) the product is

$$H(\epsilon,t,s) = \int_{\mathbb{R}^n} F(\epsilon, t+\frac{\epsilon^2}{2}(r+\frac{1}{2}s), \frac{1}{2}s - r)G(\epsilon, t+\frac{\epsilon^2}{2}(r-\frac{1}{2}s), r+\frac{1}{2}s)dr. \tag{20.1}$$

The 'full symbol', or Weyl form of this product is obtained by taking the Fourier transform in s and using the Fourier inversion formula:

$$\begin{aligned}\hat{H}(\epsilon,t,\tau) = (2\pi)^{-2n}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} &\hat{F}(\epsilon, t+\frac{\epsilon^2}{2}(r+\frac{1}{2}s), \tau_1)e^{i(\frac{1}{2}s-r)\tau_1}\\ &\times \hat{G}(\epsilon, t+\frac{\epsilon^2}{2}(r-\frac{1}{2}s),\tau_2)e^{i(r+\frac{1}{2}s)\tau_2}e^{-i\tau s}drdsd\tau_1 d\tau_2.\end{aligned} \tag{20.2}$$

Introducing $t_1 = t + \frac{\epsilon^2}{2}(r+\frac{1}{2}s)$ and $t_2 = t+\frac{\epsilon^2}{2}(r-\frac{1}{2}s)$ in place of r and s as variables of integration, so $r + \frac{1}{2}s = 2(t_1 - t)/\epsilon^2$ and $r - \frac{1}{2}s = 2(t_2-t)/\epsilon^2$ and $drds = dt_1dt_2$, gives

$$\begin{aligned}\hat{H}(\epsilon,t,\tau) = (2\pi)^{-2n}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} &\hat{F}(\epsilon,t_1,\tau_1)\hat{G}(\epsilon,t_2,\tau_2)\\ \times \exp&\left(\frac{2i}{\epsilon^2}\big(\omega(t_1-t,\tau_1-\tau,t_2-t,\tau_2-\tau)\big)\right)dw_1dw_2,\\ &\omega(t_1,t_2,\tau_1,\tau_2) = t_1\cdot\tau_2 - t_2\cdot\tau_1,\ dw = dtd\tau.\end{aligned} \tag{20.3}$$

Here $\omega : \mathbb{R}^{2n}\times\mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is the standard (antisymmetric) symplectic form on $\mathbb{R}^{2n}$ and dw is the corresponding (Lebesgue) volume form on $\mathbb{R}^{2n}$. In fact the formula makes sense for an arbitrary symplectic vector space, W, i.e. is invariant under the application of the same symplectic transformation in all three copies of $\mathbb{R}^{2n}$. Thus it can be written

$$\begin{aligned}h(\epsilon,w) = M(f,g)(\epsilon,w) = (2\pi)^{-\dim W}\int_W\int_W & f(\epsilon,w_1)g(\epsilon,w_2)\\ \times\exp&\left(\frac{2i}{\epsilon^2}\big(\omega(w_1-w,w_2-w)\big)\right)dw_1dw_2,\\ M : \mathcal{C}^\infty([0,1];\mathcal{S}(W))\times\mathcal{C}^\infty&([0,1];\mathcal{S}(W)) \longrightarrow \mathcal{C}^\infty([0,1];\mathcal{S}(W)).\end{aligned} \tag{20.4}$$

Consider various 'symbol spaces' associated to $\mathbb{R}^p$, and ultimately any vector space. First the Fréchet topologies on 'symbols with bounds' on $\mathbb{R}^p$, namely

$$\|a\|_{m,k} = \sup_{(t,\tau)\in\mathbb{R}^p, |\alpha|\le k} \|(1+|z|)^{-m+|\alpha|}|D_z^\alpha a| \tag{20.5}$$

is a sequence of norms. Denote by $S^m_\infty(\mathbb{R}^p)$ the subspace of $\mathcal{C}^\infty(\mathbb{R}^p)$ on which all these norms are bounded then

