TOPICS IN ANALYSIS
SMOOTH OPERATOR ALGEBRAS AND K-THEORY

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0.5L; Revised: 6-2-2009; Run: March 17, 2009

Abstract. These are ongoing notes for Lectures on MWF in Evans 45 for the course Math 276-2: Topics in Analyis. I will add here such notes as I manage to write up as I go along and some references.

Contents

Preface

I would like to thank the members of the audience for this course, especially those who survived at least nearly to the (I hope not too bitter) end:

Kiril Datchev
Boris Ettinger
Jesse Gell-Redman
Baoping Liu
Paul Loya
Raphaël Ponge
Frédéric Rochon
Fang Wang
Maciej Zworksi

I would especially like to thank Kiril, Jesse and Paul who asked excellent questions and also worked on improving and completing various proofs – which are incorporated into the manuscript as you will see. Most significantly I would like to thank Frédéric Rochon; a considerable part of this manuscript is based on joint work with him and I probably should have spent more of the time that went into writing up these notes on completing various manuscripts that we have 'In preparation'.

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Additional topics

- Geometric forms of determinant, determinant bundle and gerbe.
- Hopkins and Singer?
- Dirac and Bunke.
- Segal's classifying spaces
- Loop groups and representations.
- Primitive determinant bundle over whole of loop group.
- Higher Gerbes.

Introduction

In this course I plan to describe aspects of smooth K-theory.

I will start with a discussion of the algebra of smoothing operators in its various forms and properties including finite-dimensional approximation, the Fredholm determinant, group of invertible perturbations of the identity and hence to definitions of odd and even K-theory.

Subsequently I will discuss:-

1. The loop group and delooping sequence and the Chern character.

2. Semiclassical quantization and Bott periodicity. Thom isomorphism and Atiyah-Singer theorem.

3. The quantization (looping) sequence and Quillen's line bundle.

4. Segal's representation of the loop group and the K-theory gerbe.

As time (and the enthusiasm of the audience) permits I will discuss twisting of K-theory and bordism.

1. Lecture 1: The smoothing group Wednesday, 27 August, 2008

My first goal is to introduce the infinite-dimensional but 'smooth' group, $G^{-\infty}$. This is a classifying space for odd K-theory and is central to the content of these lectures. This basic classifying group comes in many different manifestations, more or less geometric. I will first start with some words of orientation, then I discuss the 'sequential', really least geometric, version of the underlying 'Schwartz' algebra and then the group. In subsequent lectures much smoother-looking geometric versions of the algebra and group will appear, associated with (finite-dimensional) manifolds.

Complex K-theory, which is what will be discussed for the most part here, is closely connected with the algebras of $N \times N$ complex matrices $M(N, \mathbb{C})$ and more particularly the group of invertible matrices, $\mathrm{GL}(N, \mathbb{C})$ and the subgroup of unitary matrices $\mathrm{U}(N) \subset \mathrm{GL}(N, \mathbb{C})$.

I leave you to remind yourself of the basic properties of matrices, multiplication, determinant, invertibility, polar decomposition, retraction onto $\mathrm{U}(N)$ etc.

Now, the odd K-group of say a compact manifold $\mathrm{K}^1(X)$ can be defined in terms of all the smooth (or continuous) maps from X into $\mathrm{GL}(N, \mathbb{C})$ (or into $\mathrm{U}(N, \mathbb{C})$. One 'difficulty' inherent in this finite-dimensional approach to K-theory is that one needs to stabilize everything. That is, one has to consider the embedding of $\mathrm{GL}(N, \mathbb{C})$ in $\mathrm{GL}(N+1, \mathbb{C})$

$$\mathrm{GL}(N, \mathbb{C}) \ni A \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in \mathrm{GL}(N+1, \mathbb{C}). \tag{1.1}$$

Of course, this can be iterated to get $\mathrm{GL}(N, \mathbb{C}) \longmapsto \mathrm{GL}(N+M, \mathbb{C})$ by interpreting 1 as $\mathrm{Id} \in \mathrm{GL}(M, \mathbb{C})$. The need for these stabilization maps tends to make things intrisically non-smooth. Instead, the group $G^{-\infty}$ is an *a priori* stabilization which I want to take the time to discuss carefully – since it is so fundamental to what will follow.

So to the sequential version, which is a direct generalization of matrices but which I will not use directly later – although it is isomorphic to the more geometric versions that I will use, as we shall see. As basic space consider $\Psi^{-\infty}(\mathbb{N})$ where $\mathbb{N} = \{1, 2, 3, \dots\}$, which for the moment just means rapidly decreasing sequences

$$a : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{C}, \ \sup_{i,j} i^N j^N |a_{ij}| < \infty, \ \forall\ N, \tag{1.2}$$

where the map is written as a double sequence.

This is a rather standard Fréchet space – let me remind you about this. First, it is countably normed as is clear from the definition

$$\|a_{\bullet\bullet}\|_{(N)} = \sup_{i,j} i^N j^N |a_{ij}| \text{ are norms.} \tag{1.3}$$

Thus a subset is open if it contains an open ball around each of its points, with respect to one of the norms (depending on the point). The additional requirement for a Fréchet space is completeness. Indeed, $\Psi^{-\infty}(\mathbb{N})$ is a complete metric space with respect to the metric

$$d(a, b) = \sum_N 2^{-N} \frac{\|a-b\|_{(N)}}{1 + \|a-b\|_{(N)}}. \tag{1.4}$$

If you haven't seen this it is worth doing as an exercise – at least check that this is a metric and that the open sets with respect to it are the same as stated above.

In fact $\Psi^{-\infty}(\mathbb{N})$ is a *Montel space*, as are many related spaces. Namely it has the Heine-Borel property that every closed bounded set is compact. This is equivalent to the condition that any sequence which is bounded with respect to each of the seminorms is convergent – hence convergent with respect to eavh of the seminorms. This is straightforward to check, namely the boundedness with respect to $\|\cdot\|_{(N+1)}$ implies that the 'tails' of the sequence with respect to the $\|\cdot\|_{(N)}$ norm are equi-small and hence that it has a sequence which converges with respect to $\|\cdot\|_{(N)}$. A sequence which converges with respect to all the seminorms can then be found by diagonalization.

Of course the important point is that the standard matrix product extends to this space, so that there is a bilinear map

$$\Psi^{-\infty}(\mathbb{N})\times\Psi^{-\infty}(\mathbb{N})\longrightarrow\Psi^{-\infty}(\mathbb{N}),\ (a\circ b)_{ij}=\sum_{l=1}^{\infty}a_{il}b_{lj}\text{ is jointly continuous.}\tag{1.5}$$

The joint continuity of such a bilinear map reduces to estimates, for each N there exists N' and $C=C(N)$ such that

$$\|a\circ b\|_N\le C\|a\|_{N'}\|b\|_{N'}.\tag{1.6}$$

In fact it is enough to check this for large N since the norms increase with N. So take $N\ge 1$. Then the definitions of the norms imply that

$$\begin{aligned}|a_{il}|\le i^{-N}l^{-1}\|a\|_{(N)}\Longrightarrow&\\ |\sum_l a_{il}b_{lj}|&\le i^Nj^N\sum_l|a_{il}||b_{lj}|\\ &\le\|a\|_{(N)}\|b\|_{(N)}\sum_l l^{-2}\le C\|a\|_{(N)}\|b\|_{(N)}.\end{aligned}\tag{1.7}$$

Thus

$$\|ab\|_{(N)}\le C\|a\|_{(N)}\|b\|_{(N)}\ \forall\ N\ge 1.\tag{1.8}$$

Thus, $\Psi^{-\infty}(\mathbb{N})$ is a topological algebra.

As is well-known, the fact that the norms on the right in (1.6) are the same as the norm on the left is especially helpful – although it is not necessary for continuity. It has an important consequence for the unital extension of the algebra. That is, let us formally add an identity – since I haven't made these matrices act on anything yet, this is a formal identity. It really means changing the product on $\Psi^{-\infty}(\mathbb{N})$ so that it looks like $\operatorname{Id}+\Psi^{-\infty}(\mathbb{N})$ using the natural identity

$$(\operatorname{Id}+a)(\operatorname{Id}+b)=\operatorname{Id}+ab+a+b.\tag{1.9}$$

Clearly then it makes sense to ask that $\operatorname{Id}+a$ be invertible in the sense that

$$\exists\ b\in\Psi^{-\infty}(\mathbb{N})\text{ such that }(\operatorname{Id}+a)(\operatorname{Id}+b)=\operatorname{Id},\text{ i.e. }ab+a+b=0.\tag{1.10}$$

Definition 1. The group $G^{-\infty}(\mathbb{N})\subset\Psi^{-\infty}(\mathbb{N})$ consists of those elements a for which $\operatorname{Id}+a$ is invertible in the sense of (1.10).

Next time I shall prove at least part of the following

Proposition 1. *The group $G^{-\infty}(\mathbb{N})$ is an open dense subset of $\Psi^{-\infty}(\mathbb{N})$ in which the product and the map $a\longmapsto b=(\operatorname{Id}+a)^{-1}-\operatorname{Id}$ are continuous. There is an entire analytic function*

$$\Psi^{-\infty}(\mathbb{N})\ni a\longmapsto\det_{Fr}(a)=\det(\operatorname{Id}+a)\in\mathbb{C}\tag{1.11}$$

for which $G^{-\infty}$ is the complement of the null space.

The definition and properties of the Fredholm determinant will have to wait a little longer. Of course the main thing to observe here is how close this is to the finite dimensional case of $\mathrm{GL}(N, \mathbb{C}) \subset M(N, \mathbb{C})$ – the main difference is that in the finite-imensional case the algebra is unital.

