10. Lecture 7: Semiclassical quantization Friday, 19 September, 2008

We will show that there is a way to directly 'pass from' $\mathcal{S}(\mathbb{R}^2)$ with its commutative product to $\Psi^{-\infty}_{\text{iso}}(\mathbb{R})$, which is the same space with the operator product. In fact this will 'work' much more generally, but it should be understood at the outset that this is not a map. It does define maps at various levels but 'semiclassical quantization' in this sense is not itself a map.

Going back to the definition of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R})$ recall that I just defined this directly in terms of the operator product. We have already discussed smooth families of such operators. For one parameter families this just corresponds to $\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}))$. We will give this space another family of products in which the product depends on the parameter. Namely the 'semiclassical operator product' is initially only defined for $\epsilon > 0$ since the integrals look singular.

I will try to motivate it after writing it down. First let me change coordinates, to 'Weyl coordinates' on $\mathbb{R}^2$ which emphasize the diagonal

$$f(z,z') = F(\frac{1}{2}(z+z'), z-z'),\ F = Wf,\ F(t,s) = f(t+\frac{1}{2}s, t-\frac{1}{2}s). \tag{10.1}$$

Clearly, W is a linear isomorphism of $\mathcal{S}(\mathbb{R}^2)$.

Definition 3. A smooth family $A \in \mathcal{C}^\infty((0,1];\Psi^{-\infty}_{\text{iso}}(\mathbb{R}))$ is said to be a semiclassical family of smoothing operators if it is of the form

$$A_\epsilon u(z) = \epsilon^{-1}\int_{\mathbb{R}} F(\epsilon, \frac{\epsilon}{2}(z+z'), \frac{z-z'}{\epsilon})u(z')dz' \text{ with } F \in \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2)). \tag{10.2}$$

Note that as an operator A_ϵ only makes sense for $\epsilon > 0$ but the kernel function F is required to be smooth down to $\epsilon = 0$, thus the singularity at $\epsilon = 0$ is of a very particular kind. Since F is determined by its restriction to $\epsilon > 0$, it is actually determined by the family of operators A_ϵ.

Lemma 8. *If A and B are semiclassical families of smoothing operators then so is the composite $A_\epsilon \circ B_\epsilon$.*

Proof. Suppose A corresponds to the kernel function $F \in \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2))$ and B to G in the sense of the definition above. The composite operator, for each $\epsilon > 0$ has kernel in the ordinary sense

$$\begin{gathered}(A_\epsilon \circ B_\epsilon)u(z) = \int_{\mathbb{R}} c(z,z')u(z')dz', \\ c(\epsilon,z,z') = \epsilon^{-2}\int_{\mathbb{R}} F(\epsilon, \frac{\epsilon}{2}(z+z''), \frac{z-z''}{\epsilon})G(\epsilon, \frac{\epsilon}{2}(z''+z'), \frac{z''-z'}{\epsilon})dz''.\end{gathered} \tag{10.3}$$

Thus the kernel function defined, H, defined from (10.2) by c is

$$\begin{gathered}H(\epsilon,t,s) = \epsilon c(\epsilon, \epsilon^{-1}t + \frac{\epsilon}{2}s, \epsilon^{-1}t - \frac{\epsilon}{2}s) = \\ \epsilon^{-1}\int_{\mathbb{R}} F(\epsilon, \frac{t}{2} + \frac{\epsilon^2}{4}s + \frac{\epsilon}{2}z'', \epsilon^{-2}t + \frac{1}{2}s - \frac{z''}{\epsilon})G(\epsilon, \frac{\epsilon}{2}z'' + \frac{1}{2}t - \epsilon^2\frac{s}{4}, \frac{z''}{\epsilon} - \epsilon^{-2}t + \frac{1}{2}s)dz''.\end{gathered} \tag{10.4}$$

Changing variable of integration $z'' = \epsilon r + \epsilon^{-1}t$ this reduces to

$$H(\epsilon,t,s) = \int_{\mathbb{R}} F(\epsilon, t + \frac{\epsilon^2}{2}(r+\frac{1}{2}s), \frac{1}{2}s - r)G(\epsilon, t + \frac{\epsilon^2}{2}(r - \frac{1}{2}s), r + \frac{1}{2}s)dr. \tag{10.5}$$

The absolute convergence, and rapid decay of the result, is clear for $\epsilon > 0$ and for ϵ small follows uniformly from fact that the integrand is bounded by

$$
\begin{aligned}
(10.6)\quad C_N(1+|t+\epsilon^2(r+s/2|)^{-N}(1+|r-s/2|)^{-N}(1+|t+\frac{\epsilon^2}{2}(r-s/2)|)^{-N} \\
\times (1+|r+s/2|)^{-N} \\
\le C'(1+|t|)^{-N}(1+|r|)^{-N}(1+|s|)^{-N}.
\end{aligned}
$$

Derivatives can be estimated in the same way. Thus (10.6) defines a continuous bilinear map

$$
\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2)) \times \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2)) \longrightarrow \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2)). \tag{10.7}
$$

This shows that the semiclassical smoothing operators form an algebra. □

I will denote this algebra as $\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R})$, with the parameter suppressed into a suffix – remember it is by no means simply a smooth parameter as $\epsilon \downarrow 0$.