(1) For each m, $S^m_\infty(\mathbb{R}^p)$ is a Fréchet space, increasing with m. In particular these are complete metric spaces.
(2) $\mathcal{S}(\mathbb{R}^p)$ is dense in $S^m_\infty(\mathbb{R}^p)$ with respect to the topology of $S^{m'}_\infty(\mathbb{R}^p)$ for any $m' > m$.
(3) Pull back gives an action of $\mathrm{GL}(p,\mathbb{R})$ on these spaces, which therefore make sense on any finite dimensional vector space.
(4) Consider the quadratic compactification ${}^q\overline{\mathbb{R}^p}$ of $\mathbb{R}^p$, with quadratic boundary defining function ρ_q^2 (e.g. $\rho_q^2 = (|z|^2+1)^{-1}$. This is a compact manifold with boundary which is diffeomorphic to a ball and has interior canonically diffeomorphic to $\mathbb{R}^p$,

$$Q : \mathbb{R}^p \longrightarrow {}^q\overline{\mathbb{R}^p}. \tag{20.6}$$

It is defined to have the property

$$Q^*\mathcal{C}^\infty({}^q\overline{\mathbb{R}^p}) = \big\{u \in \mathcal{C}^\infty(\mathbb{R}^p);\ u = \tilde{u}(\frac{1}{|z|^2}, \frac{z}{|z|}) \text{ in } z \neq 0,\ \tilde{u} \in \mathcal{C}^\infty([0,\infty) \times \mathbb{S}^{p-1})\big\}. \tag{20.7}$$

The quadratic compactification is invariant under linear isomorphisms, i.e. the action of $\mathrm{GL}(p,\mathbb{R})$ on $\mathbb{R}^p$ extends to act as diffeomorphisms on ${}^q\overline{\mathbb{R}^p}$.
(5) Similarly the radial compactification, $\overline{\mathbb{R}^p}$, with boundary defining function ρ (e.g. $\rho = \rho_q$) is a compact manifold with boundary, again diffeomorphic to a ball, with compactifying map giving a commutative diagram of smooth maps

$$\begin{tikzcd} \mathbb{R}^p \arrow[rr, "R"] \arrow[dr, "Q"'] & & \overline{\mathbb{R}^p} \arrow[dl, "\beta"] \\ & {}^q\overline{\mathbb{R}^p} & \end{tikzcd} \tag{20.8}$$

with β a parabolic blow-down map for the boundary. The analogue of (20.7) is

$$R^*\mathcal{C}^\infty(\overline{\mathbb{R}^p}) = \{u \in \mathcal{C}^\infty(\mathbb{R}^p);\ u = u'(\frac{1}{|z|}, \frac{z}{|z|}) \text{ in } z \neq 0,\ u' \in \mathcal{C}^\infty([0,\infty) \times \mathbb{S}^{p-1})\}. \tag{20.9}$$

The radial compactification is again invariant under invertible linear transformations and in addition translations on $\mathbb{R}^p$ lift to be smooth on it.
(6) Then (with the pull-back maps suppressed)

$$\rho_q^{-m}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^p}) \subset \rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^p}) \subset S^m(\mathbb{R}^p) \tag{20.10}$$

are linearly invariant.
(7)

Theorem 5. *The bilinear form M defines continuous bilinear maps, consistent under the natural inclusions,*

$$\begin{aligned} M:\mathcal{C}^\infty([0,1];\rho_q^{-m}\mathcal{C}^\infty({}^q\overline{W}))\times\mathcal{C}^\infty([0,1];\rho_q^{-m'}\mathcal{C}^\infty({}^q\overline{W}))\\ \longrightarrow\mathcal{C}^\infty([0,1];\rho_q^{-m-m'}\mathcal{C}^\infty({}^q\overline{W}))\\ M:\mathcal{C}^\infty([0,1];\rho^{-m}\mathcal{C}^\infty(\overline{W}))\times\mathcal{C}^\infty([0,1];\rho^{-m'}\mathcal{C}^\infty(\overline{W}))\\ \longrightarrow\mathcal{C}^\infty([0,1];\rho^{-m-m'}\mathcal{C}^\infty(\overline{W}))\\ M:\mathcal{C}^\infty([0,1];S^m_\infty(W))\times\mathcal{C}^\infty([0,1];S^{m'}_\infty(W))\longrightarrow\mathcal{C}^\infty([0,1];S^{m+m'}_\infty(W)). \end{aligned} \tag{20.11}$$

Note that this 'consistency' is the reason for introducing the spaces S^m. The map in the last line is defined by density from $\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2n})$, when m and m' are both increased by $\epsilon >$. Then the map itself follows by restriction (and of course has to be shown to be continuous). Then the other two maps are by restriction – when f and g are in the appropriate space from (20.10) then so is $M(f,g)$ and it depends continuosly on them in the stronger topology.