The condition on a (non-unital) Fréchet algebra that the group of invertibles, in this case $G^{-\infty}(\mathbb{N})$, is an open subset of the Fréchet algebra, here $\Psi^{-\infty}(\mathbb{N})$, and in particular that $\mathrm{Id} + a$ is invertible for the elements of a small open ball around the origin, is often expressed by saying it is a 'good' Fréchet algebra. This seems so lame to me that I refuse to follow such usage. If necessary I will refer to this condition by saying the Fréchet algebra is 'Neumann-Fréchet' since it is at least the analogue of the convergence of the Neumann series for $(\mathrm{Id} + a)^{-1}$ when a is small (and generally can be proved precisely this way).

2. Lecture 2: Finite rank approximation
Friday, 29 August, 2008

From last time recall the definition of the sequential version of the 'smoothing group'

$$
\begin{aligned}
(2.1)\quad G^{-\infty}(\mathbb{N}) = \big\{a \in \Psi^{-\infty}(\mathbb{N}); \exists\ b \in \Psi^{-\infty}(\mathbb{N}) \text{ satisfying}\\
(\mathrm{Id}+a)(\mathrm{Id}+b) = \mathrm{Id}+a+b+ab = \mathrm{Id} = (\mathrm{Id}+b)(\mathrm{Id}+a)\big\}.
\end{aligned}
$$

It is not quite obvious here that the existence of the right inverse, the first identity, implies the existence of a left inverse as in the second identity, and the equality of the two. In fact this is true as we will check later, but for the moment we just require the existence of a two-sided inverse.

The main thing for today is to see that it satisfies variants of the 'obvious' properties of $\mathrm{GL}(N,\mathbb{C})$.

Proposition 2. *The group $G^{-\infty}(\mathbb{N})$ is an open, dense, (path) connected subset of $\Psi^{-\infty}(\mathbb{N})$ in which the product and the map $a \longmapsto b = (\mathrm{Id}+a)^{-1} - \mathrm{Id}$ are continuous.*

To prove these results we will use finite dimensional approximation, so really the same stabilization that was the reason for looking at a group like this in the first place. Let Π_k be the projection on the space $\Psi^{-\infty}(\mathbb{N})$ which 'cuts off the tails' after k terms:

$$
(2.2)\qquad (\Pi_k(a))_{ij} = \begin{cases} a_{ij} & \text{if } 1 \le i,j \le k \\ 0 & \text{if } i > k \text{ or } j > k. \end{cases}
$$

Clearly $\Pi_k : \Psi^{-\infty}(\mathbb{N}) \longrightarrow \Psi^{-\infty}(\mathbb{N})$ is linear and continuous – in fact it decreases each of the norms

$$
(2.3)\qquad \|\Pi_k a\|_{(N)} \le \|a\|_{(N)}
$$

and $\Pi_k^2 = \Pi_k$.

Proposition 3. *A set $K \subset \Psi^{-\infty}(\mathbb{N})$ is precompact (has compact closure) if and only if each of the norms $\|\bullet\|_{(N)}$ is bounded on K and on such a set*

$$
(2.4)\qquad \|\Pi_k a - a\|_{(N)} \to 0 \text{ uniformly as } k \to \infty\ \forall\ N.
$$

Proof. Note that the difference $a - \Pi_k a$ has all entries with $i,j \le k$ vanishing. Thus from the definitions of the norms,

$$
(2.5)\qquad \|a - \Pi_k a\|_{(N)} \le k^{-1}\|a\|_{(N+1)}
$$

since at least one of $i,j \ge k$ on all non-zero elements. This shows that (2.4) follows from the assumption that all the norms are bounded on K. This in turn implies sequential precompactness (which is precompactness for a metric space) of a set satisfying these conditions by the usual diagonalization process. That is, given a sequence $a(n)$ in K, $\Pi_k a(n)$ is in a bounded subset of a finite dimensional space, so we can extract successive subsequences such that each $\Pi_k a(n_k)$ converges. Passing to the diagonal subsequence and relabelling it as $a(n)$ it follows that we may assume that $\Pi_k a(n) \longrightarrow \Pi_k a$ for each k and some fixed double sequence a_{ij}. It follows from (2.5) that in fact $a \in \Psi^{-\infty}(\mathbb{N})$ and that $a(n)$ converges to it in $\Psi^{-\infty}(\mathbb{N})$.

The converse is similar, maybe a little easier, and anyway of less interest in what follows, so I leave it as an exercise. □

To prove that $G^{-\infty}(\mathbb{N})$ is open we will also use another property of $\Psi^{-\infty}(\mathbb{N})$.

Lemma 1. *The algebra $\Psi^{-\infty}(\mathbb{N})$ has the 'corner property' that for any a, b, $c \in \Psi^{-\infty}(\mathbb{N})$ and any N,*

$$\|abc\|_{(N)} \le C\|a\|_{(N)}\|b\|_{(1)}\|c\|_{(N)},\ N \ge 1. \tag{2.6}$$

Here C is actually independent of N but that is not really the point. As you will see when we get to the geometric realizations of this setup, (2.6) corresponds to the 'smoothing property' of these operators

Proof. This is just the same sort of estimate as before:

$$i^N j^n |(abc)_{ij}| \le \sum_{l,m} i^N |a_{il}||b_{lm}|j^N|c_{mj}| \le (\sum_{l,m} l^{-2}m^{-2})\|a\|_{(N)}\|b\|_{(1)}\|c\|_{(N)}. \tag{2.7}$$

□

Proof of Proposition 2. So, we want to show that for each point $a \in G^{-\infty}(\mathbb{N})$ there is an open ball centred at a with respect to one of the norms which is contained in $G^{-\infty}$. We will use a Neumann series argument. Clearly the group product is continuous, since it is $(a,b) \longmapsto a+b+ab$. Thus, if $\operatorname{Id}+a \in G^{-\infty}$ and U is a neighourhood of $\operatorname{Id} \in G^{-\infty}$ then $(\operatorname{Id}+a)U$ is a neighbourhood of $\operatorname{Id}+a$. Thus it suffices to show that

$$\{a \in \Psi^{-\infty}(\mathbb{N}); \|a\|_{(1)} < 1\} \subset G^{-\infty}(\mathbb{N}). \tag{2.8}$$

To see this consider the Neumann series for the inverse

$$(\operatorname{Id}+a)^{-1} = \operatorname{Id} + \sum_{j=1}^{\infty}(-1)^j a^j. \tag{2.9}$$

This is Cauchy with respect to the norm $\| \bullet \_(1)$ provided $\|a\|_{(1)} < 1$. Of course, to get (2.8) we need to show that it is Cauchy with respect to all the norms, since that implies that it is Cauchy with respect to the distance. This is where Lemma 1 comes in, since if $\|a\|_{(1)} = c < 1$ then from (2.6)

$$\|a^{j+2}\|_{(N)} \le C\|a\|^2_{(N)}c^j \tag{2.10}$$

which implies that the sequence is Cauchy with respect to each $\|\bullet\|_{(N)}$. Thus the sequence in (2.9) does indeed converge. The limit is a two-sided inverse to $\operatorname{Id}+a$.

The continuity of the inverse map follows from this argument and the continuity of the product is clear.

Next we want to show that $G^{-\infty}(\mathbb{N})$ is connected. Here we can use the finite dimensional approximation to good effect. Since we know that $\Pi_k a \to a$ as $k \to \infty$ and now that $G^{-\infty}(\mathbb{N})$ is open, it follows that $ta + (1-t)\Pi_k a \in G^{-\infty}(\mathbb{N})$ for $t \in [0,1]$ if k is large enough. Thus $\operatorname{Id}+\Pi_k a \in G^{-\infty}(\mathbb{N})$ is connected to a. From the uniqueness of the inverse in a group, $\operatorname{Id}_{k\times k}+\Pi_k a \in \operatorname{GL}(k,\mathbb{C})$ when thought of as a finite dimensional matrix. Here we are using the fact that we can embed $\operatorname{GL}(k,\mathbb{C})$ in $G^{-\infty}$ by subtracting the identity in $\operatorname{GL}(k,\mathbb{C})$ from it, extending the resulting matrix as zero for $i,j > k$ and then adding the formal identity to it afterwards.

So, the connectedness of $G^{-\infty}(\mathbb{N})$ follows from the connectedness of each of the $\operatorname{GL}(k,\mathbb{C})$ (well, we only need this for k large enough). This of course is well known. One way to see it is to use a little spectral theory. If $a \in \operatorname{GL}(k,\mathbb{C})$ then aa^* is positive definite, in particular is selfadjoint with positive eigenvalues, so has a positive square root and defining u by $a = (aa^*)^{\frac{1}{2}}u$ makes u unitary. Moreover the curve $t(a^*a)^{\frac12}+(1-t)\operatorname{Id}_{k\times k}$ connects the positive definite factor to the identity through positive, hence invertible, elements. Thus it is enough to show that $\mathrm{U}(k)$, the group of unitary matrices is connected. The spectral decomposition of u gives an orthonormal basis of eigenvectors on each of which u acts as $e^{i\theta}$ for some $\theta\in[0,2\pi)$. Rotating this to 1 on each eigenspace connects u to the identity. Thus each $\mathrm{U}(k)$ and hence $G^{-\infty}(\mathbb{N})$ is connected. $\square$

As for $k\times k$ matrices, it is nice to know that invertibility is determined by the non-vanishing of a 'character', which is to say a multiplicative map defined on $\Psi^{-\infty}(\mathbb{N})$ in the sense that

$$\det\left((\operatorname{Id}+a)(\operatorname{Id}_b)\right)=\det(\operatorname{Id}+a)\det(\operatorname{Id}+b). \tag{2.11}$$

This is often called the 'Fredholm determinant'.

