14. Lecture 11: Adiabatic periodicity map
Monday, 29 September

Last time I started the 'easier', or perhaps better to say 'routine', part of the proof of Theorem 2 – giving the adiabatic diagonal sequence, from top left to bottom right in

(14.1)
$$\begin{tikzcd}
\{A \in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k); \sigma_{ad}(A) = \mathrm{Id}\} \arrow[dr, hook] & & \{A \in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k); R(A) = \mathrm{Id}\} \arrow[dl, hook'] \\
& G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) \arrow[dl, "R"'] \arrow[dr, "\sigma_{\mathrm{ad}}"] & \\
G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k}) & & G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k).
\end{tikzcd}$$

Observe that this already gives us a map – following the arguments of last lecture –

$$[X; G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)]_{\mathrm{c}} \longrightarrow [X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})]_{\mathrm{c}}. \tag{14.2}$$

So once we see the same properties for the other sequence we conclude that this must be an isomorphism. In fact for the moment I will only do this for $d = 1$.

Addendum to Lecture 11: $\mathrm{GL}(N, \mathbb{C})$ and the Bott element From Paul Loya

Here we present the proof on Sept. 26-th that the restriction map to $\varepsilon = 1$:

$$R : G^{-\infty}_{ad,iso}(\mathbb{R}, \mathbb{R}^k) \to G^{-\infty}_{iso}(\mathbb{R}^{k+1}),$$

is surjective at the level of homotopies using the Bott element.

Preparing for the Bott element: Lemmas from Lecture 14

The following lemma is Lemma 16 in Lecture 14.

Lemma 15 (**Finite Rank Approximation**). *Let* Π *be the orthogonal projection onto an* N*-dimensional subspace of* $\mathcal{S}(\mathbb{R}^d)$ *and choose an identification of linear maps on the range of* Π *with* $M(N, \mathbb{C})$*, and consider the map*

$$M(N, \mathbb{C}) \ni A \longmapsto \mathrm{Id} - \Pi + A\Pi \in \mathrm{Id} + \Psi^{-\infty}_{iso}(\mathbb{R}^d),$$

where $A\Pi$ *on the right is the matrix* A *acting on the range of* Π *through the chosen identification of linear maps on the range of* Π *with* $M(N, \mathbb{C})$*. This map restricts to a map*

$$\mathrm{GL}(N, \mathbb{C}) \ni A \longmapsto \mathrm{Id} - \Pi + A\Pi \in G^{-\infty}_{iso}(\mathbb{R}^d),$$

and for any topological space X*, induces a map*

$$[X, \mathrm{GL}(N, \mathbb{C})]_c \to [X, G^{-\infty}_{iso}(\mathbb{R}^d)]_c$$

that is defined independent of the choice of the N*-dimensional subspace of* $\mathcal{S}(\mathbb{R}^d)$ *chosen and the choice of identification of linear maps on the range of* Π *with* $M(N, \mathbb{C})$*. Moreover, any element of* $[X, G^{-\infty}_{iso}(\mathbb{R}^d)]_c$ *is in the image of this map for a sufficiently large* N*.*

The following lemma is Lemma 17 in Lecture 14.

Lemma 16. *If Π_1 is the orthogonal projection onto a* 1*-dimensional subspace of $\mathcal{S}(\mathbb{R}^d)$, then the map*[1]

$$G^{-\infty}_{iso}(\mathbb{R}^k) \ni h \longmapsto \mathrm{Id} - \Pi_1 + h\Pi_1 \in G^{-\infty}_{iso}(\mathbb{R}^{k+d})$$

induces an isomorphism

$$[X, G^{-\infty}_{iso}(\mathbb{R}^k)]_c \to [X, G^{-\infty}_{iso}(\mathbb{R}^{k+d})]_c$$

that is defined independent of the choice of the 1*-dimensional subspace.*

Proof of the theorem: The Magical Bott element

Recall that the Bott element is an operator $B \in \mathcal{H}^{-\infty}_{ad}(\mathbb{R}) = \{\text{involutions in}^2\ \gamma_1 + \Psi^{-\infty}_{ad}(\mathbb{R}, \mathbb{C}^2)\}$, which has the property that

$$B_+|_{\varepsilon=1} \sim \begin{pmatrix} 1 & 0 \\ 0 & \Pi_1 \end{pmatrix}$$

and

$$B_-|_{\varepsilon=1} \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 - \Pi_1 \end{pmatrix}$$

where $\Pi_1 \in \Psi^{-\infty}_{iso}(\mathbb{R})$ is the projection onto a one-dimensional subspace of $\mathcal{S}(\mathbb{R}^1)$ (say the ground state of the harmonic oscillator).

