15. Lecture 12: Atiyah's rotation Wednesday, 1 October

Let me try to clarify what we have done so far as regards the proof of Bott periodicity. Last time we proved that there is a map

$$p_{\mathrm{ad}} : [X; G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)]_{\mathrm{c}} \longrightarrow [X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})]_{\mathrm{c}} \tag{15.1}$$

obtained by semiclassical quantization. Indeed let me quickly recall how this map is defined – I have generalized from homotopy classes of smooth maps from a compact manifold to homotopy classes of compactly-supported smooth map from a general manifold, but this makes very little difference to the argument. To define (15.1) take a representative, $f : X \longrightarrow G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$ and choose, as we can, a family $\tilde{f} : X \longrightarrow \mathrm{Id} + \Psi^{-\infty}_{\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d : \mathbb{R}^k)$ which has adiabatic symbol f. Last time I showed that in fact $\tilde{f}$ is invertible for $\epsilon \in (0, \epsilon_0]$, $\epsilon_0 > 0$, with inverse given by a similar family. We can expand the parameter so that $\epsilon_0 = 1$ and then (15.1) is obtained by restriction to $\epsilon = 1$. It doesn't matter where we restrict, provided $\tilde{f}_\epsilon$ is invertible for that ϵ and smaller. So, we showed that this induces a map (15.1).

Now, we also showed that the map (15.1) is surjective for $d = 1$. To do this we proved that for a given map with compact support $g : X \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})$ we can first make a smooth homotopy, and then 'lift' it to a family which takes the form

$$\tilde{g} = \mathrm{Id} + A(t,\tau)g^{-1}(x) + A'(t,\tau)g(x) + A''(t,\tau) \in G^{-\infty}_{\mathrm{sus}(2),\mathrm{iso}}(\mathbb{R}^k) \tag{15.2}$$

and where the perturbation is compactly supported on $\mathbb{R}^2$. Note that this family has the nice property that $g = \mathrm{Id}$ implies $\tilde{g} = \mathrm{Id}$ which is important now that we want to treat compactly supported families. Thus we proved (modulo properties of the Bott element which have not yet been checked) that

$$p_{\mathrm{ad}}(\tilde{g}) = [g], \tag{15.3}$$

thus establishing surjectivity.

Now we want to prove injectivity of p_{ad}, which reduces to the weak contractibility of a certain subgroup of $G^{-\infty}_{\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d : \mathbb{R}^k)$. This can be done by rather tedious, if imaginative, computation but Atiyah realized that it is a consequence of the multiplicativity of K-theory. Now, I have not discussed this multiplicativity but we can just proceed directly and then sort out what we have done afterwards.

Stripped down in this way, Atiyah's idea goes as follows. Look at $\tilde{g}$ in (15.2). It is itself a compactly supported smooth map

$$\tilde{g} : X \times \mathbb{R}^2 \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k). \tag{15.4}$$

So, we can (after homotopy to finite rank) apply the same construction to it. Let me call the result

$$G(t',\tau',t,\tau,x) = \mathrm{Id} + A(t',\tau')\tilde{g}^{-1}(x,t,\tau) + A'(t',\tau')\tilde{g}(x,t,\tau) + A''(t',\tau'). \tag{15.5}$$

So this is a function on $\mathbb{R}^4 \times X$ with compact support. We recover $\tilde{g}$ (if you like up to homotopy) by quantizing it in the variables (t',τ'). However, let us make a rotation between (t',τ') and (t,τ), substituting in (15.5)

$$
\begin{aligned}
t' &\longmapsto t'\cos\theta + t\sin\theta,\\
\tau' &\longmapsto \tau'\cos\theta + \tau\sin\theta,\\
t &\longmapsto -t'\sin\theta + t\cos\theta,\\
\tau &\longmapsto -\tau'\sin\theta + \tau\cos\theta.
\end{aligned}
\tag{15.6}
$$

This linear change of variables leads to a smooth map

$$\tilde G : [0,\pi/2]\times\mathbb{R}^4\times X \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k). \tag{15.7}$$

So, we can think of this as an homotopy in the variable θ and in particular, we can quantize it in (t',τ') uniformly in θ, $\mathbb{R}^2$ and X. At $\theta=\pi/2$ the (t',τ') variables are replaced by (t,τ) in the sense that

$$
\begin{aligned}
&\tilde G(\pi/2,t,\tau,t',\tau',x) =\\
&\qquad \mathrm{Id} + A(-t,-\tau)\tilde g^{-1}(x,t',\tau') + A'(-t,-\tau)\tilde g(x,t',\tau') + A''(-t,-\tau).
\end{aligned}
\tag{15.8}
$$

Thus, at $\theta=\pi/2$ we are simply quantizing $\tilde g$ and $\tilde g^{-1}$. However, by construction the quantization (which we may choose) of $\tilde g$ is g and hence that of $\tilde g^{-1}$ is g^{-1}.