Proposition 4. *There is an entire analytic function*

$$\Psi^{-\infty}(\mathbb{N})\ni a\longmapsto \det_{Fr}(a)=\det(\operatorname{Id}+a)\in\mathbb{C} \tag{2.12}$$

such that $G^{-\infty}$ is the complement of its null space and if $a=\Pi_k a$ then

$$\det_{Fr}(a)=\det(\operatorname{Id}+a)=\det(\operatorname{Id}_{k\times k}+\Pi_k a) \tag{2.13}$$

reduces to the usual determinant.

So the proof I have in mind is the first use I will make of differential forms on $G^{-\infty}$. There are other, possibly simpler, proofs but this one has the virtue of linking up with the Chern classes later on – in fact that is what we are discussing here, the first odd chern class.

I will therefore launch into a brief discussion of analysis on $G^{-\infty}(\mathbb{N})$. This is fairly straightforward since $G^{-\infty}(\mathbb{N})$ is open in $\Psi^{-\infty}(\mathbb{N})$; it is therefore a complete metric space, so we certainly know what continuity means. For differentiability I will take a strong definition – there are lots of possibilities on Fréchet manifolds but many of them coincide here. So, first note that as an open set of a linear space, the tangent space at any point can be identified with $\Psi^{-\infty}(\mathbb{N})$ itself. For a function $f:U\longrightarrow\mathbb{C}$ where $U\subset\Psi^{-\infty}(\mathbb{N})$ is open, to be differentiable at a we will require the existence of a continuous linear map $Df(a):\Psi^{-\infty}(\mathbb{N})\longrightarrow\mathbb{C}$ such that

$$\begin{gathered}f(a+b)-f(a)-Df(a)\cdot b=o(\|b\|_{(N)})\text{ for }N\text{ sufficiently large}\\ \Longleftrightarrow\\ \exists\, N\text{ such that }\forall\,\delta>0\ \exists\,\epsilon>0\text{ for which}\\ \|b\|_{(N)}<\epsilon\Longrightarrow\|f(a+b)-f(a)-Df(a)\cdot b\|_{(N)}\le\delta\|b\|_{(N)}.\end{gathered} \tag{2.14}$$

Note that if N is large enough and $\epsilon>0$ is small enough then $a+b\in U$ if $\|b\|_{(N)}<\epsilon$.

The special properties of $G^{-\infty}(\mathbb{N})$ allow us to require as part of the definition of once continuous differentiability, which or course requires differentiability at each point, that (2.14) hold everywhere with the same N and that the derivative

$$Df:U\times\Psi^{-\infty}(\mathbb{N})\longrightarrow\mathbb{C}\text{ be continuous with respect to }\|\bullet\|_{(N)} \tag{2.15}$$

on both factors. This does not make much sense unless U contains open $\|\bullet\|_{(N)}$-balls around each of its points – which of course is the case for $G^{-\infty}(\mathbb{N})$.

3. Lecture 3: K-groups and loop groups Wednesday, 3 September, 2008

Reconstructed, since I did not really have notes – because I was concentrating too hard on the 3 lectures on blow-up at MSRI!

(1) Odd K-theory
(2) Loop group
(3) Even K-theory
(4) Delooping sequence

Having defined the group $G^{-\infty}(\mathbb{N})$ and shown that it is open (and dense) in $\Psi^{-\infty}(\mathbb{N})$ we can define the odd K-theory of a space simply as the set of smooth equivalence classes of smooth maps. For the moment let us just consider a compact smooth manifold X then a map

$$f : X \longrightarrow G^{-\infty}(\mathbb{N}) \hookrightarrow \Psi^{-\infty}(\mathbb{N}) \tag{3.1}$$

is smooth if it is differentiable to infinite order. For a map into a fixed topological vector space this is quite a simple condition. Namely forget for that X is compact, then we certainly know what a continous map is. Differentiability at a point $\bar{x} \in X$ is the existence of the derivative, which is to be a continuous linear map,

$$Df(\bar{x}) : T_{\bar{x}}X \longrightarrow \Psi^{-\infty}(\mathbb{N}) \tag{3.2}$$

such that in local coordinates near $\bar{x}$, for given $\delta > 0$ there exists $\epsilon > 0$ such that

$$\|f(x) - f(\bar{x}) - Df(\bar{x})(x - \bar{x})\|_{(N)} \le \delta|x - \bar{x}| \text{ in } |x - \bar{x}| < \epsilon. \tag{3.3}$$

Then we require that $Df(\bar{x})$ exists everywhere so defines a map

$$Df : TX \longrightarrow \Psi^{-\infty}(\mathbb{N}) \tag{3.4}$$

which we then require to be continuous and differentiable. Proceeding inductively we can require the existence of higher derivatives by the same procedure, where differentiablilty in the linear variables is trivially true. Thus the kth derivative is required to be a map

$$D^kF(\bar{x}) : T_{\bar{x}}X \times T_{\bar{x}}X \cdots \times T_{\bar{x}}X \longrightarrow \Psi^{-\infty}(\mathbb{N}) \tag{3.5}$$

for each point $\bar{x} \in X$ which is the derivative of the $k - 1$st derivative and which is continuous in all variables.

Examples are immediately provided by smooth maps $X \longrightarrow \mathrm{GL}(N, \mathbb{C})$ in the usual finite-dimensional sense, for any N because of the smooth inclusion

$$\mathrm{GL}(N, \mathbb{C}) \longrightarrow G^{-\infty}(\mathbb{N}). \tag{3.6}$$

So, having defined smoothness on compact manifold – including a compact manifold with boundary, we then define a smooth homotopy between two such maps. If $f_0,\ f_1 : X \longrightarrow G^{-\infty}(\mathbb{N})$ are smooth maps then they are said to be *smoothly homotopic* if there exists a smooth map

$$F : [0, 1]_t \times X \longrightarrow G^{-\infty}(\mathbb{N}) \tag{3.7}$$

such that

$$F(0, x) = f_0(x),\ F(1, x) = f_1(x)\ \forall\ x \in X. \tag{3.8}$$

Definition 2. The odd K-theory of a compact manifold X is defined to be the set of equivalence classes under smooth homotopy of smooth maps $f : X \longrightarrow G^{-\infty}(\mathbb{N})$:

$$K^{-1}(X) = \{f : X \longrightarrow G^{-\infty}(\mathbb{N})\}/\sim . \tag{3.9}$$

The same definition applies to compact manifolds with boundaries, or with corners. For non-compact manifolds I will require the smooth maps to have 'compact support', meaning they reduce to the identity outside some compact set. The maps F in the homotopies are then also require to be equal to the identity outside a compact set, although of course the set is not itself fixed. I will use the slightly non-standard notation

$$K_{\mathrm{c}}^{-1}(X) = \{f : X \longrightarrow G^{-\infty}(\mathbb{N});\ \exists\, K \Subset X,\ f(x) = \mathrm{Id}\ \forall\, x \in X \setminus K\}/\sim \tag{3.10}$$

in this case.

Now, in fact $K^{-1}(X)$ is not just a set, but is an abelian group. That it is a group is relatively clear. The commutativity of the group structure follows from the approximation properties.

Proposition 5. *Group composition in $G^{-\infty}(\mathbb{N})$ induces the structure of an abelian group on $K^{-1}(X)$.*

Proof. Given two smooth maps $f_i : X \longrightarrow G^{-\infty}(\mathbb{N})$, $i = 1, 2$, the product $f_1 f_2(x) = f_1(x) f_2(x)$ is smooth in view of the smoothness of the product map on $G^{-\infty}(\mathbb{N})$. To see that $K^{-1}(X)$ inherits a group structure from this, we need to check that it is consistent with homotopy, i.e. is independent of the choice of representative. However, that is obvious enough since if $f_i' : X \longrightarrow G^{-\infty}(\mathbb{N})$, $i = 1, 2$ are two other representatives of the same K-classes then, by definition, there homotopies $F_i : [0, 1] \times X \longrightarrow G^{-\infty}(\mathbb{N})$, $i = 1, 2$ which are smooth and such that

$$F_i(0, x) = f_i(x),\ F_i(1, x) = f_i'(x). \tag{3.11}$$

Then, $F_1 F_2$ is a smooth homotopy between the products, so the class $[f_1 f_2] \in K^{-1}(X)$ only depends on the classes $[f_1]$, $[f_2] \in K^{-1}(X)$. This product makes $K^{-1}(X)$ into a group since the inverse of $[f]$ is clearly $[f^{-1}]$ and the other group conditions follow from $G^{-\infty}(\mathbb{N})$.

So, the remaing thing to show is that the product is commutative. To do so, we show that each element $[f] \in K^{-1}(X)$ can be represented by a smooth map $f' : X \longrightarrow \mathrm{GL}(N, \mathbb{C})$ for some N (and hence for any larger N by stabilization). This follows from the approximation result proved early. Namely, the image of f is certainly compact (since X is assumed to be so) and thus

$$\Pi_N f(x) \longrightarrow f(x) \text{ uniformly for } x \in X. \tag{3.12}$$

It follows from the openness of $G^{-\infty}(\mathbb{N})$ that for N large enough the smooth homotopy $F(t, x) = (1 - t) f(x) + t \Pi_N f(x)$ takes values in $G^{-\infty}(\mathbb{N})$ and so $\Pi_N f$ also represents $[f] \in K^{-1}(X)$.

