12. Lecture 9: Adiabatic algebra and group Wednesday, 24 September

To carry through the argument for Bott periodicity that I have been edging towards, I have decided to take a slightly higher road than I initially intended. I hope this will actually be pretty clear but the first step is to throw together what we have done so far and work with an *adiabatic algebra* of smoothing operators. This is the same as the semiclassical algebra except that 'adiabatic' refers to a situation in which the 'semiclassical degeneration' occurs in only some of the variables. The name arises from Physics and refers to a formal motion which is so slow that the system remains in equilibrium. Here this just means that some of the variables become commutative. I will get to more geometric versions of this later in the semester. In fact we might as well jump into the higher dimensional case, which really makes very little difference.

Definition 4. A one-parameter family $A \in \mathcal{C}^\infty((0,1];\Psi^{-\infty}(\mathbb{R}^{d+k}))$ is *an adiabatic family* of smoothing operators, with respect to the the first d variables, if its Schwartz kernel is of the form

(12.1)
$$a_\epsilon(\epsilon,z,z',Z,Z') = \epsilon^{-d}F(\epsilon,\frac{\epsilon(z+z')}{2},\frac{z-z'}{\epsilon},Z,Z'),\ \ F \in \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2d+2k})).$$

So the case discussed up to this point corresponds to $d=1$ and $k=0$, although we allowed matrix values – which we could include here at only notational expense. If $k=0$ but $d>1$ we are in a higher dimensional semiclassical setting.

Proposition 12. *The adiabatic operators as in* (12.1) *form an algebra, denoted* $\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)$, *under operator compostion for* $\epsilon>0$.

Proof. The proof generalizes easily from the case above where $d=1$ and $k=0$. Let me give the defining isomorphism (12.1) a name:

$$\kappa:\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)\longrightarrow \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2d+2k})) \tag{12.2}$$

which we are pretty free to regard as an identification – indeed that is what I have been doing implicitly up to this point. What we showed when $d=1$ is that operator composition for $\epsilon>0$ induces a product which can be (corrected) generalized and written out explicitly:

(12.3)
$$H=\kappa(A\circ B)=$$
$$\int_{\mathbb{R}^d\times\mathbb{R}^k} F(\epsilon,t+\frac{\epsilon^2}{2}(r+\frac12 s),\frac12 s-r,Z,Z'')G(\epsilon,t+\frac{\epsilon^2}{2}(r-\frac12 s),r+\frac12 s,Z'',Z')dZ''$$
$$F=\kappa(A),\ G=\kappa(B).$$

Recall that this just arises by noting the relationship of the Schwartz kernel, a, of A and $F=\kappa(A)$:

$$\begin{gathered}a(\epsilon,z,z',Z,Z')=\epsilon^{-d}F(\epsilon,\frac{\epsilon(z+z')}{2},\frac{z-z'}{\epsilon},Z,Z'),\\ F(\epsilon,t,s,Z,Z')=\epsilon^d a(\epsilon,\epsilon^{-1}t+\frac{\epsilon}{2}s,\epsilon^{-1}t-\frac{\epsilon}{2}s,Z,Z'),\end{gathered}\tag{12.4}$$

substituting into the formula for the product and changing variable. The same estimates as before show that this product is indeed a continuous bilinear map

$$\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2d+2k}))\times\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2d+2k}))\longrightarrow\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2d+2k})).\tag{12.5}$$

$\square$

One thing I did not get to before is the extraction of the 'Moyel product' from the formula (12.4). Notice that (apart from the explicit dependence of $\kappa(A)$ on ϵ) ϵ only occurs through ϵ^2. Computing the Taylor series therefore gives

$$\hat{H}(\epsilon,t,\tau) \simeq \sum_{j=0}^{\infty} \epsilon^{2j} \sum_{|\alpha|+|\beta|=j} c_{\alpha,\beta}(\partial_t^\alpha \partial_\tau^\beta \hat{F}(\epsilon,t,\tau)) \circ (\partial_t^\beta \partial_\tau^\alpha \hat{G}(t,\tau)). \tag{12.6}$$

where I have not (yet) computed the coefficients properly. Here the product is just the product in the suspended algebra $\Psi^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$.

