19. Lecture 16: The Chern character Friday, 10 October

We now have two 'series' of classifying spaces for K-theory. The (loop) groups based on $G^{-\infty}$ and the spaces of involutions $\mathcal{H}^{-\infty}$ and their looped (or suspended) versions. I have previously introduced the Chern forms in the first case. Today I want to introduce the basic Chern forms in the second and discuss their properties – leading to the definition of the Chern character. You might want to recall that $\mathcal{H}^{-\infty}$ is a Fredholm manifold.

Now, the basis 'zeroth' Chern form is the index, the relative dimension invariant used to analyze the components of $\mathcal{H}^{-\infty}$:

$$\operatorname{Ch}_0^{\mathcal{H}} = \operatorname{ind} : \mathcal{H}^{-\infty} \ni I \longmapsto \frac{1}{2}\operatorname{tr}(I - \gamma_1) \in \mathbb{Z}. \tag{19.1}$$

The higher forms do not require 'regularization' – the subtraction of γ_1 – because they involve derivatives, so we can think of (19.1) as a regularized version of $\frac{1}{2}\operatorname{tr}(I)$. Thus, consider the forms on $\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)$

$$\operatorname{Ch}_{2k}^{\mathcal{H}} = 2^{-2k-1}\operatorname{tr}(I(dI)^{2k}) = 2^{-2k-1}\operatorname{tr}(I dI \wedge dI \cdots \wedge dI). \tag{19.2}$$

Recall that $I^2 = \mathrm{Id}$, so $I(dI) = -(dI)I$ shows that I is an additional anticommuting factor. Antisymmetry shows that an odd number of factors of dI would lead to zero, so we only consider the even cases. Notice that $dI = da$, $I = \gamma + a$, is a 1-form valued in the underlying algebra $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k; \mathbb{C}^2)$ so the trace does exist. Moreover it follows directly that

$$d\operatorname{Ch}_{2k}^{\mathcal{H}} = 2^{-2k-1}\operatorname{tr}(dI(dI)^{2k}) = 0 \tag{19.3}$$

since all the terms dI are closed and the involution I and dI anticommute, so

$$\begin{aligned} &(dI)^{2k+1} = I^2(dI)^{2k+1} = -I(dI)^{2k+1}I \\ &\Longrightarrow \operatorname{tr}((dI)^{2k+1}) = -\operatorname{tr}(I(dI)^{2k+1}I) = -\operatorname{tr}(I^2(dI)^{2k+1}) = -\operatorname{tr}((dI)^{2k+1}) \end{aligned} \tag{19.4}$$

using the trace identity.

Why the factors of 2 in (19.2)? Consider what happens when we pull back these forms under a map $f \in \mathcal{C}^\infty_c(X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k))$. Since $\operatorname{Ch}_{2k}^{\mathcal{H}}$ is closed, it follows that under homotopy it changes by an exact form:

Exercise 13. Observe, following the discussion in Lecture 4 that the under for an homotopy f_t the parameter derivative $\frac{d}{dt} f_t^* \operatorname{Ch}^{\mathcal{H}}$ is exact and hence so is the differnece $f_1^* \operatorname{Ch}^{\mathcal{H}} - f_0^* \operatorname{Ch}^{\mathcal{H}}$.

So, in view of Lemma 21 we can assume that f is replaced by $\tilde{f}$ in (18.7). Then,

$$\begin{aligned} &\tilde{f}^* dI = \\ &d(E_+ \otimes (\mathrm{Id} - P^-(x)) + E_- \otimes (P^+(x))) - d(E_- \otimes (\mathrm{Id} - P^+(x)) + E_+ \otimes (P^-(x))) \\ &\qquad = 2\big(E_- \otimes dP^+(x) - E_+ \otimes dP^-(x)\big). \end{aligned} \tag{19.5}$$