Now, having take two classes, represented by f and g. For N large enough, these classes are represented by $\Pi_N f$ and $\Pi_N g$ which take values in $\mathrm{GL}(N, \mathbb{C})$. We can also embed $\mathrm{GL}(N, \mathbb{C})$ in $\mathrm{GL}(2N, \mathbb{C})$ by stabilization and see that each of these classe is represented by a map taking values in matrices like this

$$\begin{pmatrix} * & 0 \\ 0 & \mathrm{Id}_N \end{pmatrix} \tag{3.13}$$

which are block $N \times N$ matrices. Now consider the following homotopy which is really just 'rotation by a 2×2 matrix' for say f :
(3.14)
$$F(t,x) = \begin{pmatrix} \cos(\frac12\pi t) & \sin(\frac12\pi t) \\ -\sin(\frac12\pi t) & \cos(\frac12\pi t) \end{pmatrix} \begin{pmatrix} f(x) & 0 \\ 0 & \mathrm{Id}_N \end{pmatrix} \begin{pmatrix} \cos(\frac12\pi t) & -\sin(\frac12\pi t) \\ \sin(\frac12\pi t) & \cos(\frac12\pi t) \end{pmatrix}.$$
The outer matrices are inverses of each other (of course there are hidden Id_N's in the 2×2 matrices). At $t = 0$, F reduces to the suspended f but at $t = 1$ it is

$$\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & f(x) \end{pmatrix}. \tag{3.15}$$

Thus, f is homotopic to a map which commutes with g. The product is therefore commutative. □

For the moment, I will not go into any detail, but these abelian groups (which are written additively, so that the class [Id] = 0) behave like (and indeed form) a cohomology theory. Thus, under a smooth map between compact manifold, $h : X \longrightarrow Y$, these odd k-groups pull back:

$$h^* : H^{-1}(Y) \longrightarrow K^{-1}(X),\ h^*[f] = [f \circ h]. \tag{3.16}$$

Check for yourself that this is well-defined.

As well as this bald definition of odd K-theory, which is only really justified by subsequent properties, I want to introduct even K-theory and also the delooping sequence today. Even K-theory here will be defined in terms of the appropriate loop group as a classifying space. Loops in general are just maps from the circle. In the case of a group, for us $G^{-\infty}(\mathbb{N})$ one can restrict to smooth pointed loops, which take the value Id at the base point, $1 \in \mathbb{S}$. In fact, for analytic reasons that will appear later, it is best here to take an even smaller group the flat-pointed loop (smooth) loop group:

$$G^{-\infty}_{\mathrm{sus}}(\mathbb{N}) = \{b : \mathbb{S} \longrightarrow G^{-\infty}(\mathbb{N}); \mathcal{C}^\infty,\ b(1) = \mathrm{Id},\ \frac{d^k b}{d\theta^k}(1) = 0\ \forall\ k \ge 1\}. \tag{3.17}$$

Thus these loops not only take the value Id at the point $1 \in \mathbb{S}$ but all derivatives vanish there as well, making the loop 'flat'. I use the abbreviation 'sus' for this group to indicate that it is obtained by 'suspension' from $G^{-\infty}(\mathbb{N})$ in a way that will be clarified below.

Now, I did not do the following in the lecture, because I did not have my notes!

Lemma 2. *The suspended group $G^{-\infty}_{\mathrm{sus}}(\mathbb{N})$ is open and dense in the Fréchet algebra*
(3.18)
$$\dot{\mathcal{C}}^\infty([0,2\pi]; \Psi^{-\infty}(\mathbb{N})) = \{b : [0,2\pi]_\theta \longrightarrow \Psi^{-\infty}(\mathbb{N}); \frac{d^k b}{d\theta^k} b(\theta) = 0,\ \theta = 0,\ 2\pi\ \forall\ k \ge 0\}.$$

Proof. So, to do this properly I need to show that

(1) The space (3.18) is a Fréchet algebra
(2) There is a natural map from the group in (3.17) into it.
(3) The range is open (and in fact it is dense).

So, this is just like the relationship between $\Psi^{-\infty}(\mathbb{N})$ and $G^{-\infty}(\mathbb{N})$. □

Having defined this suspended group, we can set by direct analogy with the odd case above

$$K^{-2}(X) = \{f : X \longrightarrow G^{-\infty}_{\mathrm{sus}}(\mathbb{N}); \mathcal{C}^\infty\}/\sim \tag{3.19}$$

with the equivalence relation being smooth homotopy in the same sense. Thus f_0 and f_1 are homotopic if there exists

$$F:[0,1]\times X\longrightarrow G^{-\infty}_{\text{sus}}(\mathbb{N}); F(0,x)=f_0(x),\ F(1,x)=f_1(x)\ \forall\ x\in X\}. \tag{3.20}$$

I need to expand a bit on the things I said in the later part of the lecture. Namely there is a natural injection

$$K^{-2}(x)\longrightarrow K^{-1}(\mathbb{S}\times X) \tag{3.21}$$

which corresponds to the fact that an element of $G^{-\infty}_{\text{sus}}(\mathbb{N})$ is already a smooth map of $\mathbb{S}$ into $G^{-\infty}(\mathbb{N})$ and hence a map from X into $G^{-\infty}_{\text{sus}}(\mathbb{N})$ can be regarded as a map from $\mathbb{S}\times X$ into $G^{-\infty}(\mathbb{N})$. In the lecture I did not prove injectivity but I did say:

Lemma 3. *For any compact manifold there is a natural short exact sequence*

$$K^{-2}(X)\longrightarrow K^{-1}(\mathbb{S}\times X)\longrightarrow K^{-1}(X). \tag{3.22}$$

Proof. □

The relationship between the circle $\mathbb{S}$ and the interval $[0,2\pi]$, which already appears (at the moment implicitly) above gives rise to the delooping sequence. This comes above by cutting the circle at 1. So, consider in place of (3.17) the group

$$\tilde{G}^{-\infty}_{\text{sus}}(\mathbb{N})= \{\tilde{b}:[0,2\pi]_\theta\longrightarrow G^{-\infty}(\mathbb{N});\mathcal{C}^\infty,\ \tilde{b}(0)=\mathrm{Id},\ \frac{d^k\tilde{b}}{dt^k}(t)=0,\ t=0,2\pi\ \forall\ k\ge 1\}. \tag{3.23}$$

Thus, these are smooth maps from the interval, hence 'open loops', which take the value Id at 0 and which are flat at both ends. However the value at the far end, $t=2\pi$, is not specified. The topology on this group is of the same type as is (not yet) discussed above.

Proposition 6. *The natural maps, given by the identification $\mathbb{S}=\mathbb{R}/2\pi\mathbb{N}$ with fundamental domain $[0,2\pi]$ and by restriction to 2π, give a short exact sequence of groups*

$$\{1\}\longrightarrow G^{-\infty}_{\text{sus}}(\mathbb{N})\longrightarrow \tilde{G}^{-\infty}_{\text{sus}}(\mathbb{N})\xrightarrow{|_{2\pi}} G^{-\infty}(\mathbb{N})\longrightarrow\{1\}. \tag{3.24}$$

Proof. Identifying $\mathbb{S}=\mathbb{R}/2\pi\mathbb{Z}$ gives a smooth map $[0,2\pi]\longrightarrow\mathbb{S}$, explicitly $\theta\longmapsto e^{i\theta}$, under which 0 and 2π are both identified with $1\in\mathbb{S}$. Thus, elements of $G^{-\infty}_{\text{sus}}(\mathbb{N})$, being maps on $\mathbb{S}$, pull back to $[0,2\pi]$. In fact, comparing (3.17) and (3.23) this map is injective into $\tilde{G}^{-\infty}_{\text{sus}}(\mathbb{N})$ and has range the subset on which $\tilde{b}(2\pi)=\mathrm{Id}$. The restricition map in (3.24) is just evaluation at $\theta=2\pi$ so exactness at $\tilde{G}^{-\infty}_{\text{sus}}(\mathbb{N})$ also follows. The surjectivity of this map follows from the connectedness of $G^{-\infty}(\mathbb{N})$. In fact the argument above, by approximation gave a piecewise smooth curve from any given point of $G^{-\infty}(\mathbb{N})$ to Id. To prove surjectivity we need to show that this curve, which can be assumed to be from $[0,2\pi]$ can be chosen smooth and flat at the ends. If it is smooth, reparameterization makes it flat. Namely consider a map $\psi:[0,2\pi]\longrightarrow[0,2\pi]$ with is smooth and constant near the ends with $\psi(0)=0$, $\psi(2\pi)=2\pi$. Then if $b':[0,2\pi]\longrightarrow G^{-\infty}(\mathbb{N})$ is smooth, $b'\circ\psi$ is smooth and flat at the ends. The same construction allows a piecewise smooth curve to be mad smooth, by making it flat at the special points. This completes the proof of the exactness of (3.24). □

Add some words about the contractibility of $\tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{N})$ and why this might be important.

4. Lecture 4: Delooping and Chern forms
Friday, 5 September, 2008

(1) Contractibility of $\tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{N})$.
(2) Odd Chern forms.
(3) Even Chern forms – only started.
(4) Transgression under delooping; next time.

Last time I defined two versions of the loop group on $G^{-\infty}(\mathbb{N})$ and discussed the delooping sequences. The central group in the sequence consists of open loops. Identifying the circle with the quotient of the interval $[0, 2\pi]$ there is in fact no contuity condition corresponding to $0 = 2\pi$ so this group can be written as

(4.1)
$$\tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{N}) = \{b : [0, 2\pi]_s \longrightarrow G^{-\infty}(\mathbb{N}); b(0) = \mathrm{Id},\ \frac{d^k b}{dt^k}(0) = 0,\ \frac{d^k b}{dt^k}(2\pi) = 0\ \forall\ k \ge 1\}.$$

Thus, the value at the end, $s = 2\pi$, of the loop is 'free' but the curve is required to be flat there and also to be flat as it approaches Id at $s = 0$.

