13. Lecture 10: Bott periodicity Friday, 26 September

I will start the notes, if not the lecture, with an extended reply to a question from the end of the last lecture.

Question 1. (Jesse Gell-Redman) What has this got to do with index theory?

Answer 1. My first answer is that we need to develop K-theory in order to understand the index theorem, however I am trying to do more than that. This question is out or order of course, I don't mean parliamentary order here, just logical order. Still, let me run ahead a bit, taking this as an opportunity to indicate where I am trying to go – since for one thing you might not wish to come along!

The second answer is that 'this' meaning Theorem 2 really *is* an index theorem, or at least is closely related to one. Let me try to describe this relationship, even though I will use some as-yet-undefined objects. The index theorem most closely related to Theorem 2 is Fedosov's index theorem on isotropic operators. Well, the original theorem was about the numerical index but let me jazz it up to the families theorem. First, an isotropic pseudodifferential operator corresponds to equal scaling for z and D_z on $\mathbb{R}^n$, as I have indicated earlier in relation to isotropic smoothing operators. So, whatever they are, isotropic pseudodifferential operators of order 0 are bounded operators on $L^2(\mathbb{R}^n)$ and they have symbols. Since they are 'isotropic' the symbol is just a homogeneous function on $\mathbb{R}^{2n} \setminus \{0\}$, or equivalently on $\mathbb{S}^{2n-1}$, with values in $M(N;\mathbb{C})$. The operator is Fredholm if and only if the symbol is invertible. Let X be a parameter space, then the symbol of an elliptic family of such operators, parameterized by X, is a smooth map

$$a : X \times \mathbb{S}^{2n-1} \longrightarrow \mathrm{GL}(N;\mathbb{C}) \hookrightarrow G^{-\infty}. \tag{13.1}$$

This defines a K-class, $[a] \in K^{-1}(X \times \mathbb{S}^{2n-1})$. If we had the Künneth formula at our disposal, which we do not, we would know that $K^{-1}(X \times \mathbb{S}^{2n-1}) \equiv K^{-1}(X) \otimes K^0(\mathbb{S}^{2n-1}) \oplus K^0(X) \otimes K^{-1}(\mathbb{S}^{2n-1})$, where $K^0(X)$ is the soon-to-be-introduced group based on vector bundles, or projections. Now, both K-groups of the sphere are $\mathbb{Z}$ so this means that $K^{-1}(X \times \mathbb{S}^{2n-1}) \equiv K^{-1}(X) \oplus K^0(X)$. This can be understood more directly here in terms of two maps

$$\mathrm{K}^{-1}(X) \xleftarrow{S^*} \mathrm{K}^{-1}(X \times \mathbb{S}^{2n-1}) \xrightarrow{\mathrm{cl}} \mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}^{2n}). \tag{13.2}$$

The map on the left is just pull-back by choosing a point, say the South Pole, on the sphere. The map on the right is a version of the 'clutching construction' which in this context just means a map made explicitly with matrices which turns an isomorphism into a bundle. The maps in (13.2) are each isomorphisms when restricted to the null space of the other, so the K-space splits as indicated.

Now, the elliptic family of symbols can be quantized to a family of operators which are not only Fredholm but have constant rank null spaces. The null spaces then form a bundle over X as do the null spaces of the adjoints and the formal difference of these (we will get to this next week) define an element of $K^0(X)$; this is the index (in K-theory) and it only depends on the class of the symbol $[a]$. This gives us the little diagram

(13.3)
$$\begin{tikzcd} K^{-1}(X) \\ K^{-1}(X\times\mathbb{S}^{2n-1}) \arrow[u, "S^*"'] \arrow[d, "\mathrm{ind}_{\mathrm{iso}}"] \\ K^0(X). \end{tikzcd}$$

where I put in the upward map because one consequence of the discussion below is that the null space of this isotropic index map is the same as the map on the right in (13.2). Now we can add the clutching construction above and another variant of the clutching construction both above and below to get a bigger diagram

(13.4)
$$\begin{tikzcd} K^{-1}(X) \\ K^{-1}(X\times\mathbb{S}^{2n-1}) \arrow[u, "S^*"] \arrow[r, "\mathrm{cl}"] \arrow[d, "\mathrm{ind}_{\mathrm{iso}}"'] & \mathrm{K}^0_{\mathrm{c}}(X\times\mathbb{R}^{2n}) \arrow[r, "\mathrm{cl}", "\simeq"'] \arrow[dl, "p^{\mathrm{even}}_{\mathrm{sl}}"', "\simeq"] & \mathrm{K}^1_{\mathrm{c}}(X\times\mathbb{R}^{2n+1}) \arrow[dl, "p^{\mathrm{odd}}_{\mathrm{sl}}"', "\simeq"] \\ K^0(X) \arrow[r, "\mathrm{cl}", "\simeq"'] & \mathrm{K}^{-1}_{\mathrm{c}}(X\times\mathbb{R}) \end{tikzcd}$$

where I have added in two more maps. Namely, the 'odd' semiclassical Bott periodicity map – on the far right – that we are currently discussing and its even brother in the middle that we will soon get to. The $\simeq$'s indicate isomorphisms.