Since $E_+E_- = 0$, the big wedge product in (19.2) decomposes into two pieces:

$$\Big(2\big(E_- \otimes dP^+(x) - E_+ \otimes dP^-(x)\big)\Big)^{2k} = 2^{2k}\big(E_-(dP^+)^{2k} - E_+(dP^-)^{2k}\big). \tag{19.6}$$

Now, recall that the differential of a projection satisfies

$$PdP + (dP)P = dP \Longrightarrow PdP = dP(\mathrm{Id} - P),\ (\mathrm{Id} - P)dP = dP(P). \tag{19.7}$$

Thus, inserting the identity as $P^{\pm} + (\operatorname{Id} - P^{\pm})$ into the corresponding term and expanding out we find

$$(19.8)\quad E_-(dP^+)^{2k} = E_-\big(P^+(dP^+)(\operatorname{Id} - P^+)(dP^+)P^+)^k + (\operatorname{Id} - P^+)(dP^+)P^+(dP^+)(\operatorname{Id} - P^+)\big).$$

Inserting this (for both signs) back into (19.2), note that the factor of I swithces half the sings, then the trace identity shows that the two terms in (19.8) give the same contribution. Thus in fact

$$(19.9)\quad \tilde{f}^* \operatorname{Ch}^{\mathcal{H}}_{2k} = \operatorname{tr}\big((P^+(x)(dP^+(x))(dP^+(x))P^+(x))^k - (P^-(x)(dP^-(x))(dP^-(x))P^-(x))^k\big)$$

with the constants cancelling, which is why they were included in the first place.

Exercise 14. Show that if $P(x)$ is a smooth family of projections, valued in $M(N,\mathbb{C})$ for some N then the curvature of the connection defined on the bundle $E_P = \operatorname{Ran}(P)$ by $\nabla u = P(x)(du)$ for any section, is precisely $P(x)(dP(x))(dP(x))P(x)$.

Thus in fact, (19.9) shows that $\operatorname{Ch}^{\mathcal{H}}_{2k}$ represents the difference of the trace of the kth powers of the curvature of the two bundles. The standard definition of the Chern character is then

$$(19.10)\qquad \operatorname{Ch}^{\mathcal{H}} = \sum_{k=0}^{\infty} \frac{1}{(2\pi i)^k k!} \operatorname{Ch}^{\mathcal{H}}_{2k} = \frac{1}{2} \operatorname{tr}\big(\exp(\frac{I(dI)^2 I}{4\pi i})\big).$$

The $2\pi i$'s are included to make the first Chern class, here $\operatorname{Ch}^{\mathcal{H}}_2$, integral – this is the usual normalization for the curvature of a line bundle. It is sometimes omitted, but will then crop up somewhere else. This normalization corresponds to the multiplicativity.

For pairs of vector bundles, or 'superbundles', $V_+ \ominus V_-$, on a compact manifold the super tensor product is

$$(19.11)\quad (V_+^{(1)} \ominus V_-^{(1)}) \otimes (V_+^{(2)} \ominus V_-^{(2)}) = \Big((V_+^{(1)} \otimes V_+^{(2)}) \oplus (V_-^{(1)} \otimes V_-^{(2)})\Big) \ominus \Big((V_+^{(1)} \otimes V_1^{(2)}) \oplus (V_-^{(1)} \otimes V_+^{(2)})\Big).$$

Proposition 18. *The universal Chern character (in deRham cohomology) pulls back to define a homomorphism of Abelian groups*

$$(19.12)\qquad \operatorname{Ch}^{\mathcal{H}} : \mathrm{K}^0_c(X) \longrightarrow \mathrm{H}^{even}_c(X)$$

which is multiplicative under the super tensor product

$$(19.13)\qquad \operatorname{Ch}^{\mathcal{H}}([f][g]) = \operatorname{Ch}^{\mathcal{H}}([f]) \wedge \operatorname{Ch}^{\mathcal{H}}([g]).$$

Proof. Compute after arranging that the projections all commute. □

We also want to understand the behaviour of the Chern character under the clutching and periodicity maps. To do the latter, for instance to see what happens under pull-back for (18.17) we need first consider the 'suspended' versions of the