Proof. Not included for the moment – ultimately it can be proved by some form of the lemma of stationary phase. Much more is proved in the paper of Hörmander [4]. There are other sources which are maybe a bit more accessible. □

(8) The corresponding associative filtered algebras will be denoted $\Psi^m_{\text{qisy}}(W)$, $\Psi^m_{\text{isy}}(W)$ and $\Psi^m_{\infty\text{-isy}}(W)$. Note that for a symplectic vector space these are not naturally algebras of operators, just algebras. However, in the case of $W=\mathbb{R}^{2n}$ they are all operators on $\mathcal{S}(\mathbb{R}^n)$ and then $\Psi^m_{\text{qiso}}(\mathbb{R}^n)=\Psi^m_{\text{qisy}}(\mathbb{R}^{2n})$ etc. Moreover the action on $\mathcal{S}(\mathbb{R}^n)$ extends and restricts, much as for the product itself to give

$$\begin{gathered} \Psi^m_{\text{qiso}}(\mathbb{R}^n)\times\rho_q^{-k}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^n})\longrightarrow\rho_q^{-k-m}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^n}),\\ \Psi^m_{\text{iso}}(\mathbb{R}^n)\times\rho^{-k}\mathcal{C}^\infty(\overline{\mathbb{R}^n})\longrightarrow\rho^{-k-m}\mathcal{C}^\infty(\overline{\mathbb{R}^n}),\\ \Psi^m_{\infty\text{-iso}}(\mathbb{R}^n)\times S^k_\infty(\mathbb{R}^n)\longrightarrow S^{k+m}_\infty(\mathbb{R}^n). \end{gathered} \tag{20.12}$$

(9) The various algebras of isotropic pseudodifferential operators are what we get by setting $\epsilon=1$ (or up to invertible linear change of variable, any other $\epsilon>0$). The 'classical' space of isotropic pseudodifferential operators have a leading symbol map

$$\sigma_{m,\text{iso}}:\Psi^m_{\text{iso}}(\mathbb{R}^n)\longrightarrow\mathcal{C}^\infty(\mathbb{S}^{2n-1};(d\rho)^{-m}) \tag{20.13}$$

which should be thought of as a section of a certain trivial line bundle over the sphere at infinity – namely the products $\rho^{-m}f$, $f\in\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}}$ modulo the $\rho^{-m+1}f$). This symbol is multiplicative in the obvious sense

$$\sigma_{m+m',\text{iso}}(AB)=\sigma_{m,\text{iso}}(A)\sigma_{m',\text{iso}}(B),\ A\in\Psi^m_{\text{iso}}(\mathbb{R}^n),B\in\Psi^m_{\text{iso}}(\mathbb{R}^n) \tag{20.14}$$

and gives a short exact sequence

$$\Psi^{m-1}_{\text{iso}}(\mathbb{R}^n)\hookrightarrow\Psi^m_{\text{iso}}(\mathbb{R}^n)\xrightarrow{\sigma_{m,\text{iso}}}\mathcal{C}^\infty(\mathbb{S}^{2n-1};(d\rho)^{-m}). \tag{20.15}$$

(10) For the quadratic isotropic algebra we get the same thing, with an improved 'error estimate'

$$\Psi^{m-2}_{\mathrm{qiso}}(\mathbb{R}^n) \hookrightarrow \Psi^m_{\mathrm{qiso}}(\mathbb{R}^n) \xrightarrow{\sigma_{m,\mathrm{qiso}}} \mathcal{C}^\infty(\mathbb{S}^{2n-1}; (d\rho_q)^{-m}). \tag{20.16}$$

For the corresponding algebras on a symplectic vector space the same properties hold, now with $\mathbb{R}^{2n}$ replaced by W and the sphere $\mathbb{S}^{2n-1}$ replace by the 'sphere of W' which is $\mathbb{S}W = (W \setminus 0)/\mathbb{R}^+$, so for instance (20.16) becomes the exact sequence

$$\Psi^{m-2}_{\mathrm{qisy}}(W) \hookrightarrow \Psi^m_{\mathrm{qisy}}(W) \xrightarrow{\sigma_{m,\mathrm{qiso}}} \mathcal{C}^\infty(\mathbb{S}W; (d\rho_q)^{-m}). \tag{20.17}$$

(11) The adjoint (or transpose) is an involution on each of the algebras described above and it follows by duality that $\Psi^m_{\infty\text{-iso}}(\mathbb{R}^n)$ acts on $\mathcal{S}'(\mathbb{R}^n)$.

