17. Lecture 14: Even periodicity map
Monday, 6 October

Question 2. (Jesse Gell-Redman) The construction of the (odd) periodicity map looks a bit fishy, would you care to clarify it?

Answer 2. Well, he was politer than that. Let me put things together, maybe a little more carefully than I did before. First let me try to clarify a couple of things – the sense in which we are free to switch the number of variables in which our smoothing operators act and are also free to work with finite dimensional matrices.

Lemma 19. *For any N and for any selection of N of the elements of the standard basis of eigenfunctions of the harmonic oscillator, $e_{i_1}, \dots, e_{i_N}$ the inclusion*

$$\mathrm{GL}(N;\mathbb{C}) \ni g_{ij} \longmapsto \hat{g} \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}),\ \hat{g}e_{i_l} = \sum_{p=1}^{N} g_{pl}e_{i_p},\ \hat{g}e_j = e_j,\ j \neq i_l, \tag{17.1}$$

is a group homomorphism which induces a map on homotopy classes

$$[X; \mathrm{GL}(N,\mathbb{C})]_c \longrightarrow [X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R})]_c \tag{17.2}$$

which is independent of the choice of basis and such that every element in the target group is in the image for N sufficiently large.

Proof. That the stabilizing map (17.1) is a group homorphism is clear enough and so it induces a map (17.2). Any two choices of basis are conjugate. To see this it suffices to change one element at a time to another eigenfunction which is not in the current set, and then finally relabel. The relabelling is given by an element of $\mathrm{GL}(N,\mathbb{C})$ acting by conjugation and the switching is given by a rotation between the two elements. In either case on the image space this is conjugation by a fixed (in terms of X) element of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R})$. Since this group is connected, the conjugation can be removed by a homotopy, constant in X. This proves that the induced maps (17.2) are all the same. □

One can easily go further somewhat further, as we will later, and conclude that as long as $\mathrm{GL}(N,\mathbb{C})$ is made to act on an N-dimensional subspace of the range of Π_K, for some k, and the identity on a complementary space, and on the range of $\mathrm{Id} - \Pi_k$, the same map (17.2), at the level of homotopy, results.

Lemma 20. *If $0 \neq e \in \mathcal{S}(\mathbb{R}^l)$ and $\pi_e = e \otimes \bar{e} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^l)$ is the orthogonal projection onto e then the group homomorphism*

$$G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^d) \ni \mathrm{Id} + a \longmapsto \mathrm{Id} + \pi_e \otimes a \in G^{-\infty}(\mathbb{R}^{d+k}) \tag{17.3}$$

is a weak homotopy equivalence (Is it an homotopy equivalence?) which induces an isomorphism, for any manifold X,

$$[X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^d)]_c = [X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})]_c \tag{17.4}$$

which is independent of the choice of e.

Note that e is fixed but arbitrary. As in the proof above, we can deform any of the maps (17.3) to any other by rotating e.

Note that the embeddings above are constant, i.e. we are not permitting twisting which depends on the point in X. It is also worth re-emphasizing that in both these maps the identity is 'increased in size', I have been regarding these maps as inclusions but one does need to be a little careful about this.

So, what is the inverse of the adiabatic periodicity map exactly? It is based on the lifting statement for restriction to $\epsilon = 1$

$$R : G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R} : \mathbb{R}^k) \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+k}). \tag{17.5}$$

Namely, take a smooth map $h : X \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+k})$ which is constant at the identity outside a compact set. We can 'lift' this back into one of the groups (17.1), that is there is a homotopy h_t where h_0 is in the image of the group and $h_1 = h$. So, consider h_0 instead.

Now, what do we know about $b = \gamma_1 + \delta$, $\delta \in \mathcal{C}^\infty_c(\mathbb{R}^2; M(2,\mathbb{C}))$, the Bott element. We showed that this involution is the semiclassical symbol of an involution $B = \gamma_1 + D$, $D \in \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R}; \mathbb{C}^2)$. More precisely, $D_\epsilon \in \mathcal{C}^\infty((0,1]; \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}; \mathbb{C}^2))$ is a smooth family of 2×2 matrices of isotropic smoothing operators on $\mathbb{R}$, forming a semiclassical family with symbol δ and such that $\gamma_1 \operatorname{Id} + D_\epsilon$ is an involution for all $\epsilon \in (0,1]$. Moreover, last time I finally computed the relative index of this family, showing it was 1, and hence that say $\gamma \operatorname{Id}_1 + D_{\frac{1}{2}}$ can be deformed to have

$$R(B) = \gamma \operatorname{Id}_1 + 2\Pi_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{17.6}$$

(which has the same relative index) through involutions – here Π_1 is projection onto the ground state of the harmonic oscillator (or onto some other element of $\mathcal{S}(\mathbb{R})$ of your choice). Thus, modifying the parameter in $\epsilon > \frac{1}{4}$ a bit we have a semiclassical family, B, of 2×2 matrices, in 1 dimension, which has semiclassical symbol b and takes the value (17.6) at $\epsilon = 1$.