Chern character. These are just obtained as case of the loop groups of $G^{-\infty}$ using the evaluation maps

$$\begin{gathered}\mathrm{ev}_p : \mathbb{R}^p \times \mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^k) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k),\\ \mathrm{Ch}^{\mathcal{H}(sus(p))}_{2k-p} = \int_{\mathbb{R}^p} \mathrm{ev}_p^* \mathrm{Ch}^{\mathcal{H}}_{2k},\ \mathrm{Ch}^{\mathcal{H}(\mathrm{sus}(p))} = \int_{\mathbb{R}^p} \mathrm{ev}_p^* \mathrm{Ch}^{\mathcal{H}},\end{gathered} \tag{19.14}$$

giving a $2k-p$ form, or sum of such, on $\mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^k)$. The pull-back of this form under a smooth map into $\mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^k)$ is the same as interpreting this as a map from $\mathbb{R}^p \times X$ into $\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)$, pulling back $\mathrm{Ch}^{\mathcal{H}}_{2k}$ and then pushing forward to X, i.e. integrating over $\mathbb{R}^p$.

Thus, under the map (18.17) we can pull back the whole once-suspended Chern character to $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)$.

Lemma 23. *The pull back of the once-suspended Chern character*

$$\begin{aligned}\mathrm{cl_{oe}}^* \mathrm{Ch}^{\mathcal{H}(\mathrm{sus})} &= \sum_k \frac{1}{2^{2k+2}\pi i k!} \int_{\mathbb{R}} \mathrm{tr}(I(dI(t))^{2k})\\ &= \frac{1}{2\pi i}\int_0^1 (g^{-1}dg)\exp\big(s(1-s)\frac{(g^{-1}dg)^2}{2\pi i}\big)ds.\end{aligned} \tag{19.15}$$

Proof. From (18.17),

$$dI(t) = \begin{pmatrix} -\sin(\Theta(t)) & \cos(\Theta(t))g\\ \cos(\Theta(t))g^{-1} & \sin(\Theta(t))\end{pmatrix}\Theta'(t)dt + \begin{pmatrix} 0 & \sin(\Theta(t))dg\\ -\sin(\Theta(t))g^{-1}(dg)g^{-1} & 0\end{pmatrix}, t \le 0, \tag{19.16}$$

where we can ignore the part in $t > 0$ since it is only a function of t. Proceeding term by term, only one factor of dt can occur so there are $2k$ terms, depending on which factor of dt is selected. If dt is taken from the p slot then then there are $p-1$ factor of the second term in (19.16) before it and $2k-p-1$ after. These anticommute with I and using the trace identity and antisymmetry it follows that these terms are all the same. Thus, after taking the trace, we are reduced to computing

$$\frac{2k}{2^{2k+2}\pi i k!}\int_{-\infty}^0 \mathrm{tr}\left(I(t)\frac{dI}{dt}\Theta'(t)dt \begin{pmatrix} 0 & -\sin^{2k-1}(\Theta(t))(dg(g^{-1})^{2k-2}dg\\ \sin^{2k-1}(\Theta(t))g^{-1}(g^{-1}dg)^{2k-1}g^{-1} & 0\end{pmatrix}\right) \tag{19.17}$$

Computing directly,

$$I(t)\frac{dI}{dt} = \begin{pmatrix} 0 & -g\\ g^{-1} & 0\end{pmatrix} \tag{19.18}$$

so again the two terms from (19.17) are equal and reduce to

$$\begin{aligned}-\frac{1}{2^{2k+2}\pi i(k-1)!}&\int_0^\pi \sin^{2k-1}(\Theta)d\Theta\,\mathrm{tr}((g^{-1}dg)^{2k-1}\\ &= \frac{1}{2^{2k+2}\pi i(k-1)!}\int_0^1 (1-t^2)^{k-1}dt\,\mathrm{tr}((g^{-1}dg)^{2k-1}\end{aligned} \tag{19.19}$$