Theorem 3. *The restriction map to $\varepsilon = 1$:*

$$R : G^{-\infty}_{ad,iso}(\mathbb{R}, \mathbb{R}^k) \to G^{-\infty}_{iso}(\mathbb{R}^{k+1}),$$

is surjective at the level of homotopies. That is, for any topological space X and any element $[g] \in [X, G^{-\infty}_{iso}(\mathbb{R}^{k+1})]_c$ there is an element $[\tilde{g}] \in [X, G^{-\infty}_{ad,iso}(\mathbb{R}, \mathbb{R}^k)]_c$ such that $[R\tilde{g}] = [g]$.

Proof. By finite rank approximation, any element of $[X, G^{-\infty}_{iso}(\mathbb{R}^{k+1})]_c$ is homotopic to an invertible matrix through Lemma 15 and by further stabilization we may assume that

$$g = F_0 \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix}, \quad \text{where} \quad g_0 : X \to \mathrm{GL}(N, \mathbb{C}),$$

Id_N is the $N \times N$ identity matrix, and

$$F_0 : \mathrm{GL}(2N, \mathbb{C}) \to G^{-\infty}_{iso}(\mathbb{R}^{k+1})$$

is the map in Lemma 15 defined by some choice of $2N$-dimensional subspace of $\mathcal{S}(\mathbb{R}^{k+1})$ — it's not important now what subspace we choose although at the end of this proof we'll take a subspace in $\mathcal{S}(\mathbb{R}^{k+1}) = \mathcal{S}(\mathbb{R}^k \times \mathbb{R}^1)$ spanned by $2N$ independent functions in $\mathcal{S}(\mathbb{R}^k)$ times a function in $\mathcal{S}(\mathbb{R}^1)$. The reason we take g in terms of a 2×2 matrix (of $N \times N$ matrices) is because the Bott element is given in terms of 2×2 matrices. We shall find a map

$$\tilde{g} : X \to G^{-\infty}_{ad,iso}(\mathbb{R}, \mathbb{R}^k)$$

such that $[R\tilde{g}] = [g]$. To define $\tilde{g}$, let

$$F_1 : \mathrm{GL}(2N, \mathbb{C}) \to G^{-\infty}_{iso}(\mathbb{R}^k),$$

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[1] On the right-hand side, as operators on $\mathcal{S}(\mathbb{R}^{k+d}) = \mathcal{S}(\mathbb{R}^k \times \mathbb{R}^d)$, Π_1 only acts on the $\mathbb{R}^d$ factor and h on the $\mathbb{R}^k$ factor.

[2] Recall that $\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and note that $\Psi^{-\infty}_{ad}(\mathbb{R}, \mathbb{C}^2)$ consists of 2×2 matrices of operators in $\Psi^{-\infty}_{ad}(\mathbb{R})$.

be the map in Lemma 15 induced by some choice of $2N$-dimensional subspace of $\mathcal{S}(\mathbb{R}^k)$, and then define

$$\boxed{\tilde{g} = F_1 g_1,}$$

where

$$g_1 = \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} (\mathrm{Id}_N \otimes B_+) + \mathrm{Id}_N \otimes B_-.$$

Let's pause to think about this! Note that $B_\pm$ are 2×2 matrices whose entries are operators in $\mathrm{Id} + \Psi_{ad}^{-\infty}(\mathbb{R})$, so $\mathrm{Id}_N \otimes B_\pm$ are $2N \times 2N$ matrices of the same sort. Using that $B = \gamma_1$ modulo a 2×2 matrix of operators in $\Psi_{ad}^{-\infty}(\mathbb{R})$ one can check that $g_1 = \mathrm{Id}_{2N} + R$ where R is a $2N \times 2N$ matrix of operators in $\Psi_{ad}^{-\infty}(\mathbb{R})$. Moreover, g_1 is invertible with inverse

$$\begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0^{-1} \end{pmatrix} (\mathrm{Id}_N \otimes B_+) + \mathrm{Id}_N \otimes B_-.$$

