1. Lecture 1: The smoothing group Wednesday, 27 August, 2008

My first goal is to introduce the infinite-dimensional but 'smooth' group, $G^{-\infty}$. This is a classifying space for odd K-theory and is central to the content of these lectures. This basic classifying group comes in many different manifestations, more or less geometric. I will first start with some words of orientation, then I discuss the 'sequential', really least geometric, version of the underlying 'Schwartz' algebra and then the group. In subsequent lectures much smoother-looking geometric versions of the algebra and group will appear, associated with (finite-dimensional) manifolds.

Complex K-theory, which is what will be discussed for the most part here, is closely connected with the algebras of $N \times N$ complex matrices $M(N, \mathbb{C})$ and more particularly the group of invertible matrices, $\mathrm{GL}(N, \mathbb{C})$ and the subgroup of unitary matrices $\mathrm{U}(N) \subset \mathrm{GL}(N, \mathbb{C})$.

I leave you to remind yourself of the basic properties of matrices, multiplication, determinant, invertibility, polar decomposition, retraction onto $\mathrm{U}(N)$ etc.

Now, the odd K-group of say a compact manifold $\mathrm{K}^1(X)$ can be defined in terms of all the smooth (or continuous) maps from X into $\mathrm{GL}(N, \mathbb{C})$ (or into $\mathrm{U}(N, \mathbb{C})$. One 'difficulty' inherent in this finite-dimensional approach to K-theory is that one needs to stabilize everything. That is, one has to consider the embedding of $\mathrm{GL}(N, \mathbb{C})$ in $\mathrm{GL}(N+1, \mathbb{C})$

$$\mathrm{GL}(N, \mathbb{C}) \ni A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in \mathrm{GL}(N+1, \mathbb{C}). \tag{1.1}$$

Of course, this can be iterated to get $\mathrm{GL}(N, \mathbb{C}) \longmapsto \mathrm{GL}(N+M, \mathbb{C})$ by interpreting 1 as $\mathrm{Id} \in \mathrm{GL}(M, \mathbb{C})$. The need for these stabilization maps tends to make things intrisically non-smooth. Instead, the group $G^{-\infty}$ is an *a priori* stabilization which I want to take the time to discuss carefully – since it is so fundamental to what will follow.

So to the sequential version, which is a direct generalization of matrices but which I will not use directly later – although it is isomorphic to the more geometric versions that I will use, as we shall see. As basic space consider $\Psi^{-\infty}(\mathbb{N})$ where $\mathbb{N} = \{1, 2, 3, \dots\}$, which for the moment just means rapidly decreasing sequences

$$a : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{C}, \ \sup_{i,j} i^N j^N |a_{ij}| < \infty, \ \forall \ N, \tag{1.2}$$

where the map is written as a double sequence.

This is a rather standard Fréchet space – let me remind you about this. First, it is countably normed as is clear from the definition

$$\|a_{\bullet\bullet}\|_{(N)} = \sup_{i,j} i^N j^N |a_{ij}| \text{ are norms.} \tag{1.3}$$

Thus a subset is open if it contains an open ball around each of its points, with respect to one of the norms (depending on the point). The additional requirement for a Fréchet space is completeness. Indeed, $\Psi^{-\infty}(\mathbb{N})$ is a complete metric space with respect to the metric

$$d(a, b) = \sum_N 2^{-N} \frac{\|a-b\|_{(N)}}{1 + \|a-b\|_{(N)}}. \tag{1.4}$$

If you haven't seen this it is worth doing as an exercise – at least check that this is a metric and that the open sets with respect to it are the same as stated above.

In fact $\Psi^{-\infty}(\mathbb{N})$ is a *Montel space*, as are many related spaces. Namely it has the Heine-Borel property that every closed bounded set is compact. This is equivalent to the condition that any sequence which is bounded with respect to each of the seminorms is convergent – hence convergent with respect to eavh of the seminorms. This is straightforward to check, namely the boundedness with respect to $\|\cdot\|_{(N+1)}$ implies that the 'tails' of the sequence with respect to the $\|\cdot\|_{(N)}$ norm are equi-small and hence that it has a sequence which converges with respect to $\|\cdot\|_{(N)}$. A sequence which converges with respect to all the seminorms can then be found by diagonalization.