Thus, the pull-back of the Chern character can be written

$$\int_0^1 (\frac{g^{-1}dg}{4\pi i}) \exp\big((1-t^2)(\frac{g^{-1}dg}{4\pi i})^2\big)dt. \tag{19.20}$$

Here the variable of integration has been changed to $\cos t$ and the integral divided into two equal parts, by symmetry. It is more conventional, to replace t by say $s = (1-t)/2$ and hence arrive at (19.15). □

To compute the pull-back of the odd Chern character under the clutching map (18.16) we can start by noting that the image $g(t) = \mathrm{cl}_{\mathrm{eo}}(I)$ is the product of two invertibles, $g(t) = \mathcal{U}_0(t)\mathcal{U}(t)$ where $\mathcal{U}_0$ does not depend on I – it is just there to make the leading part the identity. Moreover

$$\begin{gathered} g^{-1}(t)dg(t) = i\mathcal{U}^{-1}\sin(\Theta(t))dI + \mathcal{U}^{-1}\mathcal{U}_0^{-1}\frac{d\mathcal{U}_0}{dt}\mathcal{U} + \mathcal{U}^{-1}\frac{d\mathcal{U}}{dt} \\ i\mathcal{U}^{-1}\sin(\Theta(t))dI + i\mathcal{U}^{-1}\big(\cos(2\Theta(t)) - i\sin(2\Theta(t))\gamma_1\big)(I-\gamma)a\Theta'(t)dt. \end{gathered} \tag{19.21}$$

Expanding out $\mathrm{tr}((g^{-1}dg)^{2k+1})$ one again gets a contribution of one dt from each factor and by virtue of the trace identity these are all the same. Thus

$$\begin{aligned} &\mathrm{tr}((g^{-1}dg)^{2k+1}) \\ &= (2k+1)i^{2k}\sin^{2k}(\Theta(t))\,\mathrm{tr}\left(\big(\mathcal{U}^{-1}\mathcal{U}_0^{-1}\frac{d\mathcal{U}_0}{dt}\mathcal{U} + \mathcal{U}^{-1}\frac{d\mathcal{U}}{dt}\big)dt(\mathcal{U}^{-1}dI)^{2k}\right). \end{aligned} \tag{19.22}$$

Consider the last $2k$-fold wedge product as the k-fold product of

$$\mathcal{U}^{-1}dI \wedge \mathcal{U}^{-1}dI = dI \wedge dI \tag{19.23}$$

since $\mathcal{U}^{-1}dI = (dI)\mathcal{U}$. Using the trace identity to move the first factor of $\mathcal{U}^{-1}$ and the same identity again we arrive at:-

(19.24)

$$\begin{aligned} \mathrm{tr}((g^{-1}dg)^{2k+1}) &= (2k+1)i^{2k}\sin^{2k}(\Theta(t))\,\mathrm{tr}\left(\big(\mathcal{U}_0^{-1}\frac{d\mathcal{U}_0}{dt} + \mathcal{U}^{-1}\frac{d\mathcal{U}}{dt}\big)dt(dI)^{2k}\right) \\ &= -2(2k+1)i^{2k+1}\sin^{2k}(\Theta(t))\cos(\Theta(t))\Theta'(t)dt\,\mathrm{tr}\big(I(dI)^{2k}\big) + d_I\mathcal{T}_{2k}(I). \end{aligned}$$

(I think!) Here the exact form comes from the terms with no 'I' factor. It looks as though I have not made a good choice of normalization here, as regards the i's for a start.

Exercise 15. So, it remains to work out the constants here after integration, i.e.

$$C_{2k} = -2(2k+1)i^{2k+1}\int_0^\pi \sin^{2k}(\Theta)\cos(\Theta)d\Theta \tag{19.25}$$

and to insert these into the formula for the odd Chern character to see whether we do indeed recover the even Chern character!

Then look at the diagrams (18.18) and (18.19) and conclude what happens to the Chern character under the periodicity maps.