18. Lecture 15: Vector bundles and $\mathrm{K}^0_{\mathrm{c}}(X)$ Wednesday, 8 October

We start with an involution which is a finite rank perturbation of γ_1, $\gamma_1 + a$, $\Pi_k a = a\Pi_k = a$. Thus, restricting to $\mathbb{C}^2 \otimes \Pi_k$ which we can identify with any other $2k$-dimensional vector space we have an involution

$$I = I_+ - I_- \text{ acting on } \mathbb{C}^2 \otimes \operatorname{Ran}(\Pi_k) \equiv \mathbb{C}^{2k}. \tag{18.1}$$

Then consider a further slice $\mathbb{C}^2 \otimes (\Pi_{3k} - \Pi_k)$. Here we can identify $\operatorname{Ran}(\Pi_{3k} - \Pi_k)$ with $\mathbb{C}^{2k}$ and so write the restriction of $\gamma_1 \otimes \operatorname{Id}$ as

$$\gamma_1 \otimes (I_+ + I_-). \tag{18.2}$$

So the part of the involution in $\mathbb{C}^2 \otimes \operatorname{Ran}(\Pi_{3k})$ is

$$\begin{gathered} (I_+(x) - I_-(x)) \oplus E_+ \otimes (I_+ + I_-) \oplus -E_- \otimes (\Pi_{3k} - \Pi_k), \\ E_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{gathered} \tag{18.3}$$

Now, the I_- part of the first block can be rotated rotated with the I_- part of the second block and thus there is an homotopy leading from (18.2) to

$$\begin{aligned} &(I_+(x) + I_-(x)) \oplus E_+ \otimes (I_+ - I_-) \oplus -E_- \otimes (\Pi_{3k} - \Pi_k) \\ &= (E_+ + E_-) \oplus \Pi_k + E_+ \otimes (I_+ - I_-) \oplus -E_- \otimes (\Pi_{3k} - \Pi_k). \end{aligned} \tag{18.4}$$

This computation proves:-

Lemma 21. *Any $f \in \mathcal{C}^\infty_c(X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^d)$ is homotopic through such maps to one of the form*

$$\tilde f(x) = \gamma_1 \otimes (\operatorname{Id} - \Pi_{3k}) + \begin{pmatrix} I(x) & 0 \\ 0 & -\operatorname{Id} \end{pmatrix} \otimes (\Pi_{3k} - \Pi_k) + \begin{pmatrix} \operatorname{Id} & 0 \\ 0 & \operatorname{Id} \end{pmatrix} \otimes \Pi_k \tag{18.5}$$

where $I(x)$ is a smooth family of involutions acting on the $2k$-dimensional space which is the range of $\Pi_{3k} - \Pi_k$.

In consequence $\tilde f$ commutes with γ_1 and has positive and negative projections of the form

$$\begin{gathered} \tilde f_+(x) = E_+ \otimes (\operatorname{Id} - \Pi_{3k} + I_+ + \Pi_k) + E_- \otimes \Pi_k \\ \tilde f_-(x) = E_- \otimes (\operatorname{Id} - \Pi_k) + E_+ \otimes I_-, \end{gathered} \tag{18.6}$$

which therefore commute (with each other of course and) with E_+ and E_-. One really might as well write this in the more symmetric form

$$\begin{gathered} \tilde f_+(x) = E_+ \otimes (\operatorname{Id} - P^-(x)) + E_- \otimes (P^+(x)), \\ \tilde f_-(x) = E_- \otimes (\operatorname{Id} - P^+(x)) + E_+ \otimes (P^-(x)), \\ \Pi_l P^\pm = P^\pm \Pi_l = (P^\pm)^2 \text{ and } P^+P^- = P^-P^+ = 0 \end{gathered} \tag{18.7}$$

where $l = 3k$. Then (18.6) shows that we can take $P^+ = \Pi_k$, $k \le l$; by considering $-I$ it follows similarly that one can arrange by homotopy that $P^- = \Pi_k$ instead. Note that it follows from (18.7) that

$$\begin{gathered} \operatorname{ind}(\tilde f(x)) = \\ \frac{1}{2} \operatorname{tr}\big((E_- + E_+) \otimes P^+(x) - (E_+ + E_-) \otimes P^-(x)\big) = \operatorname{rank}(P^+) - \operatorname{rank}(P^-). \end{gathered} \tag{18.8}$$

This gives us the basic relationship between vector bundles and smooth families of involutions, namely $P^+ \ominus P^-$ is a 'superbundle' – the formal difference of two bundles – which also determines the element of $\mathrm{K}^0_{\mathrm{c}}(X)$ fixed by $\tilde{f}$.

Said a different way, the space $\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)$ of involutions itself has an involution acting on it, namely

$$\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k) \ni \gamma_1 + a \longmapsto \gamma_1(\gamma_1 + a)\gamma_1 \in \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k). \tag{18.9}$$

This is however 'trivial' as far as homotopy is concerned. Namely

Lemma 22. *Any map $f \in \mathcal{C}^\infty_c(X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k))$ is homotopic to some*

$$\begin{gathered}\tilde{f} \in \mathcal{C}^\infty_c(X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)) \text{ satisfying} \\ \gamma_1 \tilde{f}(x) = \tilde{f}(x)\gamma_1 \text{ and } a = \Pi_k a = a\Pi_k\end{gathered} \tag{18.10}$$

for some k.