So, there is your index theorem. The main claim is that this diagram commutes, so the index for isotropic operators is equal to the product going around the right. In this context the Bott periodicity maps are 'topological' and the index map is 'analytic'. Of course the semiclassical definition makes the periodicity maps rather analytic too, but that is one thing I am trying to get at! So, how to prove it? The Atiyah-Singer approach was to give enough properties of these maps that they forced into uniqueness, the general principle being that if you have a natural construction – so it is universal in X – and it is non-trivial and has a few more properties then there is only one possibility. The proof I will give later is more analytic, as you might guess. Basically we can deform the isotropic pseudodifferential operators, following the clutching construction, into families of projections valued in smoothing operators and then into the group of invertible perturbations by smoothing operators – and this corresponds precisely to the three maps along the top.

To make the picture more symmetric I can add in the odd version of the isotropic index theorem, for elliptic self-adjoint operators or suspended operators, and get a bigger commutative diagram:

(13.5)

$$\begin{tikzcd}
\mathrm{K}^{-1}(X) & & & \mathrm{K}_{\mathrm{c}}^{-1}(X\times\mathbb{R}) \\
\mathrm{K}^{-1}(X\times\mathbb{S}^{2n-1}) \arrow[u, "S^*"] \arrow[r, "\mathrm{cl}"] \arrow[dr, "\mathrm{ind}_{\mathrm{iso}}^{\mathrm{even}}"'] & \mathrm{K}_{\mathrm{c}}^{0}(X\times\mathbb{R}^{2n}) \arrow[r, "\mathrm{cl}", "\simeq"'] \arrow[d, "p_{\mathrm{sl}}^{\mathrm{even}}", "\simeq"'] & \mathrm{K}_{\mathrm{c}}^{1}(X\times\mathbb{R}^{2n+1}) \arrow[d, "p_{\mathrm{sl}}^{\mathrm{odd}}"', "\simeq"] & \mathrm{K}_{\mathrm{c}}^{-1}(X\times\mathbb{R}\times\mathbb{S}^{2n-1}) \arrow[u, "S^*"'] \arrow[l, "\mathrm{cl}"'] \arrow[dl, "\mathrm{ind}_{\mathrm{iso}}^{\mathrm{odd}}"] \\
& \mathrm{K}^{0}(X) \arrow[r, "\mathrm{cl}", "\simeq"'] & \mathrm{K}_{\mathrm{c}}^{-1}(X\times\mathbb{R}) &
\end{tikzcd}$$

What I am really after in the course is not only to do these things, and of course the geometric versions of them which include the Atiyah-Singer theorem, but also to do it in such a way as to carry the Chern charcter along. Diagrams such as (13.5) need to be 'subsumed' into a smooth K-theory.

Jesse, does this start to answer your question?

Just so that we don't get too lost, let me very briefly outline the proof of Theorem 2 – which I should finish on Monday.

(1) The group $G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)$ is open in $\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d;\mathbb{R}^k)$: – I have not given the appropriate product estimates in this algebra. Instead I show for perturbations near 0 in the algebra the operators on L^2 for each $\epsilon\in(0,1]$ are invertible and then show that the inverse is in the group.

(2) The map $\sigma_{\mathrm{ad}}: G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)\longrightarrow G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$ is surjective and any compact map into the image lifts:- Invertibility of the adiabatic symbol implies invertibility of the operator for small $\epsilon>0$ and uniformly on compact sets. Modify the family in $\epsilon>0$ to get the lifting property.

(3) The subgroup $\{A\in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k);\sigma_{\mathrm{ad}}(A)=0\}$ is weakly contractible:- Show that on compact sets one can 'cut the family off' near $\epsilon=0$ preserving invertibility. Then we are reduced to contractibility of the half-free loop group shown earlier.