It follows that

$$\tilde{g} = F_1 g_1 \in G^{\infty}_{ad,iso}(\mathbb{R}, \mathbb{R}^k).$$

Now we claim that

$$\tilde{g}|_{\varepsilon=1} \sim g = F_0 \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix}.$$

To see this, recall that

$$B_+|_{\varepsilon=1} \sim \begin{pmatrix} 1 & 0 \\ 0 & \Pi_1 \end{pmatrix} \quad \text{and} \quad B_-|_{\varepsilon=1} \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 - \Pi_1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} g_1|_{\varepsilon=1} &= \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} (\mathrm{Id}_N \otimes B_+)|_{\varepsilon=1} + \mathrm{Id}_N \otimes B_-|_{\varepsilon=1} \\ &\sim \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & \mathrm{Id}_N \Pi_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_N - \mathrm{Id}_N \Pi_1 \end{pmatrix} \\ &= \mathrm{Id}_{2N} - \mathrm{Id}_{2N} \Pi_1 + \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} \Pi_1. \end{aligned}$$

Hence,

$$\tilde{g}|_{\varepsilon=1} = F_1 g_1|_{\varepsilon=1} \sim \mathrm{Id} - \Pi_1 + \left(F_1 \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} \right) \Pi_1.$$

In other words, if

$$F_2 : [X, G^{-\infty}_{iso}(\mathbb{R}^k)]_c \to [X, G^{-\infty}_{iso}(\mathbb{R}^{k+1})]$$

is the isomorphism induced by the map, which we also denote by F_2,

$$G^{-\infty}_{iso}(\mathbb{R}^k) \ni h \longmapsto \mathrm{Id} - \Pi_1 + h\Pi_1 \in G^{-\infty}_{iso}(\mathbb{R}^{k+1}) \tag{14.3}$$

found in Lemma 16 with $d = 1$, then we see that

$$\tilde{g}|_{\varepsilon=1} = F_2 \left(F_1 \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} \right).$$

To summarize, we are left to show that

$$\left[F_2 \left(F_1 \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} \right) \right] = \left[F_0 \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} \right].$$

Thus, our theorem is finished off by proving that the composition

$$[X, \mathrm{GL}(2N, \mathbb{C})] \xrightarrow{F_1} [X, G^{-\infty}_{iso}(\mathbb{R}^k)] \xrightarrow{F_2} [X, G^{-\infty}_{iso}(\mathbb{R}^{k+1})]$$

is exactly the same as the map

$$[X, \mathrm{GL}(2N, \mathbb{C})] \xrightarrow{F_0} [X, G^{-\infty}_{iso}(\mathbb{R}^{k+1})];$$

in other words, we need to prove that the following diagram commutes:

$$\begin{tikzcd} {[X, \mathrm{GL}(2N, \mathbb{C})]} \arrow[r, "F_1"] \arrow[rr, bend right, "F_0"'] & {[X, G^{-\infty}_{iso}(\mathbb{R}^k)]} \arrow[r, "F_2"] & {[X, G^{-\infty}_{iso}(\mathbb{R}^{k+1})]} \end{tikzcd} .$$

To prove this we just have to look at what the maps F_0, F_1, and F_2. Let Π be the orthogonal projection onto any $2N$-dimensional subspace of $\mathcal{S}(\mathbb{R}^k)$. Then F_1 is the map induced by

$$\mathrm{GL}(2N, \mathbb{C}) \ni A \longmapsto \mathrm{Id} - \Pi + A\Pi \in G^{-\infty}_{iso}(\mathbb{R}^k).$$