Notice what the product looks like at $\epsilon = 0$. The limiting rescaled kernel of the product is simply

$$
H(0,t,s) = \int_{\mathbb{R}} F(0,t,\frac{1}{2}s - r)G(0,t,r+\frac{1}{2}s)dr. \tag{10.8}
$$

This is a product on $\mathcal{S}(\mathbb{R}^2)$ so we have found another one! However, notice that it is commutative – changing variable from r to $-r$ effectively reverses the product. Not surprisingly this product can actually be reduced to the usual pointwise product on $\mathcal{S}(\mathbb{R}^2)$ by the simple expedient of taking the Fourier transform in s. For any semiclassical family, (10.2), we define the *semiclassical symbol* to be the Fourier transform of the limit at $\epsilon = 0$:

$$
\sigma_{\mathrm{sl}}(A)(t,\tau) = \int_{\mathbb{R}} F(0,t,s)e^{-is\tau}ds. \tag{10.9}
$$

Then for the product

$$
\sigma_{\mathrm{sl}}(AB)(t,\tau) = \int_{\mathbb{R}} H(0,t,s)e^{-is\tau}ds \tag{10.10}
$$

and then (10.8) becomes

$$
\begin{aligned}
(10.11)\quad \sigma_{\mathrm{sl}}(AB)(t,\tau) = \\
\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i(\frac{1}{2}s-r)\tau}F(0,t,\frac{1}{2}s-r)e^{-i(\frac{1}{2}s+r)\tau}G(0,t,r+\frac{1}{2}s)drds = \sigma_{\mathrm{sl}}(A)\sigma_{\mathrm{sl}}(B).
\end{aligned}
$$

Proposition 11. *The algebra of semiclassical smoothing operators with symbol homomorphism to the commutative algebra $\mathcal{S}(\mathbb{R}^2)$ gives a short exact (and multiplicative) sequence*

$$
0 \longrightarrow \epsilon\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R}) \hookrightarrow \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R}) \twoheadrightarrow \mathcal{S}(\mathbb{R}^2) \longrightarrow 0. \tag{10.12}
$$

So, how do we get our (weak) homotopy equivalence? We simply 'turn on' the non-commutativity.

Exercise 5. (Will be done on Monday). Show that if $a \in \mathcal{S}(\mathbb{R}^2; M(N,\mathbb{C}))$ is such that $(\mathrm{Id} + a(t,\tau))^{-1}$ exists for all $(t,\tau) \in \mathbb{R}^2$ then if $A \in \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^N)$ is a (matrix-valued) semiclassical family with $\sigma_{\mathrm{sl}}(A) = a$ (which exists by (10.12)) then $\mathrm{Id} + A_\epsilon \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N)$ for $\epsilon > 0$ small. This works uniformly on compact sets, so if $f : X \longrightarrow$

$\mathcal{S}(\mathbb{R}^2; M(N,\mathbb{C})) \hookrightarrow G^{-\infty}_{\mathrm{sus}(2)}$ then the quantized map $f_\epsilon : X \longrightarrow G^{-\infty}$ (where these are different realizations of $G^{-\infty}$) for $\epsilon > 0$ small, is well defined up to homotopy. This leads to the homotopy equivalence

$$Q_{\mathrm{sl}} : \pi_j(G^{-\infty}_{\mathrm{sus}(2)}) \longrightarrow \pi_j(G^{-\infty}) \ \forall \ j, \tag{10.13}$$

which will prove Bott periodicity for us (with a bit more work).

Exercise 6. Consider the differential operators on $\mathbb{R}$ with polynomial coefficients

$$P = \sum_{k,j=0}^{N} c_{kj} x^k D_x^j, \ D_x = -i\frac{d}{dx}. \tag{10.14}$$

Give x and D_x 'homogeneity one' and so filter these operators by the combined order – this is the isotropic filtration.

Now, show that if A is a semiclassical family of smoothing operators then so is $\epsilon^N PA$ if P has total order N in this sense. Compute the semiclassical symbol of $\epsilon^N PA$.

Exercise 7. Show that the definition of semiclassical families of smoothing operators extends directly to operators on $\mathbb{R}^n$ simply by reinterpretation of the formulæ.

Exercise 8. In preparation for what I will do on Monday, if A and B are semiclassical smoothing families as defined above, we have shown that the function $H \in \mathcal{C}^\infty([0,1]_\epsilon; \mathcal{S}(\mathbb{R}^2))$ fixing its kernel is determined by the corresponding functions F and G for A and B. Show that the Taylor series of H at $\epsilon = 0$ is determined by the Taylor series of A and B and derive a formula for it – you will get a variant of the 'Moyal product' (although several diffent things go under this name). The most important thing for us is the second term in the expansion (well, given that we already know the first term!)