(4) The map $R: G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)\longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})$, given by restriction to $\epsilon=1$, is surjective and any compact map into the image lifts:- This is the most involved part. The main thing to show is that the semiclassical quantization of the Bott element, introduced last week, is a rank one perturbation of the matrix projection at infinity, and so can be deformed to

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}+\Pi_1\begin{pmatrix}0&0\\0&1\end{pmatrix} \tag{13.6}$$

where P_1 is the projection onto the group state of the harmonic oscillator. Then it follows that the element

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}+g(x)\Pi_1\begin{pmatrix}0&0\\0&1\end{pmatrix}+(1-\Pi_1)\begin{pmatrix}0&0\\0&1\end{pmatrix} \tag{13.7}$$

is in the image, where everything has been tensored by $M(N,\mathbb{C})$ and g is an arbitrary map $X\longrightarrow\mathrm{GL}(N,\mathbb{C})$ of compact support. However, any element in the image space is homotopy to one of these, so we have the lifting property.

(5) The subgroup $\{A \in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k); R(A) = \mathrm{Id}\}$ is weakly contractible:- This is where we use Atiyah's clever rotation, and this follows rather miraculously from the previous step it.

So, to work. Let me go through the simpler parts of the proof of Theorem 2 first, probably leaving the last step until Monday. For convenience I will break the result up into pieces and I will likely not go through the 'easier' part in as much detail in the lecture as in the notes.

Lemma 13. *The group $G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d; \mathbb{R}^k)$ is open in $\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d; \mathbb{R}^k)$.*

Proof. There may be a more direct proof than the one I will give here – if you find one please let me know! Since we are in a group we know that the issue is only the invertibility of $\mathrm{Id} + A$ where A lies in some small metric ball around the origin. As we know this just means that for one of the norms on $\mathcal{C}^\infty([0,1]; \mathcal{S}(\mathbb{R}^{2d+2k})$, and for some $\epsilon > 0$, $\|A\|_{(N)} < \epsilon$ implies the existence of $B \in \Psi^{-\infty}_{\mathrm{ad,sl}}(\mathbb{R}^d; \mathbb{R}^k)$ such that $(\mathrm{Id} + B) = (\mathrm{Id} + A)^{-1}$. We will get this by using the 'old-fashioned' method of invertibility acting on $L^2(\mathbb{R}^{k+d})$.

Thus, we first need to show that each element of $\Psi^{-\infty}_{\mathrm{ad,sl}}(\mathbb{R}^d; \mathbb{R}^k)$ defines a uniformly bounded operator on L^2 for $\epsilon \in (0,1]$. Note that ϵ is a parameter so the only problem is at $\epsilon = 0$ were the operator blows up. Just to make sure there is no confusion, we are considering A_ϵ as an operator on say $L^2(\mathbb{R}^{d+k})$ through the usual integral formula

$$(A_\epsilon u)(z, Z) = \epsilon^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^k} A(\epsilon, \frac{\epsilon(z+z')}{2}, \frac{z-z'}{\epsilon}, Z, Z') u(\epsilon, z', Z') dz' dZ' \tag{13.8}$$

and we want to get a uniform estimate on the L^2 norm as $\epsilon \downarrow 0$. This follows from Schur's lemma (not the one in representation theory of course) which says that the norm satisfies

$$\|A_\epsilon\|^2_{L^2} \le \sup_{z,Z} \int_{\mathbb{R}^{d+k}} |a(\epsilon, z, z', Z, Z')| dz' dZ' \times \sup_{z', Z'} \int_{\mathbb{R}^{d+k}} |a(\epsilon, z, z', Z, Z')| dz dZ, \tag{13.9}$$

assuming I have not missed out a constant. So, we just need to show that the right side is small if some norm on A is small. There is symmetry between the two terms so it suffices to consider the first and to see that

$$\begin{aligned} &\sup_{z,Z} \int_{\mathbb{R}^{d+k}} |a(\epsilon, z, z', Z, Z')| dz' dZ' \\ &= \sup_{z,Z} \int_{\mathbb{R}^{d+k}} \epsilon^{-d} |A(\epsilon, \frac{\epsilon(z+z')}{2}, \frac{z-z'}{\epsilon}, Z, Z')| dz' dZ' \\ &\sup_{z,Z} \int_{\mathbb{R}^{d+k}} |A(\epsilon, -\epsilon^2 s/2 + \epsilon z, s, Z, Z')| ds dZ' \le C \|A\|_{(N)} \end{aligned} \tag{13.10}$$

where we have just made the change of variable of integration from z' to $s = (z - z')/\epsilon$. Here N is just large enough to ensure convergence of the integrals.