Proof by Jesse and Paul, not proofread yet. First suppose that

$$f_i : X \to \mathcal{H}^{-\infty}_{iso}\left(\mathbb{R}^d\right),\ i = 0, 1,$$

are maps with $f_1 \sim f_2$. Then there is a map

$$\begin{aligned} F : [0,1] \times X &\to \mathcal{H}^{-\infty}_{iso}\left(\mathbb{R}^d\right) \\ F(0,x) &= f_0(x) \\ F(1,x) &= f_1(x) \end{aligned}$$

f By the above lemma, there is a homotopy from F to a map $\widetilde{F}$ so that $\widetilde{F}$ has a decomposition

$$\widetilde{F}(t,x) = E_+ \otimes (Id - 2P_-(t,x)) - E_- \otimes (Id - 2P_+(t,x)),$$

and that furthermore P_- can be chosen so that $P_- \equiv \pi_k$ for some big k, so in particular $P_-(0,x) = P_-(1,x)$. It follows that the $P_+(t,\cdot)$ define isomorphic bundles for all t by an open and closed argument (openness is always true, and the closed part follows from the constancy of the rank.)

For the converse, suppose we have an equivalence of bundles

$$P^0_- \oplus P^1_+ \oplus S = P^1_- \oplus P^0_+ \oplus S = \mathbb{C}^l, \tag{18.11}$$

over a space X. Then we choose an identification of $\mathbb{C}^l$ with a subspace of $\mathcal{S}(\mathbb{R}^d)$ so that π_l is projection thereon, and define

$$f^i = E_+ \otimes \left(Id - 2P^i_-\right) - E_- \otimes \left(Id - 2P^i_+\right),$$

for $i = 0, 1$. The lemma then follows by using (18.11) and rotating blocks as follows.

$$\begin{aligned} f^0 &= E_+ \otimes \left(Id - 2P^0_-\right) - E_- \otimes \left(Id - 2P^0_+\right) \\ &= E_+ \otimes \left((Id - 2P^0_-)\pi_l\right) - E_- \otimes \left((Id - 2P^0_+)\pi_l\right) \\ &\quad + E_+ \otimes \left((Id - 2P^0_-)(Id - \pi_l)\right) - E_- \otimes \left((Id - 2P^0_+)(Id - \pi_l)\right), \end{aligned}$$

so just deal with the middle line, so that we only consider $f^0\left(E_+ \otimes \pi_k + E_- \otimes \pi_k\right)$, which is

$$\begin{aligned} &= E_+ \otimes \left(Id_{\mathbb{C}^l} - 2P^0_-\right) - E_- \otimes \left(id_{\mathbb{C}^l} - 2P^0_+\right) \\ &= E_+ \otimes \left(P^1_+ + S - P^0_-\right) - E_- \otimes \left(P^1_- + S - P^0_+\right) \end{aligned}$$

Everything here is in blocks, so you can rotate the two S's into one another, which switches their signs. This and another substitution gives

$$\begin{aligned}
&= E_+ \otimes \left(P_+^1 - S - P_-^0\right) - E_- \otimes \left(P_-^1 - S - P_+^0\right)\\
&= E_+ \otimes \left(2P_+^1 - Id_{\mathbb{C}^l}\right) - E_- \otimes \left(2P_-^1 - Id_{\mathbb{C}^l}\right)\\
&= E_+ \otimes \left(Id_{\mathbb{C}^l} - 2P_-^1\right) - E_- \otimes \left(Id_{\mathbb{C}^l} - 2P_+^1\right)
\end{aligned}$$

Adding this back to the part we ignored gives the homotopy we wanted. □

This is a direct consequence of Lemma 21.

Proposition 16. *Any map $\tilde{f} \in \mathcal{C}_c^\infty(X; \mathcal{H}_{\mathrm{iso}}^{-\infty}(\mathbb{R}^k))$ satisfying* (18.10) *is of the form* (18.7) *and two such maps $\tilde{f}_i$ are homotopy if and only if there is a vector bundle S over X which is identified with $\mathbb{C}^p$ outside a compact set and a bundle isomorphism*

$$\mathrm{Ran}(P_1^+) \oplus \mathrm{Ran}(P_2^-) \oplus S \longrightarrow \mathrm{Ran}(P_2^+) \oplus \mathrm{Ran}(P_1^-) \oplus S \tag{18.12}$$

which is the natural identification outside a compact set; here the ranges of the projections are considered as vector bundles over X.

Proof. □

The adiabatic Bott element constructed ealier

$$B = \gamma_1 \otimes \mathrm{Id} + D,\ D \in \Psi_{\mathrm{sl,iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2) \tag{18.13}$$

is an involution, $B^2 = \mathrm{Id}$, and satisfies

$$\begin{aligned}
\sigma_{\mathrm{sl}}(B) &= \gamma_1 \otimes \mathrm{Id} + \delta(t,\tau) = b(t,\tau)\\
R(B) &= \gamma_1 \otimes (\mathrm{Id} - \Pi_1) + \mathrm{Id} \otimes \Pi_1 \in M(2,\mathbb{C}) + \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2)
\end{aligned} \tag{18.14}$$

which is (18.7) with $P^- = 0$, $P^+ = \Pi_1$, $l = 1$.