Therefore, at the homotopy group level, $F_2 F_1$ is the map (see (14.3) for F_2) induced by

$$\begin{aligned} \mathrm{GL}(2N, \mathbb{C}) \ni A \longmapsto \mathrm{Id} - \Pi_1 + \big(\mathrm{Id} - \Pi + A\Pi\big)\Pi_1 &\in G^{-\infty}_{iso}(\mathbb{R}^{k+1}) \\ = \mathrm{Id} - \Pi\,\Pi_1 + A\Pi\,\Pi_1 &\in G^{-\infty}_{iso}(\mathbb{R}^{k+1}) \end{aligned}$$

On the other hand, $\Pi\,\Pi_1$ is the orthogonal projection onto a $2N$-dimensional subspace of $\mathcal{S}(\mathbb{R}^{k+1})$ (namely, the space of functions of the form $f(x)\,g(y)$ where $(x, y) \in \mathbb{R}^k \times \mathbb{R}^1$ with f and g in the range of Π and Π_1, respectively). Hence, we can take the map F_0 to be induced by

$$\mathrm{GL}(2N, \mathbb{C}) \ni A \longmapsto \mathrm{Id} - \Pi\,\Pi_1 + A\Pi\,\Pi_1 \in G^{-\infty}_{iso}(\mathbb{R}^{k+1}),$$

which is exactly $F_2 F_1$. This completes the proof. $\square$

Extra: Bott periodicity for the general linear group

This section is not needed for the Bott element but might be useful to be written down. In this section we prove that $\pi_0(\mathrm{GL}(N, \mathbb{C})) = \{0\}$ and $\pi_1(\mathrm{GL}(N, \mathbb{C})) = \mathbb{Z}$.

Lemma 17. *There is a smooth map*

$$T : \mathbb{S}^{2N-1} \setminus \{-e_1\} \to \mathrm{GL}(N, \mathbb{C}) \quad , \quad v \mapsto T_v$$

such that $T_{e_1} = \mathrm{Id}$ *and for all* $v \in \mathbb{S}^{2N-1} \setminus \{-e_1\}$, $T_v v = e_1$.

Proof. Given $v \in \mathbb{S}^{2N-1}$, for all $x \in \mathbb{C}^N$, define

$$T_v(x) = x + (x \cdot v)(e_1 - v) + \frac{(e_1 \cdot v)(x \cdot v) - x \cdot e_1}{1 + e_1 \cdot v}(e_1 + v).$$

Here "$\cdot$" denotes the usual Hermitian inner product on $\mathbb{C}^N$ (linear in the first slot and conjugate linear in the second slot). Using this formula, it's easy to show that T_v depends smoothly on $v \in \mathbb{S}^{2N-1} \setminus \{-e_1\}$, $T_{e_1} = \mathrm{Id}$, and $T_v v = e_1$. $\square$

Using this smooth map T_v we prove prove the following theorem. (I haven't seen a proof of this theorem that uses the linear map T_v. Has anyone?)

Theorem 4. *If $0 < k < N$, then the inclusion map*

$$\mathrm{GL}(k,\mathbb{C}) \to \mathrm{GL}(N,\mathbb{C}) \; ; \quad A \mapsto \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & A \end{pmatrix}$$

induces an isomorphism between homotopy spaces

$$[\mathbb{S}^1, \mathrm{GL}(k,\mathbb{C})] \to [\mathbb{S}^1, \mathrm{GL}(N,\mathbb{C})].$$

Proof. By iteration we may assume that $k = N-1$. Our theorem follows immediately from the following two claims, which we'll prove using the lemma: For any $N > 1$,

(1) Any element of $[\mathbb{S}^1, \mathrm{GL}(N,\mathbb{C})]$ has a representative of the form $\begin{pmatrix} 1 & 0 \\ 0 & g(x) \end{pmatrix}$, where $g : \mathbb{S}^1 \to \mathrm{GL}(N-1,\mathbb{C})$.

(2) Two maps $g_0, g_1 : \mathbb{S}^1 \to \mathrm{GL}(N-1,\mathbb{C})$ are homotopic if and only if the maps $\begin{pmatrix} 1 & 0 \\ 0 & g_0(x) \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & g_1(x) \end{pmatrix}$ are homotopic as maps into $\mathrm{GL}(N,\mathbb{C})$.