So, from this it follows that if $\|A\|_{(N)} < \epsilon$ for some $\epsilon > 0$ then the family $\mathrm{Id} + A_\epsilon$ has an inverse as operators uniformly on L^2, $\mathrm{Id} + B_\epsilon$, where B_ϵ is small with A. So, it remains to show that this inverse actually comes from an element

$B \in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k)$. To do this we just need to construct the inverse near $\epsilon = 0$ since we already know what happens for $\epsilon > \epsilon_0 > 0$. The adiabatic symbol,

$$\sigma_{\mathrm{ad,iso}}(A) \in \Psi^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^d) \tag{13.11}$$

is small with $\|A\|_{(N)}$ and hence

$$(\mathrm{Id} + \sigma_{\mathrm{ad,iso}}(A))^{-1} = \mathrm{Id} + \sigma_{\mathrm{ad,iso}}(B) \text{ in } G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^d). \tag{13.12}$$

Now we can choose $B_0 \in \Psi^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^d)$ with this symbol, as usual, and we only need to invert

$$(\mathrm{Id} + B_0)(\mathrm{Id} + A) = \mathrm{Id} + \epsilon A'. \tag{13.13}$$

To do this we can use Neumann series to remove the Taylor series at $\epsilon = 0$:

$$\sum_j (-1)^j \epsilon^j (A')^j. \tag{13.14}$$

Sum this series using Borel's lemma and then we are back to a trivial case $\mathrm{Id} + A''$ where $A'' \in \epsilon^\infty \mathcal{C}^\infty([0,1]; \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})$, which is automatically invertible for small ϵ with inverse of the same type. Of course, we could have done this from the start. However, summing the Taylor series invovles norms of all orders and the problem is uniformity. However, we already know the existence of the L^2 inverse uniformly down to $\epsilon = 0$. Here we have shown that this inverse, being unique, is in fact an element of $G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k)$ without directly getting a bound on B.

This might make one wonder about the continuity of the inverse map, $A \longrightarrow (\mathrm{Id} + A)^{-1} - \mathrm{Id}$. However, the construction above works uniformly on compact sets and so the sequential continuity of the map follows – and we are in a complete metric space so all is well. ☐

The last part of this proof is very close to proving the properties of the 'adiabatic' sequence.

Lemma 14. *The adiabatic symbol gives a surjective map*

$$\sigma_{\mathrm{ad,iso}} : G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) \longrightarrow G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k), \tag{13.15}$$

and any smooth map $X \longrightarrow G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$ on a manifold, reducing to the identity outside a compact set, can be lifted under (13.15) *and the elements mapping to* Id *under* (13.15) *form a weakly contractible subgroup.*

Proof. The argument in the proof of the Lemma above shows that if

$$b \in G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$$

then any element $B \in \mathrm{Id} + \Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k)$ which has $\sigma_{\mathrm{ad,iso}}(B) = b$ is invertible on some smaller interval $[0, \epsilon_0]$ which depends on the choice of B. However, ϵ is just a parameter so we can 'expand' it by choosing a diffeomorphism $[0, \epsilon_0] \longrightarrow [0, 1]$ which is the identity near 0. Thus in fact the adiabatic symbol map is surjective. The same argument works uniformly on compact sets gives the lifting property.

The uniquess of the lift, up to homotopy, is the weak contractibility of the kernel-group. That is, we need to show that a smooth map

$$f : X \longrightarrow \left\{ A \in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k); \sigma_{\mathrm{ad,iso}}(A) = \mathrm{Id} \right\} \tag{13.16}$$

is smoothly homotopic to the identity, where if X is not compact both the map and the homotopy are required to restrict to the identity outside some compact subset of X.

This looks very like the contractibility of the half-open loop group and it may be that there is a global retraction of a similar sort to used there. At the moment I do not know it, so we actually reduce to that case using the compactness of the supports. So, given a map as in (13.16) we can insert a cutoff, choosing

$$\rho \in \mathcal{C}^\infty([0,1]),\ \rho(\epsilon) = 1 \text{ in } \rho < \frac{1}{4},\ \rho(\epsilon) = 0 \text{ in } \epsilon > \frac{1}{2}, 0 \le \rho(\epsilon) \le 1 \tag{13.17}$$

and consider the family

$$f_t(x) = \operatorname{Id} + \epsilon t \rho(\epsilon/\delta) + \epsilon(1 - \rho(\epsilon/\delta))A(\epsilon, x),\ \ f(x) = \operatorname{Id} + \epsilon A(\epsilon, x). \tag{13.18}$$

The uniformity in the construction of inverses above (and the factor of ϵ) shows that $\delta > 0$ is chosen small enough then this is an homotopy in the group in (13.16). At $t = 1$ it is f and at $f = 0$ it is in the flat loop group, since it reduces to the identity near $\epsilon = 0$. The earlier contraction argument therefore allows it to be retracted to the identity. $\square$