Completion of proof of Proposition 15. To prove the even semiclassical lifting property we can take an element in the form (18.7). Consider

$$\begin{aligned}
\tilde{B} = \gamma_1 \otimes (\mathrm{Id} - P^+(x) - P^-(x)) + B \otimes P^+(x) - B \otimes P^-(x)\\
\in \mathcal{C}_c^\infty(X; M(2,\mathbb{C}) + \Psi_{\mathrm{ad,iso}}^{-\infty}(\mathbb{R} : \mathbb{R}^k).
\end{aligned} \tag{18.15}$$

I think this quantizes to the right thing and so proves the surjectivity of R in (17.15). Injectivity follows using Atiyah's rotation again. □

Now, let me consider the clutching constructions. Perhaps I will take the time to do this carefully, for the moment I have just written these down and am hoping for the best!

First, from even to odd. There is an actual map

$$\begin{aligned}
\mathrm{cl}_{\mathrm{eo}} : \mathcal{H}_{\mathrm{iso}}^{-\infty}(\mathbb{R}^k) \ni I = \gamma_1 + a \longmapsto&\\
\big(\cos(\Theta(t)) - i\sin(\Theta(t))\gamma_1\big)\big(\cos(\Theta(t)) + i\sin(\Theta(t))I\big)&\\
= \mathrm{Id} + i\sin(\Theta(t))\big(\cos(\Theta(t)) - i\sin(\Theta(t))\gamma_1\big)a \in G_{\mathrm{sus,iso}}^{-\infty}(\mathbb{R}^k; \mathbb{C}^2).&
\end{aligned} \tag{18.16}$$

Here $\Theta \in \mathcal{C}^\infty(\mathbb{R})$ is non-decreasing, vanishes for t sufficiently negative and is equal to π for t positive. Similarly, from odd to even

$$\text{(18.17)} \quad \mathrm{cl}_{\mathrm{oe}} : G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k) \ni g \longmapsto$$
$$I(t) = \begin{cases} \begin{pmatrix} \cos(\Theta(t)) & \sin(\Theta(t))g \\ \sin(\Theta(t))g^{-1} & -\cos(\Theta(t)) \end{pmatrix} & t \le 0 \\ \begin{pmatrix} \cos(2\pi - \Theta(-t)) & \sin(2\pi - \Theta(-t)) \\ \sin(2\pi - \Theta(-t)) & -\cos(2\pi - \Theta(-t)) \end{pmatrix} & t > 0 \end{cases} \in \mathcal{H}^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^k).$$

Proposition 17. *The clutching maps in* (18.16) *and* (18.17) *induce isomorphisms in K-theory giving commutative diagrams for any manifold X :*
(18.18)

$$\begin{tikzcd} {[X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)]_c} \arrow[r, "\mathrm{cl}_{\mathrm{eo}}"] \arrow[d, equal] & {[X; G^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^k; \mathbb{C}^2)]_c} \arrow[r, "\mathrm{cl}_{\mathrm{oe}}"] \arrow[d, equal] & {[X; \mathcal{H}^{-\infty}_{\mathrm{sus(2),iso}}(\mathbb{R}^k; \mathbb{C}^2)]_c} \arrow[d, equal] \arrow[ll, bend right, "\mathrm{p_{ad}}"'] \\ \mathrm{K}^0_c(X) \arrow[r, "\mathrm{cl}_{\mathrm{eo}}"] & \mathrm{K}^1_c(\mathbb{R} \times X) \arrow[r, "\mathrm{cl}_{\mathrm{oe}}"] & \mathrm{K}^0_c(\mathbb{R}^2 \times X) \arrow[ll, bend left, "\mathrm{p_{ad}}"] \end{tikzcd}$$

and

$$\text{(18.19)} \quad \begin{tikzcd} {[X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)]_c} \arrow[r, "\mathrm{cl}_{\mathrm{oe}}"] \arrow[d, equal] & {[X; \mathcal{H}^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^k)]_c} \arrow[r, "\mathrm{cl}_{\mathrm{oe}}"] \arrow[d, equal] & {[X; G^{-\infty}_{\mathrm{sus(2)\ iso}}(\mathbb{R}^k; \mathbb{C}^2)]_c} \arrow[d, equal] \arrow[ll, bend right, "\mathrm{p_{ad}}"'] \\ \mathrm{K}^{-1}_c(X) \arrow[r, "\mathrm{cl}_{\mathrm{eo}}"] & \mathrm{K}^0_c(\mathbb{R} \times X) \arrow[r, "\mathrm{cl}_{\mathrm{oe}}"] & \mathrm{K}^{-1}_c(\mathbb{R}^2 \times X) \arrow[ll, bend left, "\mathrm{p_{ad}}"] \end{tikzcd}$$