Let $f : \mathbb{S}^1 \to \mathrm{GL}(N,\mathbb{C})$ be a continuous map. Since $f(\mathbb{S}^1)$ is a compact subset of $\mathrm{GL}(N,\mathbb{C})$, an open set in the set of all $N \times N$ matrices, it follows that any $N \times N$ matrix sufficiently close to the image $f(\mathbb{S}^1)$ must lie in $\mathrm{GL}(N,\mathbb{C})$. Using this fact plus a standard compactness argument, it is straightforward to show[3] that f is homotopic to a map (again denoted by f) such that the first column $w_1(x)$ of f is not a positive real multiple of $-e_1$. Hence, $v(x) = w_1(x)/\|w_1(x)\|$ is never equal to $-e_1$. By Lemma 17 we have $T_{v(x)} w_1(x) = \|w_1(t)\|$, so for all $x \in \mathbb{S}^1$,

$$T_{v(x)} f(x) = \begin{pmatrix} \|w_1(x)\| & * \\ 0 & g(x) \end{pmatrix},$$

where $*$ is unimportant components and $g : \mathbb{S}^1 \to \mathrm{GL}(N-1,\mathbb{C})$. We may homotopy the first column to e_1, so

$$f(x) \sim T_{v(x)}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & g(x) \end{pmatrix}$$

Now $v(x) \in \mathbb{S}^{2n-1} \setminus \{-e_1\} \cong \mathbb{R}^{2n-1}$ so we can homotopy $v(x)$ to the constant vector e_1 within $\mathbb{S}^{2n-1} \setminus \{-e_1\}$. Since $T_{e_1} = \mathrm{Id}$ it follows that $T_{v(x)}^{-1} \sim \mathrm{Id}$. This proves Claim 1.

We now prove 2. Certainly the "only if" part holds, so assume that $\begin{pmatrix} 1 & 0 \\ 0 & g_0(x) \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & g_1(x) \end{pmatrix}$ are homotopic as maps into $\mathrm{GL}(N,\mathbb{C})$, which means there is a continuous map $F : \mathbb{S}^1 \times [0,1] \to \mathrm{GL}(N,\mathbb{C})$ such that

$$F(x,0) = f_0(x) = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x) \end{pmatrix} \quad \text{and} \quad F(x,1) = f_1(x) = \begin{pmatrix} 1 & 0 \\ 0 & g_1(x) \end{pmatrix}. \tag{14.4}$$

By a similar argument as we stated in the previous paragraph we may assume that $w_1(x,t)$, the first column of $F(x,t)$, is never a positive multiple of $-e_1$. Hence, $v(x,t) = w_1(x,t)/\|w_1(x,t)\|$ is never equal to $-e_1$. By Lemma 17, we have

$$T_{v(x,t)} F(x,t) = \begin{pmatrix} 1 & * \\ 0 & g(x,t) \end{pmatrix},$$

[3] If $f_{N1}(x)$ denotes the N-th row, 1-st column element of $f(x)$, all you have to do is replace this function by a new function such that $f_{N1}(x) \neq 0$ for $x \neq 1$.

where $g(x,t) \in \mathrm{GL}(N-1,\mathbb{C})$. Since $v(x,0) = e_1 = v(x,1)$ and $T_{e_1} = \mathrm{Id}$, it follows that $g(x,0) = g_0(x)$ and $g(x,1) = g_1(x)$. Thus, $g : \mathbb{S}^1 \times [0,1] \to \mathrm{GL}(N-1,\mathbb{C})$ provides a homotopy between g_0 and g_1. $\square$

Corollary 3. $\pi_0(\mathrm{GL}(N,\mathbb{C})) = \{0\}$ *and* $\pi_1(\mathrm{GL}(N,\mathbb{C})) = \mathbb{Z}$.

Proof. The second statement follows from Theorem 4 (with $k = 1$) and the fact that $\pi_1(\mathrm{GL}(1,\mathbb{C})) = [\mathbb{S}^1, \mathrm{GL}(1,\mathbb{C})] = \mathbb{Z}$ (which can be proved using for example the winding number). The proof of Theorem 4 also works if we replace $\mathbb{S}^1$ with a point, so the first statement follows from the fact that $\pi_0(\mathrm{GL}(1,\mathbb{C})) = \{0\}$. $\square$