A little later I will need (only in the 1-dimensional case in fact) the two leading terms. The first leads to the product law for the adiabatic symbol

$$\begin{gathered} \hat{H}(\epsilon,t,\tau) = \hat{F}(\epsilon,t,\tau) \circ \hat{G}(\epsilon,t,\tau) \\ +\epsilon^2 \sum_{j=1}^{d} \big(\partial_{t_j} \hat{F}(\epsilon,t,\tau) \circ \partial_{\tau_j} \hat{G}(\epsilon,t,\tau) - \partial_{\tau_j} \hat{F}(\epsilon,t,\tau) \circ \partial_{t_j} \hat{G}(\epsilon,t,\tau)\big) + O(\epsilon^4) \\ \hat{H}(\epsilon,t,\tau,Z,Z') = \int_{\mathbb{R}^d} H(\epsilon,t,s,Z,Z') e^{-is\cdot\tau} ds. \end{gathered} \tag{12.7}$$

This implies the analogous symbolic property to 1-dimensional case:-

$$\begin{gathered} \sigma_{\mathrm{ad}} : \Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) \longrightarrow \Psi^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k), \\ \sigma_{\mathrm{ad}}(A)(t,\tilde{a}) = \int_{\mathbb{R}^d} \kappa(A)(0,t,s,Z,Z') e^{-is\cdot\tau} ds \end{gathered} \tag{12.8}$$

satisfies

$$\sigma_{\mathrm{ad}}(AB) = \sigma_{\mathrm{ad}}(A) \circ \sigma_{\mathrm{ad}}(B) \text{ in } \Psi^{-\infty}_{\mathrm{sus}(2),\mathrm{iso}}(\mathbb{R}^k). \tag{12.9}$$

This sum in (12.7) correspnd to the Poisson bracket, as it should! Of course I hardly need pause to say that σ_{ad} gives a short exact sequence

$$\epsilon\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) \hookrightarrow \Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) \xrightarrow{\sigma_{\mathrm{ad}}} \Psi^{-\infty}_{\mathrm{sus}(2),\mathrm{iso}}(\mathbb{R}^k). \tag{12.10}$$

Now, one thing I have been pushing, rather relentlessly, in these lectures so far is that one should take these sorts of algebras 'seriously'. In particular look at the corresponding group and see what you get. Let me do again what we did earlier, perhaps with a little more care. Namely the algebra $\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k)$ does not have a unit. So simply append a unit by taking the direct product and considering

$$\Psi^{-\infty,\dagger}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) = \mathbb{C} + \Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) \tag{12.11}$$

where the product is the obvious one and in particular, $\mathrm{Id} = 1 + 0$ is the unit. Less abstractly one can consider $\mathbb{C}$ as being the complex multiples of the identity as operators on $\mathcal{S}(\mathbb{R}^{d+k})$ depending trivially on the parameter ϵ. Then one can consider the group

(12.12)

$$G^{-\infty,\dagger}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) = \{A \in \Psi^{-\infty,\dagger}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k); \exists\, B \in \Psi^{-\infty,\dagger}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k),\ AB = BA = \mathrm{Id}\}.$$

In fact it follows that if $A = z\,\mathrm{Id} + A'$ is invertible in this sense then $z \in \mathbb{C}^*$ and $\mathrm{Id} + z^{-1}A'$ is invertible. Thus we really do not lose anything by considering the group of the type we have been considering all along:-

$$G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) = \{\mathrm{Id} + A' \in G^{-\infty,\dagger}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k)\} \hookrightarrow \Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k). \tag{12.13}$$

This is all just formal. What is important, as I indicated earlier, is that this group happens to be open in terms of the inclusion (12.13) and hence is a nice topological (and of course smooth) group. We need to check this, but in fact lots of amusing things happen here so let me list this more formally.

Theorem 2. *The inclusion in* (12.13) *is open and the two maps, the adiabatic symbol* (12.8) *and the restriction map*

$$R:\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)\xrightarrow{\big|_{\epsilon=1}}\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k}) \tag{12.14}$$

lead to a commutative diagram where the lower two maps are surjective, and admit compact lifting, and the upper two spaces are weakly contractible:

(12.15)

$$\begin{tikzcd}
\{A\in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k);\sigma_{ad}(A)=\mathrm{Id}\} \arrow[dr, hook] & & \{A\in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k);R(A)=\mathrm{Id}\} \arrow[dl, hook'] \\
& G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k) \arrow[dl, "R"'] \arrow[dr, "\sigma_{\mathrm{ad}}"] & \\
G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k}) & & G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k).
\end{tikzcd}$$

Recall that weak contractibility here means that for any smooth map from a compact manifold into the space there is an homotopy to a constant map – in this case taking the value Id. The diagonal sequences are therefore exact. Note that compact lifting would usually be stated, at least in the topological literature, in the form that these sequences are 'Serre fibrations'. It means precisely that if $f:X\longrightarrow G$ is a smooth map into one of the bottom two spaces, then it can be lifted to $\tilde{f}:X\longrightarrow G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)$ so that $R\tilde{f}=f$ or $\sigma_{\mathrm{ad}}\tilde{f}=f$ respectively. Of the five or so things to be proved here, three are reasonably straightforward and the remaining part, amounting to the (Serre) exactness of the 'R' sequence, depends heavily on the construction I did last time. I will postpone the proof, probably until next time.