Proposition 7. *The is a smooth global retraction*

$$\begin{gathered} R : [0, 1]_t \times \tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{N}) \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{N}), \\ R(1, b) = b,\ R(0, b) = \mathrm{Id}\ \ \forall\ b \in \tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{N}). \end{gathered} \tag{4.2}$$

Proof. The idea is to simply shorten the curves but we need to be a little careful in order to maintain the flatness conditions. First choose two smooth functions

$$\begin{gathered} \psi_i : [0, 1] \longrightarrow [0, 1], i = 0, 1 \text{ with} \\ \psi_1(1) = 1 \text{ in } s > 3/4,\ \psi_1(s) = 0 \text{ in } s < 1/2, \\ \psi_0(s) = 0 \text{ in } s > 3/4 \text{ and } s < 1/4,\ \psi_0(s) + \psi(s) = 1 \text{ in } s \ge 1/2. \end{gathered} \tag{4.3}$$

Then consider a smooth function $f : [0, 2\pi] \longrightarrow [0, 2\pi]$ with $f(s) = 0$ in $s > 1/20$, $f(s) = 2\pi$ in $s > 2\pi - 1/20$ and define

$$\chi : [0, 1]_t \times [0, 2\pi]_s \longrightarrow [0, 2\pi] \text{ by } \chi(t, s) = f(s)\psi_0(t) + s\psi_1(t). \tag{4.4}$$

Clearly $\chi(0, s) = 0$, $\chi(1, s) = s$ for all $s \in [0, 2\pi]$. Moreover, $\chi(t, 0) = 0$ forall t, $\chi(t, 2\pi) = 2\pi$, for $t \ge 1/2$ and for $t \le 1/2$,

$$\frac{d^k \chi}{ds^k}(t, 2\pi) = 0\ \forall\ k > 1. \tag{4.5}$$

Then the desired homotopy is given by

$$R(t, a)(s) = a(\chi(t, s)) \in \tilde{G}^{-\infty}(\mathbb{N}) \tag{4.6}$$

where the flatness at $s = 0$ follows from the flatness of a at $s = 0$ and the flatness at $s = 2\pi$ follows from that of a for $t \ge 1/2$ and from that of χ for $t \le 1/2$. Thus (4.2) follows, proving the Proposition. □

Now, let me turn, or return, to the Chern forms. As in a Lie group, the canonical map $g : G^{-\infty}(\mathbb{N}) \longrightarrow \Psi^{-\infty}(\mathbb{N})$ which embeds the group as an open dense subset of the algebra trivializes the tangent bundle to the group, so we can identify

$$dg : TG^{-\infty} = G^{-\infty} \times \Psi^{-\infty}(\mathbb{N}). \tag{4.7}$$

The 'name' chosen for this identification, dg, is supposed to be suggestive but can be confusing. Really the 'g' here just tells you at which point of the group you are supposed to be and the 'd' indicates the identification of tangent spaces. However, it does give magical formulæ which fortunately are correct.

So, the higher tensor spaces, multi-tangent bundles, are just the formal tensor products. This means that if we want to have a cotensor at a point of $G^{-\infty}$ it will just be a continuous multilinear map

$$\Psi^{-\infty}(\mathbb{N}) \times \Psi^{-\infty}(\mathbb{N}) \cdots \Psi^{-\infty}(\mathbb{N}) \longrightarrow \mathbb{C}. \tag{4.8}$$

The continuity of such multilinear maps automatically generates a completed tensor product of the dual spaces, so we do not have to worry about formalizing this at the moment. In short then a k-form on $G^{-\infty}$ should be a smooth map

$$G^{-\infty}(\mathbb{N}) \times \Psi^{-\infty}(\mathbb{N}) \times \Psi^{-\infty}(\mathbb{N}) \cdots \Psi^{-\infty}(\mathbb{N}) \longrightarrow \mathbb{C} \tag{4.9}$$

which is linear in each of the last k factors and which is totally antisymmetric in these. Before worrying too much about differentials etc, let's just check that we can manufacture some.

The simplest forms one could think of would be those 'independent' of the first factor in (4.9) – although such independence is illusionary since the trivialization of the tangent bundle introduces a degree of twisting. Thus, since we have the product in the algebra and the trace functional at our disposal we can just consider

$$\Psi^{-\infty}(\mathbb{N}) \times \Psi^{-\infty}(\mathbb{N}) \cdots \Psi^{-\infty}(\mathbb{N}) \ni (b_1, \dots, b_k) \longmapsto \sum_{\Sigma_k \ni i} \operatorname{sgn}(i) \operatorname{tr}(b_{i_1} b_{i_2} \dots b_{i_k}). \tag{4.10}$$

Here I have explicitly introduced the exterior product by antisymmetrizing in all the variables. So, at the identity of the group this can be written

$$\operatorname{tr}(dg \wedge dg \wedge \dots dg)(b_1, \dots, b_k) \text{ at } g = \operatorname{Id} \in G^{-\infty}(\mathbb{N}). \tag{4.11}$$

This does indeed define a global form on $G^{-\infty}$ but not a very interesting one as it turns out. Rather we need to introduce factors of g^{-1} to map everything back to the identity. So we consider the k-form

$$\operatorname{tr}(g^{-1}dg \wedge \dots g^{-1}dg) = \operatorname{tr}\left((g^{-1}dg)^k\right). \tag{4.12}$$

Written out at any point on the group it just looks like (4.10) with g^{-1}'s inserted between the factors and then antisymmetrized in the tangent variables. Clearly (4.12) is a rather simpler formula, especially when we suppress the wedge product as well!

Now, from antisymmetry alone this form vanishes identically if k is even. You can think of this as 'moving' the first factor to last – which is okay because of the properties of the trace – but in doing so one has to pass over an odd number of terms each of which reverses the sign, so overall it is equal to its negative. Thus we only consider the odd case and write

$$\operatorname{Ch}^{\text{odd}}_{2k+1} = \operatorname{Ch}_{2k+1} = \operatorname{tr}\left((g^{-1}dg)^{2k+1}\right),\ k = 0, 1, 2, \dots. \tag{4.13}$$

The 'odd' here is redundant, since the forms are only in odd degree anyway.

Note that the insertion of the factors of g^{-1} makes this form left-invariant. That is, consider the map from $G^{-\infty}$ to itself given by multiplication on the left by $h \in G^{-\infty}(\mathbb{N})$, fixed but arbitrary. Ther is of course a similar right multiplication map, which conventionally has an inverse inserted

$$\begin{gathered} L_h : G^{-\infty}(\mathbb{N}) \longrightarrow G^{-\infty}(\mathbb{N}),\ g \longmapsto hg \\ R_h : G^{-\infty}(\mathbb{N}) \longrightarrow G^{-\infty}(\mathbb{N}),\ g \longmapsto gh^{-1} \end{gathered} \tag{4.14}$$

are both global diffeomorphisms.

Lemma 4. *All the (odd) Chern forms in* (4.13) *are are bi-invariant.*

Proof. Trivial enough. Namely $(hg)^{-1} = g^{-1}h^{-1}$ and $d(hg) = hdg$. Thus even the product $g^{-1}dg = (hg)^{-1}d(hg)$ is left-invariant. On the other hand under the right action, $R_h^*(g^{-1}dg) = h(g^{-1}dg)h^{-1}$. Thus the forms are obviously left-invariant and right-invariance follows from the invariance properties of the trace:

$$R_h^* \operatorname{Ch}_{2k+1} = \operatorname{tr}\left(h(g^{-1}dg)^{2k+1}h^{-1}\right) = \operatorname{Ch}_{2k+1}. \tag{4.15}$$

□

Most importantly of all, of course, is that

Lemma 5. *The (odd) Chern forms are closed.*

Proof. The operator d is perfectly well-defined as in the finite-dimensional case. Let me just leave this as an exercise for the moment! In fact d makes good sense on smooth forms valued in any vector space, such as $\Psi^{-\infty}(\mathbb{N})$. Thus we see,

$$g^{-1} : G^{-\infty}(\mathbb{N}) \longleftrightarrow G^{-\infty}(\mathbb{N}),\ dg^{-1} = -g^{-1}dgg^{-1}. \tag{4.16}$$

As usual, this just follows by differntiating the identity $g^{-1}g = \operatorname{Id}$. Thus the product $g^{-1}dgg^{-1}$ is closed, in fact is exact. Similarly of course, dg is closed – being the differential of a linear map. So,

$$\operatorname{tr}\left((g^{-1}dg)^{2k+1}\right) = \operatorname{tr}\left(g^{-1}dg(g^{-1}dgg^{-1}dg)^k\right). \tag{4.17}$$

Since tr is a continuous linear functional $d\operatorname{tr}(F) = \operatorname{tr}(dF)$ so we see that

$$d\operatorname{Ch}_{2k+1} = \operatorname{tr}\left((dg^{-1})dg(g^{-1}dg)^k\right) = -\operatorname{tr}\left(g^{-1}dgg^{-1}dg(g^{-1}dg)^k\right) = 0 \tag{4.18}$$

since we are back to the case of an even number of factors. □

Where does this lead us? Each of these Chern forms defines a cohology class on $G^{-\infty}(\mathbb{N})$ – of course we have not yet checked that they are non-zero. In fact they are and so it is interesting to consider the pull-backs:

Proposition 8. *The odd Chern forms define maps for each* k

$$K^{-1}(X) \longrightarrow H^{2k+1}(X;\mathbb{C}) \tag{4.19}$$

for any compact manifold.