(12) We are mostly interested below in the algebras of operators of order 0. For these the symbol can be recovered in part by noting that

$$\Psi^0_{\mathrm{iso}}(\mathbb{R}^n) : \mathcal{C}^\infty(\overline{\mathbb{R}^n}) \longrightarrow \mathcal{C}^\infty(\overline{\mathbb{R}^n}) \tag{20.18}$$

where $\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \subset \mathcal{S}'(\mathbb{R}^n)$. Then restriction to the sphere at infinity gives

$$u \in \mathcal{C}^\infty(\overline{\mathbb{R}^n}),\ A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n),\ (Au)\big|_{\mathbb{S}^{n-1}} = \sigma_{\mathrm{iso},0}(A)\big|_{[(\mathbb{R}^n,0)]} u\big|_{\mathbb{S}^{n-1}} \tag{20.19}$$

which allows the symbol on the equatorial sphere, $\tau = 0$, to be recovered. To get the symbol at all other points of the sphere, except at the 'vertical subsphere' $t = 0$, one can take a real quadratic (homogeneous) polynomial q. Then $q(t) - q(t') = (t - t') \cdot L(\frac{t+t'}{2})$ where L is a linear map. There is a mapping property extending (20.18):

$$\Psi^0_{\mathrm{iso}}(\mathbb{R}^n) : e^{iq}\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \longrightarrow e^{iq}\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \tag{20.20}$$

and then as in (20.19)

$$(e^{-iq}Ae^{iq}u)\big|_{\mathbb{S}^{n-1}} = \sigma_{\mathrm{iso},0}(A)\big|_{[(\mathbb{R}^n,L\mathbb{R}^n)]} u\big|_{\mathbb{S}^{n-1}}\ \forall\ u \in \mathcal{C}^\infty(\overline{\mathbb{R}^n}). \tag{20.21}$$

From this one can recover the symbol everywhere on the sphere at infinity by continuity.

(13) The Fourier transform is also an isomorphism on the space of isotropic operators, thus

$$\begin{gathered} A_{\mathcal{F}}\hat{v} = \hat{f} \text{ if } Av = f,\ v \in \mathcal{S}(\mathbb{R}^n),\ A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \\ \Longrightarrow A_{\mathcal{F}} \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n),\ \sigma_{m,\mathrm{iso}}(A_{\mathcal{F}}(t,\tau) = \sigma_{m,\mathrm{iso}}(A)(\tau, -t). \end{gathered} \tag{20.22}$$

(14) For the full semiclassical (and 'classical') there is both a conventional symbol as described above and a semiclassical symbol reduced to the previous case for smoothing operators – I will discuss these more later.

(15) There is yet another generalization of the isotropic algebras that we need to consider. Namely we want to allow them to 'take values in (isotropic) smoothing operators. This is not so hard. I will denote the corresponding algebras of operators in the form $\Psi^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R}^n;\mathbb{R}^p)$. These are smoothing in the last variables. The kernels can be thought of as just Schwartz maps

$$k \in \mathcal{S}(\mathbb{R}^{2p}; \Psi^0_{\mathrm{qiso}}(\mathbb{R}^n)). \tag{20.23}$$

The composition is then given by composing in the isotropic algebra and then in the usual way as smoothing operators

$$k \circ k'(\bullet; z, z') = \int_{\mathbb{R}^p} k(\bullet; z, z'')k'(\bullet; z'', z')dz''. \tag{20.24}$$

Make sure you have a picture of how these isotropic operators, especially the ones of order zero 'work'. For the moment look at (20.24) and take $n = 1$, and for the picture $p = 1$. Then the kernels can be consider as distributions on $\mathbb{R}^2 \times \mathbb{R}^2$ where everything is Schwartz in the last two variables. Recall that we are considering the partial Fourier transform of the Schwartz kernels, so $k = k(t, \tau; z, z')$ where the product is given by (20.4) (or (20.3)) in the t, τ variables with $\epsilon = 1$. So the function k is $\mathcal{C}^\infty$ on the product of two disks and vanishes to infinite order at the boundary of the second (with the z, z' variables).

Picture: Product of two disks.