Remark 1. (Frédéric Rochon) The two diagonal sequences in (12.15) are in fact fibrations, not just Serre fibrations. So, if you know a little topology, the Serre lifting condition does in fact follows from the surjectivity. I will prove it directly anyway but this observation makes it clear why the proof of the lifting condition is no harder than the proof of surjectivity!

So, suppose we have managed to prove the theorem, then what? Basically it amounts to a weak homotopy equivalence between the bottom two spaces. That is, the diagram induces a map, which is an isomorphism,

$$p_{\mathrm{ad}}:[X;G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)]\xrightarrow{\simeq}[X;G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})] \tag{12.16}$$

for any compact manifold X. Namely, take a smooth map $f:X\longrightarrow G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$. The 'Serre property' asserts that it can be lifted to $\tilde{f}:G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)$ so the map in (12.16) is supposed to be induced by

$$[f]\longmapsto[R\tilde{f}]. \tag{12.17}$$

Of course, we need to check that this is well-defined. For fixed f two liftings $\tilde{f}(1)$ and $\tilde{f}(2)$ are such that $F = \tilde{f}(2)^{-1}\tilde{f}(1) : X \longrightarrow$ satisfies $\sigma_{\mathrm{ad}}(F) \equiv \mathrm{Id}$. So the stated weak contractibility in the Theorem implies that this is homotopic to the constant identiy map – hence $\tilde{f}(1)$ and $\tilde{f}(2)$ are homotopic. It follows that $R\tilde{f}(1)$ and $R\tilde{f}(2)$ are homotopic so the image class in (12.17) is well-defined given f. On the other hand if f_0 and f_1 are homotopic, so represent the same class $[f]$ on the left, then an homotopy $F : [0,1] \times X \longrightarrow G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$ can also be lifted and shows that the resulting image classes are the same. Thus (12.17) does lead to a well-defined map (12.16). Of course, the argument is reversible in the sense that there is a similar map defined the other way. These two maps are then inverses of each other. Recalling that we have defined

$$K^{-1-2d}(X) = [X; G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)] \ \forall \ d \geq 0, \tag{12.18}$$

with the result indepedent of the choice of k we conclude:

Corollary 1 (Bott periodicity). *For any compact manifold semiclassical quantization induces an isomorphism for any d :*

$$p_{\mathrm{ad}} : K^{-1-2d}(X) \longrightarrow K^{-1}(X). \tag{12.19}$$

In fact the Theorem and the isomorphism (12.19) extends to the case of non-compact manifolds X. We just need to consider the 'homotopy groups of maps with compact support'

$$\begin{aligned} \mathrm{K}_{\mathrm{c}}^{-1-j}(X) &= [X; G^{-\infty}_{\mathrm{sus}(j),\mathrm{iso}}(\mathbb{R})]_{\mathrm{c}} \\ &= \Big\{ f : X \longrightarrow G^{-\infty}_{\mathrm{sus}(j),\mathrm{iso}}(\mathbb{R}); f(x) = \mathrm{Id},\ x \in X \setminus K,\ K \Subset X \Big\} / \sim \end{aligned} \tag{12.20}$$

where the equivalence relation is through homotopies also reducing to the identity outside some compact subset. Then (12.19) extends to

$$p_{\mathrm{ad}} : \mathrm{K}_{\mathrm{c}}^{-1-2d}(X) \longrightarrow \mathrm{K}_{\mathrm{c}}^{-1}(X) \tag{12.21}$$

In fact,

Lemma 12.

$$\mathrm{K}_{c}^{-1-j}(X) \equiv \mathrm{K}_{c}^{-1}(X \times \mathbb{R}^j) \ \forall \ j \geq 0. \tag{12.22}$$

Proof. Left as an exercise, but said in brief as follows. Schwartz functions can always be approximated by functions of compact support. □

There are many ways to rewrite these isomorphism including the form of Bott periodicity mentioned earlier.

Corollary 2.

$$\pi_j(G^{-\infty}) = \begin{cases} \{0\} & j \ \textit{even} \\ \mathbb{Z} & j \ \textit{odd} \end{cases} \tag{12.23}$$

Proof. Assuming we know that $G^{-\infty}$ is connected and that $\pi_1(G^{-\infty}) = \mathbb{Z}$ then we just note that

$$\begin{gathered} \pi_{2j}(G^{-\infty}) = K^{-1-2j}(\mathrm{pt}) = K^{-1}(\mathrm{pt}) = \{0\}, \\ \pi_{2j+1}(G^{-\infty}) = K^{-1-2j-1}(\mathrm{pt}) = K^{-2}(\mathrm{pt}) = \pi_1(G^{-\infty}) = \mathbb{Z}. \end{gathered} \tag{12.24}$$

□