Proof. By definition an odd K-class is defined by a smooth map $f : X \longrightarrow G^{-\infty}(\mathbb{N})$. Thus we can simply pull the forms back to get

$$f^* \operatorname{Ch}_{2k+1} = \operatorname{tr}\left((f^{-1}df)^{2k+1}\right) \tag{4.20}$$

where now we can think of f as a map into $\Psi^{-\infty}({}^{\mathrm{b}}N)$ (which happens to map into $G^{-\infty}(\mathbb{N})$ of course). So, we only need to show that the cohomology class defined by this form is the same for homotopic f's. Given a homotopy $F : [0,1]_t \times X \longrightarrow G^{-\infty}$ the Chern form pulls back to $\gamma = F^* \operatorname{Ch}_{2k}$ which is a smooth closed form on $[0,1] \times X$. Then if f_0 and f_1 are the restrictions to $t = 0$ and $t = 1$ it follows that

$$f_1^* \operatorname{Ch}_{2k} - f_0^* \operatorname{Ch}_{2k} = d\tau \tag{4.21}$$

for a smooth form τ. Indeed $\gamma = dt \wedge v + v'$ where v, v' are forms on X which depend on t as a parameter. That γ is closed means that

$$\frac{\partial v'}{\partial t} - d_X v = 0,\ d_X v' = 0. \tag{4.22}$$

Hence

$$f_1^* \operatorname{Ch}_{2k} - f_0^* \operatorname{Ch}_{2k} = v'(1) - v'(0) = \int_0^1 \frac{\partial v'}{\partial t} dt = d_X \tau,\ \tau = \int_0^1 v(t) dt. \tag{4.23}$$

□

In fact it is usual to sum these forms up, to give the Chern character mapping from odd K-theory to odd cohomology – this involves questions of normalization which I will postpone for a little while.

So, next consider the even analogue of these forms. Of course there are no even forms on $G^{-\infty}(\mathbb{N})$ but these are forms on $G^{-\infty}_{\text{sus}}(\mathbb{N})$. In fact these are induced by the forms we already have on $G^{-\infty}(\mathbb{N})$ through the evaluation map

$$\text{ev} : \mathbb{S} \times G^{-\infty}_{\text{sus}}(\mathbb{N}) \longrightarrow G^{-\infty},\ (\theta, g) \longrightarrow g(\theta). \tag{4.24}$$

Since this map is smooth, we can pull the forms Ch_{2k+1} back to the product and then we can push-forward to the suspended group by integrating over the circle. Thus reduces the degree by one, so we define

$$\operatorname{Ch}^{\text{even}}_{2k} = \int_{\mathbb{S}} \text{ev}^* \operatorname{Ch}_{2k+1} \text{ on } G^{-\infty}_{\text{sus}}(\mathbb{N}). \tag{4.25}$$

Now, it is straighforward to write this form down in terms of dg, the same map on $G^{-\infty}(\mathbb{N})$ and the parameter $\theta \in \mathbb{S}$:

$$\operatorname{Ch}_{2k} = \int_{\mathbb{S}} \operatorname{tr} \left(g^{-1} \frac{\partial g}{\partial \theta} (g^{-1} dg)^{2k} \right). \tag{4.26}$$

5. Lecture 5: Harmonic oscillator Monday, 8 September, 2008

I have not really talked so far about the topology on the loop spaces. I hope to get to this today, or at least do the preparation for it, and also consider the first 'geometric form' of $G^{-\infty}$, namely the 'isotropic smoothing algebra', or Schwarz algebra, of operators on $\mathbb{R}$.

- Schwartz space
- Harmonic oscillator
- Creation and annihilation operators
- Eigenfunctions
- Hermite polynomials
- Completeness
- Convergence of eigenseries
- The algebra $\Psi^{-\infty}(\mathbb{R})$ and group $G^{-\infty}(\mathbb{R})$.
- Loop groups again.

I will assume that you are somewhat familiar with the Schwartz space $\mathcal{S}(\mathbb{R})$, but let me remind you of the definition and basic properties. In fact we might as well consider $\mathcal{S}(\mathbb{R}^n)$ for any n.

So, $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n)$ consists of all the (complex-valued) smooth functions of rapid decay, meaning that all the norms

$$\|u\|_{p,\infty} = \sup_{z\in\mathbb{R}^n,\ 0\le|\alpha|\le p} (1+|z|^2)^{\frac{p}{2}}|D^\alpha u(z)| < \infty \tag{5.1}$$

are finite. Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, so $\alpha_i \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$

$$D_z^\alpha u = i^{-|\alpha|}\frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}u(z),\ |\alpha| = \alpha_1 + \cdots + \alpha_n \tag{5.2}$$

where the powers of i are there for reasons to do with formal self-adjointness. This sequence of norms is just like those considered on sequences above. Just as there, $\mathcal{S}(\mathbb{R}^n)$ is a complete metric space with the distance

$$d(u, v) = \sum_p 2^{-p}\frac{\|u\|_{p,\infty}}{1+\|u\|_{p,\infty}} \tag{5.3}$$

where convergence of a sequence with respect to this distance means exactly the same as convergence with respect to each of the norms $\|u\|_{p,\infty}$ (with no uniformity in p). The dual space, the space of continuous linear maps

$$U : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}, \tag{5.4}$$

is the space of tempered (or temperate) distributions, $\mathcal{S}'(\mathbb{R}^n)$. There is a natural inclusion, almost always treated as an identification

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n),\ u \longmapsto U_u : \mathcal{S}(\mathbb{R}^n) \ni f \longrightarrow \int_{\mathbb{R}^n} u(x)f(x)dx. \tag{5.5}$$

Since it is treated as an identification we normally write $U_u = u$.

Now consider the algebra

$$\Psi^{-\infty}_{\text{iso}}(\mathbb{R}) = \mathcal{S}(\mathbb{R}^2) \tag{5.6}$$

where the product is

$$ab(x,x') = \int_{\mathbb{R}} a(x,x'')b(x'',x')dx''. \tag{5.7}$$

These are the Schwartz smoothing operators on $\mathbb{R}$. They act on $\mathcal{S}(\mathbb{R})$ in the obvious way, as integral operators

$$a : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R}),\ (au)(x) = \int_{\mathbb{R}} a(x,x')u(x')dx'. \tag{5.8}$$

Then (5.7) is operator composition.

The spectral theory of the harmonic oscillator

$$H = -\frac{d^2}{dx^2} + x^2 \tag{5.9}$$

on the line can be discussed in an essentially algebraic way. This is based on the two first order operators,

$$A = \frac{d}{dx} + x \text{ and } C = -\frac{d}{dx} + x, \tag{5.10}$$

respectively the annihilation and creation operator. The idenitites

$$H = CA + 1,\ [A,C] = 2[\frac{d}{dx}, x] = 2,\ Ae^{-\frac{x^2}{2}} = 0 \tag{5.11}$$

are easily checked. Since

$$\int_{\mathbb{R}} (e^{-\frac{x^2}{2}})^2 dx = \int e^{-x^2} dx = \sqrt{\pi} \tag{5.12}$$

the function

$$h_1 = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \tag{5.13}$$

has norm 1 in $L^2(\mathbb{R})$ and satisfies

$$Hh_1 = h_1. \tag{5.14}$$

This is the ground state of the harmonic oscillator. The higher eigenfunctions are obtained by applying the creation operator. Thus

$$\begin{gathered} C^k h_1(x) \text{ satisfies} \\ H(C^k h_1) = C^k h_1 + CAC^k h_1 = C^k h_1 + 2C^k h_1 + C^2 A C^{k-1} h_1 = (1+2k) C^k h_1 \end{gathered} \tag{5.15}$$

as follows from (5.11) by induction. Moreover it also follows inductively that

$$C^k h_1(x) = (2^k x^p + q_{k-1})(x))h_1(x) \tag{5.16}$$

where q_{k-1} is a polynomial of degree at most $k-1$. Certainly, $C^k h_1 \in \mathcal{S}(\mathbb{R})$. The L^2 norm can be computed by integration by parts using the fact that A and C are adjoints of each other

$$\int_{\mathbb{R}} (C^k h_1(x))^2 dx = \int_{\mathbb{R}} h_x(x) A^k C^k h_1(x) dx = 2^k k!. \tag{5.17}$$

Moreover $C^k h_1$ and $C^l h_1$ are orthogonal in L^2 by a similar argument and hence the

$$h_k = 2^{-\frac{k}{2}} (k!)^{\frac{1}{2}} C^k h_1,\ k = 0, 1, 2, \dots \tag{5.18}$$

form an orthonormal sequence of eigenfunctions of H.

In fact this is a complete orthonormal basis of $L^2(\mathbb{R})$. To see this observe from (5.16) that the span of the first k elements consists of all the products $q(x)e^{-\frac{x^2}{2}}$ where q is a polynomial of degree at most k. In particular if $u \in L^2(\mathbb{R})$ then

$$\int u(x)h_k(x)dx = 0 \;\forall\; k \Longleftrightarrow \int u(x)x^k e^{-\frac{x^2}{2}}dx = 0 \;\forall\; k. \tag{5.19}$$

Taking the Fourier transform and using Plancherel's formula and the fact that the Fouier transform of h_1 is a multiple of itself, (5.19) is equivalent to

$$\frac{d^k}{d\tau^k}v(0) = 0 \;\forall\; k,\; v(\tau) = \int e^{-ix\tau}u(x)\exp(-\frac{x^2}{2})dx. \tag{5.20}$$

Now v is entire, since the integral defining it is absolutely convergent for all $\tau \in \mathbb{C}$. It follows that $v \equiv 0$ and hence $u \equiv 0$ by Fourier inversion. This shows that the h_k form a complete orthonormal basis of $L^2(\mathbb{R})$.