The operator product takes two such functions and composes then – the composition in the second disk(s) is usual composition of Schwartz-smoothing operators. The composition in the first disk(s) is really the same, but we have taken a partial Fourier transform of everything and then this same product extends to $\mathcal{C}^\infty$ functions up to the boundary. Both parts of the product are non-commutative of course, but at a point approaching the boundary in the first fact the product becomes more and more commutative and, as I will discuss later, the Taylor series at the boundary of the product only depends on the Taylor series of the factors. So the principal symbol – on in t, τ – is a function on the circle with values in the smoothing operators on $\mathbb{R}$ (or just as well $\mathbb{R}^p$) and composes as loops:-

$$\sigma_{0,\mathrm{qiso}}(AB) = \sigma_{0,\mathrm{qiso}}(A)\sigma_{0,\mathrm{qiso}}(B) \quad \text{in } \mathcal{C}^\infty(\mathbb{S}; \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p), \text{ for } A, B \in \Psi^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R}; \mathbb{R}^p). \tag{20.25}$$

So, let me identify the looping, or quantization sequence in terms of these algebras. This involves three groups, two of which we are already familiar with:-

$$G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}) \longrightarrow \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R}; \mathbb{R}^p) \xrightarrow{\sigma_{0,\mathrm{iso}}} G^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^p). \tag{20.26}$$

In this form it is *not quite* exact. What precisely is the central group? It is made up from (20.25). First consider the subalgebra of $\Psi^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R}; \mathbb{R}^p)$ obtained by denanding that the (partially-Fourier-transformed) kernel $k \in \mathcal{C}^\infty({}^q\overline{\mathbb{R}^2} \times \overline{\mathbb{R}^2})$ – which by assumption vanishes to infinite order at the boundary in the second variable – also vanishes to infinite order at one point, $N \in {}^q\overline{\mathbb{R}^2}$, on the boundary of the first disk, say the North Pole (i.e. nothing interesting happens at the North Pole). As I say, this is a subalgebra because of the Taylor-series-locality at the boundary. Then the group $\dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R}; \mathbb{R}^p)$ is the operators (on $\mathcal{S}(\mathbb{R}^{1+p})$ or instance) of the form $\mathrm{Id} + A$ with A of this form and invertible, with inverse of the same form.

The first map in (20.26) is then inclusion. The Schwartz-smoothing operators correspond to those kernels (before and after Fourier transform) which vanish to infinite order at the whole boundary of the first disk as well as the second. The second map is just the principal symbol – given by the restriction to the boundary in the first variable (but not in the second set of variables). The identity appears here either formally, or as it turns out corresponding to the function with is constant in the first variable (if you like 1 from the Fourier transform of a delta function) and actually the identity, i.e. $\delta(z - z')$, in the second variable. Anyway, it is just the identity in the second variables. Thus the image of an element $\mathrm{Id}+A$ of the central group is $\mathrm{Id}+s_{0,\mathrm{iso}}(A)$ which is the identity plus a function on the circle, flat at N, with values in the Schwartz-smoothing operators on $\mathbb{R}^p$ and as such invertible! This gives the sequence (20.26).

Now, why go to all the gymnastics of the flatness at N? Well, otherwise we would not get the loop group out on the right for one thing. More seriously

Theorem 6. *The central group in* (20.26) *is weakly contractible and the range is precisley the component of the identity in the loop group.*

Even when we adjust the targent group the resulting sequence is not exact, but only for a silly reason. Namely we have not taken into account the higher terms in the Taylor series at the boundary. When we do this we get the *looping sequence* which is an exact sequence of groups:-

$$\{\mathrm{Id}\} \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}) \longrightarrow \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p) \xrightarrow{\sigma_{0,\mathrm{iso}}} G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso}}(\mathbb{R}^p) \xrightarrow{\mathrm{ind}} \mathbb{Z} \longrightarrow \{0\}. \tag{20.27}$$

Here $G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso}}(\mathbb{R}^p)$ is a 'star product extension' or formal quantization of the original group $G^{-\infty}_{\mathrm{sus},\mathrm{iso}}(\mathbb{R}^p)$. Namely it consists of formal power series in a formal variable ρ (which can be identified with the defining function for the boundary of ${}^q\overline{\mathbb{R}^2}$) where the leading term is an element of the suspended group:-

$$G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso}}(\mathbb{R}^p) \ni a = \sum_{j\geq 0} \rho^j a_j,\ a_0 \in G^{-\infty}_{\mathrm{sus},\mathrm{iso}}(\mathbb{R}^p),\ a_j \in \Psi^{-\infty}_{\mathrm{sus},\mathrm{iso}}(\mathbb{R}^p),\ j \geq 1 \tag{20.28}$$

and the product is given by differential operators – more about this later! However, it is important to note that invertibility of such a formal power series is just invertibility of the leading term and the lower order terms are just 'affine junk' from a topoligical point of view – they can be deformed away. However, as we shall see, from an analytic viewpoint they turn out to be important.