Now, much as above we want to turn this into an adiabatic family. First we can undo B into the projections onto its positive and negative partis, $B_\pm$ and then consider the semiclassical family of $2N \times 2N$ matrices

$$g' = (h_0^{-1}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})(h_0(x) \otimes B_+ + B_-) \in G^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R}; \mathbb{C}^{2N}). \tag{17.7}$$

The first factor makes the semiclassical family have 'unital part' – the leading matrix multiple of the identity – be Id_{2N}, giving (17.7) and from (17.6)

$$R(g') = \operatorname{Id} + \Pi_1 \otimes (h_0 - \operatorname{Id}_N) \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{17.8}$$

This is just $h_0(x)$ acting as an $N \times N$ matrix on $\mathbb{C}^N$, extended to the second part of the $\mathbb{C}^2$, plus the identity on everything else.

Now, we are free to embed $M(2N, \mathbb{C})$ to act on a finite span of the harmonic oscillator eigenfunctions in $\mathbb{S}(\mathbb{R}^k)$ however we want, and we can do this so that the $h_0(x)$ in (17.8) is the h_0 we started with. Moreover this embedding corresponds to the same sort of map as (17.3) but now giving

$$G^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R}; \mathbb{C}^{2N}) \longrightarrow G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R} : \mathbb{R}^k; \mathbb{C}^2). \tag{17.9}$$

So, combining these steps the image, g, of g' under (17.9) has $R(g) = h_0$. Modifying the family in $\epsilon > 0$ we can insert the extra homotopy to arrange that $R(g) = h$ as desired.

Let us note some stabilization results of the same type as discussed above, but for the classifying spaces $\mathcal{H}^{-\infty}(\mathbb{R}^d)$. As for the corresponding groups there are isomorphisms – for the moment fixed, but at some point I will have to discuss the possible choices –

$$\begin{aligned} &\mathcal{H}^{-\infty}(\mathbb{R}) \longrightarrow \mathcal{H}^{-\infty}(\mathbb{R}^d) \\ &\mathcal{H}^{-\infty}(\mathbb{R}^d;\mathbb{C}^N) \longrightarrow \mathcal{H}^{-\infty}(\mathbb{R}^d). \end{aligned} \tag{17.10}$$

They are obtained by relabelling basis elements or by extending the perturbation to have rank 1 in the other 'variables'. One can also think about this as going back to our original sequential group and considering the corresponding

$$\mathcal{H}^{-\infty}(\mathbb{N}) = \left\{ a \in \Psi^{-\infty}(\mathbb{N};\mathbb{C}^2); \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + a \text{ is an involution} \right\}. \tag{17.11}$$

Now, we also have adiabatic and suspended versions of these spaces:-

$$\begin{aligned} &\mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^k) = \\ &\qquad \left\{ a \in \Psi^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^k;\mathbb{C}^2); \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + a(T) \in \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)\ \forall\ T \in \mathbb{R}^p \right\} \\ &\mathcal{H}^{-\infty}_{\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d:\mathbb{R}^k) = \\ &\qquad \left\{ a \in \Psi^{-\infty}_{\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d:\mathbb{R}^k;\mathbb{C}^2); \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + a(\epsilon) \in \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k}),\ \epsilon > 0 \right\}, \end{aligned} \tag{17.12}$$

including of course those acting on $\mathbb{C}^2 \otimes \mathbb{C}^N$ instead of just $\mathbb{C}^2$.

At this stage it probably has already occurred to you that we can 'put things together'. That is, we can define a combined adiabatic-suspended-isotropic space of involutions

$$\begin{aligned} \mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d:\mathbb{R}^k) = \Big\{ &a \in \mathcal{S}(\mathbb{R}^p;\Psi^{-\infty}_{\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d:\mathbb{R}^k;\mathbb{C}^2\otimes\mathbb{C}^N)); \\ &\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + a(\epsilon) \in \mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^{d+k};\mathbb{C}^N),\ \forall\ \epsilon > 0 \Big\}. \end{aligned} \tag{17.13}$$

Here it is understood that if $d = 0$ or $p = 0$ one is reduced to the earlier cases.

Proposition 15. *The space in* (17.13) *is classifying for K-theory of the parity of p (provided* $k > 0$*), the base component (which is the whole space if p is odd) is homogeneous*

$$\begin{aligned} &\mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d:\mathbb{R}^k) = \\ &G^{-\infty}_{\mathrm{sus}(p),\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d:\mathbb{R}^k;\mathbb{C}^2)\Big/\Big(G^{-\infty}_{\mathrm{sus}(p),\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d:\mathbb{R}^k)\oplus G^{-\infty}_{\mathrm{sus}(p),\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d:\mathbb{R}^k;\mathbb{C}^2)\Big) \end{aligned} \tag{17.14}$$

and for $d = 1$ *(also for* $d > 1$ *shown later) the lower maps in the following diagram are surjective, with the lifting property for compactly-supported families, and with*

the upper spaces weakly contractible:-
(17.15)

$$\begin{tikzcd}[column sep=small]
\left\{\sigma_{\mathrm{ad}}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\} \arrow[dr] & & \left\{R=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\} \arrow[dl] \\
& \mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d:\mathbb{R}^k) \arrow[dl, "R"'] \arrow[dr, "\sigma_{\mathrm{ad}}"] & \\
\mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^{d+k}) & & \mathcal{H}^{-\infty}_{\mathrm{sus}(p+2d),\mathrm{iso}}(\mathbb{R}^k).
\end{tikzcd}$$

Proof. Everything here, except the lifting property for R is fairly straightforward, meaning it is much the same as before. For the momemnt I omit proofs from the notes for these parts. To prove the properties of R we need some more properties of involutions $\square$