Lemma 6. *If $u \in \mathcal{S}(\mathbb{R})$ it follows that the Fourier-Bessel expansion of u in terms of the h_k converges in $\mathcal{S}(\mathbb{R})$:*

$$u(x) = \sum_{k=1}^{\infty} c_k h_k,\; c_k = \int h_k(x)u(x) \Longrightarrow \sum_k k^p|c_k| < \infty \;\forall\; p. \tag{5.21}$$

Proof. This follows from Stirling's formula

$$k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k. \tag{5.22}$$

which implies the existence of positive constants R, c and C such that

$$cR^k k^{k+\frac{1}{2}} \le 2^k k! \le CR^k k^{k+\frac{1}{2}}. \tag{5.23}$$

Fix $p \in \mathbb{N}$. Then $h_{p+k} = \mu(k,p)C^p h_k$ so integrating by parts from the definition of the coefficients in (5.21),

$$c_{p+k} = \mu(k,p)\int_{\mathbb{R}} h_k(x)A^p u(x)dx. \tag{5.24}$$

Now, in terms of the seminorms on $\mathcal{S}(\mathbb{R})$,

$$|A^p u(x)| \le 2^p(1+|x|)^{-2}\|u\|_{p+2,\infty} \tag{5.25}$$

where the extra factor is to ensure integrability. Thus

$$|c_{p+k}| \le \mu(k,p)2^p\|u\|_{p+2,\infty}. \tag{5.26}$$

Combining (5.23) and (5.26) it follows that

$$|c_{p+k}| \le C_p k^{-p/2}\|u\|_{p,\infty}. \tag{5.27}$$

Thus the coefficients decrease rapidly.

Estimating directly it also follows that

$$\|h_k\|_{p,\infty} \le C_p k^{p/2+1} \tag{5.28}$$

so the sequence does indeed converge in $\mathcal{S}(\mathbb{R})$. □

Proposition 9. *The map*

$$\Psi^{-\infty}(\mathbb{N}) \ni a_{ij} \longmapsto \sum_{ij} a_{ij}h_i(x)h_j(x') \in \Psi^{-\infty}(\mathbb{R}) \tag{5.29}$$

is an isomorphism.

Proof. This requires the same sort of argument as in the previous proof, but now applied in both variables. □

So everything I have said for $\Psi^{-\infty}(\mathbb{N})$ carries over to $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R})$ and we can define the group $G^{-\infty}_{\mathrm{iso}}(\mathbb{R})$ which is similarly isomorphic, as a topological group and by a smooth isomorphism, to $G^{-\infty}(\mathbb{N})$. The trace functional is the integral over the diagonal

$$\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}) \ni a \longmapsto \operatorname{tr}(a) = \int_{\mathbb{R}} a(x,x)dx. \tag{5.30}$$

Thus the Chern character forms look the same as before but now involve lots of integrals instead of sums.

Now, if I get this far, the loop group on $\Psi^{-\infty}(\mathbb{N})$ can also be written is 'Schwartz form'. Namely we can take an isomorphism

$$(0,2\pi) \longrightarrow \mathbb{R},\ T(\theta) = \arctan((\theta-\pi)/2) \tag{5.31}$$

which identifies smooth functions on $[0,2\pi]$ which vanish with all their derivatives at the end points with $\mathcal{S}(\mathbb{R})$. Basically only the 'polynomial' behaviour of T at 0 and 2π (and the fact that it is a diffeomorphism of the open sets of course) is important here.

6. Lecture 6: Higher loop groups and determinant Wednesday, 10 September, 2008

Before picking up where I left off last time, let me outline where I will head after the coming week-long break. What we need to justify the definition of the odd K-theory of a space as the homotopy classes of maps into $G^{-\infty}(\mathbb{N})$ is to prove *Bott periodicity*:

$$\pi_j(G^{-\infty}(\mathbb{N})) = \begin{cases} \{0\} & j \text{ even} \\ \mathbb{Z} & j \text{ odd.} \end{cases} \tag{6.1}$$

We already know that $G^{-\infty}(\mathbb{N})$ is connected and using the Fredholm determinant it is reasonably easy to show the result for π_1. The general case will following from this by constructing a weak homotopy equivalence

$$G^{-\infty}_{\mathrm{sus}(2)}(\mathbb{N}) \longrightarrow G^{-\infty}(\mathbb{N}). \tag{6.2}$$

I have not defined the group on the left yet, but will do so today. It is just an iterated loop group; see (6.8).

At the end last time I talked about turning flat loops into Schwartz functions. The basic statement is simple enough. Consider the diffeomorphism of the open interval to the line

$$T : (0, 2\pi) \ni \theta \longmapsto \tan\left(\frac{\theta - \pi}{2}\right). \tag{6.3}$$

The derivative is $\frac{1}{2}\sec^2(\frac{\theta-\pi}{2}) > 0$ on $(0, 2\pi)$. As $\theta \downarrow 0$, $\theta T(\theta) \to -2$ and similarly at the other end, $(2\pi - \theta)T(\theta) \to 2$.

Lemma 7. *Pull-back under T in (6.3) gives a topological isomorphism*

$$T^* : \mathcal{S}(\mathbb{R}) \longrightarrow \dot{\mathcal{C}}^\infty([0, 2\pi]) = \left\{u \in \mathcal{C}^\infty([0, 2\pi]); \frac{d^k u}{d\theta^k}(e) = 0,\ e = 0, 2\pi,\ \forall\ k \geq 0\right\}. \tag{6.4}$$

Proof. Clearly under T^* an element $v \in \mathcal{S}(\mathbb{R})$ pulls back to be smooth in the interior. The derivatives of T all grow at most polynomially at the end points from which it follows easily that $T^*v \in \dot{\mathcal{C}}^\infty([0, 2\pi])$ and the converse is similar. □

Of course this proof could do with a bit of expansion!

Anyway, it follows from this Lemma, or a rather a generalization of it, that the suspended group can be moved to the real line.

$$\begin{gathered} b \in G^{-\infty}_{\mathrm{sus}}(\mathbb{N}) = \{b \in \dot{\mathcal{C}}^\infty([0, 2\pi]; \Psi^{-\infty}(\mathbb{N}); (\mathrm{Id} + b(\theta)) \in G^{-\infty}(\mathbb{N})\} \\ \Longleftrightarrow \\ b = T^*b',\ b' \in \mathcal{S}(\mathbb{R}; \Psi^{-\infty}(\mathbb{N})) \text{ and } \mathrm{Id} + b'(t) \in G^{-\infty}(\mathbb{N})\ \forall\ t \in \mathbb{R}. \end{gathered} \tag{6.5}$$

The point here, as usual, is that having values in $\Psi^{-\infty}(\mathbb{N})$ is really no different from usual complex-valued functions.

It is also important to note that the even Chern forms, defined on $G^{-\infty}_{\mathrm{sus}}(\mathbb{N})$ are independent of such a change of parameterization, even though it is from a compact to a non-compact space. Namely they are all given by push-forward of forms on this manifold, and this is independent of choice. At a more prosaic level the forms look like

$$\int_0^{2\pi} \operatorname{tr}(\dots \frac{dg}{d\theta} \dots) d\theta = \int_{\mathbb{R}} \operatorname{tr}(\dots \frac{dg}{dt} \dots) dt \tag{6.6}$$

with the Jacobians cancelling.

Now, one reason why this change of point of view, which is all I have really done here, is that combined with the shift last time from the sequential to isotropic forms of $G^{-\infty}$ it makes things look much more uniform. Thus we can write

$$\begin{aligned} G^{-\infty}(\mathbb{R}) &= \{a \in \mathcal{S}(\mathbb{R}^2); (\operatorname{Id} + a)^{-1} = \operatorname{Id} + a',\ a' \in \mathcal{S}(\mathbb{R}^2)\} \\ G^{-\infty}_{\text{sus}}(\mathbb{R}) &= \{b \in \mathcal{S}(\mathbb{R}^3); (\operatorname{Id} + b(t, \bullet)) \in G^{-\infty}(\mathbb{R})\ \forall\ t \in \mathbb{R}\} \\ \tilde{G}^{-\infty}_{\text{sus}}(\mathbb{R}) &= \big\{\tilde{b} \in \mathcal{C}^\infty(\mathbb{R}^3); (\operatorname{Id} + \tilde{b}(t, \bullet)) \in G^{-\infty}(\mathbb{R})\ \forall\ t \in \mathbb{R} \text{ and} \\ &\qquad \exists\ \tilde{a}_\pm \in \mathcal{S}(\mathbb{R}^3),\ a_\infty \in \Psi^{-\infty}(\mathbb{R}) \text{ such that} \\ &\qquad \tilde{b}(t) = a_-(t) \text{ in } t < 0,\ \tilde{b} = a_\infty + a_+(t) \text{ in } t > 0\big\}. \end{aligned} \tag{6.7}$$

In the second case, t is the first variable, which is just a parameter, and the other two are 'non-commutative' in the sense of the operator product. Of course the assertion here is that these two groups are, as has already been shown for the first two, isomorphic to $G^{-\infty}(\mathbb{N})$, $G^{-\infty}_{\text{sus}}(\mathbb{N})$ and $\tilde{G}^{-\infty}_{\text{sus}}(\mathbb{N})$, respectively, using the Hermite expansion and in the second two cases the compactification of $\mathbb{R}$ corresponding to T^{-1}. I leave the last case to you.

Now, since the suspended group is just the invertible Schwartz perturbations of the identity, it is reasonable to define higher loop groups in the same way:

$$G^{-\infty}_{\text{sus}(p)}(\mathbb{R}) = \{b \in \mathcal{S}(\mathbb{R}^{p+2}); (\operatorname{Id} + b(\tau, \bullet)) \in G^{-\infty}(\mathbb{R})\ \forall\ \tau \in \mathbb{R}^p\},\ p \in \mathbb{N}. \tag{6.8}$$

Having earlier shown that the spaces on $\mathbb{N}$ and $\mathbb{R}$ are the same I will simplify the notation and now write $\Psi^{-\infty}$, $G^{-\infty}$ and $G^{-\infty}_{\text{sus}(p)}$ for either case – and some more which will appear later.