Of course the important point is that the standard matrix product extends to this space, so that there is a bilinear map

$$\Psi^{-\infty}(\mathbb{N})\times\Psi^{-\infty}(\mathbb{N})\longrightarrow\Psi^{-\infty}(\mathbb{N}),\ (a\circ b)_{ij}=\sum_{l=1}^{\infty}a_{il}b_{lj}\text{ is jointly continuous.}\tag{1.5}$$

The joint continuity of such a bilinear map reduces to estimates, for each N there exists N' and $C=C(N)$ such that

$$\|a\circ b\|_N\le C\|a\|_{N'}\|b\|_{N'}.\tag{1.6}$$

In fact it is enough to check this for large N since the norms increase with N. So take $N\ge 1$. Then the definitions of the norms imply that

$$\begin{aligned}(1.7)\quad |a_{il}|\le i^{-N}l^{-1}\|a\|_{(N)}\Longrightarrow&\\ |\sum_l a_{il}b_{lj}|&\le i^Nj^N\sum_l|a_{il}||b_{lj}|\\ &\le\|a\|_{(N)}\|b\|_{(N)}\sum_l l^{-2}\le C\|a\|_{(N)}\|b\|_{(N)}.\end{aligned}$$

Thus

$$\|ab\|_{(N)}\le C\|a\|_{(N)}\|b\|_{(N)}\ \forall\ N\ge 1.\tag{1.8}$$

Thus, $\Psi^{-\infty}(\mathbb{N})$ is a topological algebra.

As is well-known, the fact that the norms on the right in (1.6) are the same as the norm on the left is especially helpful – although it is not necessary for continuity. It has an important consequence for the unital extension of the algebra. That is, let us formally add an identity – since I haven't made these matrices act on anything yet, this is a formal identity. It really means changing the product on $\Psi^{-\infty}(\mathbb{N})$ so that it looks like $\operatorname{Id}+\Psi^{-\infty}(\mathbb{N})$ using the natural identity

$$(\operatorname{Id}+a)(\operatorname{Id}+b)=\operatorname{Id}+ab+a+b.\tag{1.9}$$

Clearly then it makes sense to ask that $\operatorname{Id}+a$ be invertible in the sense that

$$\exists\ b\in\Psi^{-\infty}(\mathbb{N})\text{ such that }(\operatorname{Id}+a)(\operatorname{Id}+b)=\operatorname{Id},\text{ i.e. }ab+a+b=0.\tag{1.10}$$

Definition 1. The group $G^{-\infty}(\mathbb{N})\subset\Psi^{-\infty}(\mathbb{N})$ consists of those elements a for which $\operatorname{Id}+a$ is invertible in the sense of (1.10).

Next time I shall prove at least part of the following

Proposition 1. *The group $G^{-\infty}(\mathbb{N})$ is an open dense subset of $\Psi^{-\infty}(\mathbb{N})$ in which the product and the map $a\longmapsto b=(\operatorname{Id}+a)^{-1}-\operatorname{Id}$ are continuous. There is an entire analytic function*

$$\Psi^{-\infty}(\mathbb{N})\ni a\longmapsto\det_{Fr}(a)=\det(\operatorname{Id}+a)\in\mathbb{C}\tag{1.11}$$

for which $G^{-\infty}$ is the complement of the null space.

The definition and properties of the Fredholm determinant will have to wait a little longer. Of course the main thing to observe here is how close this is to the finite dimensional case of $\mathrm{GL}(N,\mathbb{C}) \subset M(N,\mathbb{C})$ – the main difference is that in the finite-imensional case the algebra is unital.

The condition on a (non-unital) Fréchet algebra that the group of invertibles, in this case $G^{-\infty}(\mathbb{N})$, is an open subset of the Fréchet algebra, here $\Psi^{-\infty}(\mathbb{N})$, and in particular that $\mathrm{Id} + a$ is invertible for the elements of a small open ball around the origin, is often expressed by saying it is a 'good' Fréchet algebra. This seems so lame to me that I refuse to follow such usage. If necessary I will refer to this condition by saying the Fréchet algebra is 'Neumann-Fréchet' since it is at least the analogue of the convergence of the Neumann series for $(\mathrm{Id} + a)^{-1}$ when a is small (and generally can be proved precisely this way).