Finally then see what happens. If $\tilde g$ is such that $p_{\mathrm{ad}}\tilde g=\mathrm{Id}$ then we can find $\tilde G(\theta)$ which at $\theta=0$ quantizes to $\tilde g$ and at $\theta=\pi/2$ quantizes to Id. Thus in fact $\tilde g$ is homotopic to the identity and we have proved that (15.1) is an isomorphism, well for $d=1$ and modulo the discussion of projections – which I will proceed to do.

Maybe it is worthwhile going back and checking, modulo the same issues of course, that I have now proved Theorem 2 since that *looks* more substantial. So, I claim that, for $d=1$, I can pretend to have done everything except the weak contractibility of the subgroup

$$\mathcal{N} = \{A\in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}:\mathbb{R}^k);\,R(A)=\mathrm{Id}\}. \tag{15.9}$$

That is, consider a compactly supported smooth map into it. We need to show that it can be deformed to the constant-at-the-identity map. This is almost what we have shown. Namely we have shown that if $\gamma : X\longrightarrow\mathcal{N}$ is such that $R(\gamma)=\mathrm{Id}$ then $\sigma_{\mathrm{ad}}(\gamma)$, which is what we discussed above, is homotopic to the identity in $G^{-\infty}_{\mathrm{sus(2),iso}}(\mathbb{R}^k)$. Now, quantizing this family of symbols (including the homotopy) gives us an homotopy in $G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}:\mathbb{R}^k)$ which I can call $\Gamma(t,x)$. At $t=0$ it is γ and at $t=1$ it is Id. We are almost there, but it is *not* (necessarily) an homotopy in $\mathcal{N}$. It starts there and finishes there but we have done nothing to control $R(\Gamma(t,x))$ for $t\in(0,1)$. Fortunately we have the lifting map. That is we can lift $R(\Gamma(t,x))$ to a family $\Gamma' : [0,1]\times X\longrightarrow G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}:\mathbb{R}^k)$ so that $R(\Gamma'(t,x))=R(\Gamma(t,x))$ *and* so that $\Gamma'(0,x)=\Gamma'(1,x)=\mathrm{Id}$ since $R(\Gamma(t,x)=\mathrm{Id}$ there. Then $(\Gamma'(t,x))^{-1}\Gamma(t,x)$ is a new homotopy from γ to Id which *is* in $\mathcal{N}$.

Okay, so it is on to projections to check the little facts about the Bott projection and its quantization.

Let me formalize what we were doing earlier as regards projections and set

$$\mathcal{H}^{-\infty}(\mathbb{R}^k) = \left\{a\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k;\mathbb{C}^2);\,I_\infty+a \text{ is an involution}\right\},\ I_\infty=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{15.10}$$

This is not a group, but it has certain features making it similar to $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)$. Notably it is an infinite dimensional manifold 'modelled on (two copies of) $\mathcal{S}(\mathbb{R}^{2k})$. This I will discuss next time.

Definition 5. The K-groups (not quite immediately obvious that they are groups) associated to a smooth manifold X are the compactly-supported homotopy classes of smooth maps

$$\mathrm{K}^0_{\mathrm{c}}(X) = [X; \mathcal{H}^{-\infty}(\mathbb{R})]_{\mathrm{c}}. \tag{15.11}$$

Of course we will quickly show that one could just as well take

$$\mathrm{K}^0_{\mathrm{c}}(X) = [X; \mathcal{H}^{-\infty}(\mathbb{R}^k)]_{\mathrm{c}} \tag{15.12}$$

for any k and the results are naturally isomorphic. If X is compact one can drop the 'c' suffix – and historically it is even dropped in the general case, meaning that in the literature $\mathrm{K}^0(X)$ denotes what I am calling $\mathrm{K}^0_{\mathrm{c}}(X)$.

The work I did on the Bott element now extends directly.

Proposition 13. *There is a well-defined (adiabatic) periodicity map*

$$p_{\mathrm{ad}} : \mathrm{K}^0_c(X \times \mathbb{R}^{2d}) \longrightarrow \mathrm{K}^0_c(X). \tag{15.13}$$

Proof. A class in $\mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}^{2d})$ is represented by a compactly-supported map $X \times \mathbb{R}^{2d} \longrightarrow \mathcal{H}^{-\infty}(\mathbb{R})$. The discussion in Lecture 8 shows that this can be quantized to an adiabatic family of involutions. We need to check homotopy invariance of the result but this follows the same lines. ☐

Now we have lots of groups and lots of identifications between them:-

$$\begin{gathered} p_{\mathrm{ad}} : \mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}^{2d}) = \mathrm{K}^0_{\mathrm{c}}(X),\ p_{\mathrm{ad}} : \mathrm{K}^1_{\mathrm{c}}(X \times \mathbb{R}^{2d}) = \mathrm{K}^1_{\mathrm{c}}(X), \\ \mathrm{cl} : \mathrm{K}^0_{\mathrm{c}}(X) \longrightarrow \mathrm{K}^1_{\mathrm{c}}(X \times \mathbb{R}).\, \mathrm{cl} : \mathrm{K}^1_{\mathrm{c}}(X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}) \end{gathered} \tag{15.14}$$

We do need to make sure that these maps are consistent under compostion – so we can regard them as identifications (with some care!). Typically the top row are regarded really as identifications – this is Bott periodicity – and the bottom row as maps.