Next I meant to discuss the Fredholm determinant rather briefly – actually I spent the rest of the lecture doing so!

Theorem 1. *There is a unique $\mathcal{C}^\infty$ functions, which is in fact entire analytic,* $\det(a) = \det_{Fr}(\operatorname{Id} + a)$

$$\det : \Psi^{-\infty} \longrightarrow \mathbb{C} \tag{6.9}$$

satisfying the multiplicativity condition

$$\det_{Fr}\left((\operatorname{Id} + a)(\operatorname{Id} + b)\right) = \det_{Fr}(\operatorname{Id} + a)\det_{Fr}(\operatorname{Id} + b)\ \forall\ a, b \in \Psi^{-\infty} \tag{6.10}$$

and the normalization

$$\frac{d}{dt}\det_{Fr}(\operatorname{Id} + ta)\big|_{t=0} = \operatorname{tr}(a). \tag{6.11}$$

The last condition prevents one from replacing det by a power.

Proof. The multiplicativity means that at any point $(\operatorname{Id} + b)$ the derivative can be computed:-

$$\begin{aligned} &\frac{d}{dt}\det_{\text{Fr}}(\operatorname{Id} + b + ta)\big|_{t=0} = \\ &\quad \det_{\text{Fr}}(\operatorname{Id} + b)\frac{d}{dt}\det_{\text{Fr}}(\operatorname{Id} + t(\operatorname{Id} + b)^{-1}a)\big|_{t=0} = \det_{\text{Fr}}(\operatorname{Id} + b)\operatorname{tr}\left((\operatorname{Id} + b)^{-1}a\right). \end{aligned} \tag{6.12}$$

This is just the first odd Chern form discussed earlier. Namely, the total derivative must satisfy

$$d\det_{\mathrm{Fr}}(g) = \det_{\mathrm{Fr}}(g)\,\mathrm{tr}\left(g^{-1}dg\right). \tag{6.13}$$

Of course, (6.13) is just the standard formula for the logarithmic derivative of the determinant for $N \times N$ matrices.

So, to define the function $\det_{\mathrm{Fr}}$ we can use the connectedness of $G^{-\infty}$ to choose a smooth curve

$$\chi : [0,1] \longrightarrow G^{-\infty},\ \chi(0) = \mathrm{Id},\ \chi(1) = g \tag{6.14}$$

and then set

$$\det_{\mathrm{Fr}}(g) = \exp\left(\int_\chi \mathrm{tr}\left(g^{-1}dg\right)\right) = \exp\left(\int_0^1 \mathrm{tr}\left(\chi^{-1}(t)\frac{d\chi(t)}{dt}\right) dt\right). \tag{6.15}$$

Since the integrand is smooth the integral on the right certainly exists. However, we need to show that the result does not depend on the choice of path. We will do this in two stages, first showing homotopy invariance.

Thus, suppose that $\chi(t,s)$, $t,s \in [0,1]$ is a smooth homotopy in $G^{-\infty}$ with $\chi(0,s) = \mathrm{Id}$, $\chi(1,s) = g$ fixed. The homotopy invariance follows from the fact that Ch_1 is closed, but let me prove it directly for reassurance. Thus we just compute the derivative

$$\begin{aligned}
&\frac{d}{ds}\int_0^1 \mathrm{tr}\left(\chi^{-1}(t)\frac{d\chi(t)}{dt}\right) dt \\
&= \int_0^1 \mathrm{tr}\left(-\chi^{-1}(t,s)\frac{d\chi(t,s)}{ds}\chi^{-1}(t,s)\frac{d\chi(t,s)}{dt} + \chi^{-1}(t,s)\frac{d^2\chi(t,s)}{dtds}\right) dt \\
&= \int_0^1 \mathrm{tr}\left(-\chi^{-1}(t,s)\frac{d\chi(t,s)}{dt}\chi^{-1}(t,s)\frac{d\chi(t,s)}{ds} + \chi^{-1}(t,s)\frac{d^2\chi(t,s)}{dtds}\right) dt \\
&= \int_0^1 \frac{d}{dt}\,\mathrm{tr}\left(\chi^{-1}(t,s)\frac{d\chi(t)}{ds}\right) dt = 0.
\end{aligned} \tag{6.16}$$

Here the trace identity has been used and of course the constancy at the ends.

It also follows directly that the result is independent of the parameterization of the curve. We can use this, as discussed above, to reparameterize the curve so that it is flat at both ends. Alternatively, this could have been required in the original definition. This flatness allows us to 'add' to curves and keep smoothness. Thus if

$$\begin{aligned}\chi_i : [0,1] \longrightarrow G^{-\infty} \text{ are smooth with } \frac{d^k}{dt^k}\chi_i(t') = 0\ \forall\ k \geq 1,\ t' = 0,1, \\ \chi_i(0) = \mathrm{Id},\ \chi_i(1) = g_i,\ i = 1,2\end{aligned} \tag{6.17}$$

then we can simply define

$$\chi : [0,1] = \begin{cases} \chi_1(2t) & 0 \leq t \leq \frac{1}{2} \\ a_1\chi_1(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} \tag{6.18}$$

and conclude (using left invariance of Ch_1) that

$$\det_{\mathrm{Fr}}(g_1g_2) = \det_{\mathrm{Fr}}(g_1)\det_{\mathrm{Fr}}(g_2) \tag{6.19}$$

once we have shown independence of the choice of homotopy class of path.

To show this independence it suffices, using (6.18) to see that if χ is a closed smooth curve into $G^{-\infty}$ starting and ending at the identity then

$$\int_0^1 \operatorname{tr}\left(\chi^{-1}(t)\frac{d\chi(t)}{dt}\right)dt \in 2\pi i\mathbb{Z}. \tag{6.20}$$

Since we have already shown homotopy invariance it suffices to show this where $\chi(t) = \operatorname{Id} + a(t)$ is replaced by $\operatorname{Id} + \Pi_N a \Pi_N$. This reduces to the case of the determinant on $\mathrm{GL}(N, \mathbb{C})$ where I assume it is well-known and I leave it to you. □

Below is what I meant to cover today, instead of spending so long discussing the Fredholm determinant.

So, that takes care of what was left over from last time. What I wanted to do today was consider the effect of the delooping sequence on the Chern forms. Written in terms of this isotropic version of the groups the delooping sequence is

$$G^{-\infty}_{\mathrm{sus}}(\mathbb{R}) \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{R}) \xrightarrow{R} G^{-\infty}(\mathbb{R}). \tag{6.21}$$

So far I have defined Chern forms on the first and last groups. The even Chern forms were defined by pull-back and integration and the same idea works for the middle group. However, I will call the resulting form an 'eta form'.

Thus, for each $k \geq 0$ there is a form on $\tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{R})$ given explicitly by

$$\eta_{2k} = -\int_{\mathbb{R}} \operatorname{tr}\left(g^{-1}\frac{\partial g}{\partial t}(g^{-1}dg)^{2k}\right)dt. \tag{6.22}$$

The main thing to observe here is that the integral is absolutely convergent. This follows as before from the fact that the term $\partial g/\partial t$ is Schwartz in t, since both the identity term and the constant term in the expansion as $t \to \infty$ are killed by differentiation. Thus the whole integrand, evaluated on $2k$ tangent elements is itself an element of $\mathcal{S}(\mathbb{R}^3)$.

Now, the main difference between the eta form and the Chern forms is that the latter were closed. The eta form is a 'transgression form'.

Proposition 10. *The eta forms in* (6.22) *restrict to the subgroup* $G^{-\infty}_{\mathrm{sus}}(\mathbb{R})$ *to the even Chern forms and have 'basic' derivatives in the sense that*

$$d\eta_{2k} = R^* \operatorname{Ch}_{2k+1}. \tag{6.23}$$

Proof. The first statement is immediate, since we are using the same formula, (6.22) to define both even Chern and even eta forms. To prove the second part we need to compute the derivative. Let me remind you of the proof that the even Chern forms are closed – obviously we just follow that and see what happens.

We know that the odd Chern form is closed. When pulled back under the evaluation map from $G^{-\infty}$ to $\mathbb{R} \times \tilde{G}^{-\infty}_{\mathrm{sus}}$ this becomes the condition

$$D \operatorname{tr}((g^{-1}Dg)^{2k+1}) = 0,\ D = dt\frac{\partial}{\partial t} \wedge + d. \tag{6.24}$$

Expanding out the form according in terms of the product we find

$$\mathrm{ev}^* \operatorname{Ch}_{2k+1} = dt \wedge A(t) + B(t) \Longrightarrow \frac{\partial B}{\partial t} = dA(t) \text{ and } dB(t) = 0. \tag{6.25}$$

Integrating over t we find by definition

$$\eta_{2k} = \int \operatorname{tr}(A(t))dt \Longrightarrow d\eta_{2k} = \int \operatorname{tr}(dA(t))dt = \int \operatorname{tr}(\frac{dB(t)}{dt})dt = B(\infty). \tag{6.26}$$

In the case of the Chern forms A and B are Schwartz. Here A is Schwartz, since there is a t-derivative somewhere. However, $B(t)$ involves no t derivative, so it has a possibly non-zero limit as $t \to \infty$. Clearly in fact

$$B(\infty) = \operatorname{tr}((g_\infty^{-1} dg_\infty)^{2k+1}) = R^* \operatorname{Ch}_{2k+1} . \tag{6.27}$$

□

7. Topic 1: Determinant and eta
In place of lecture for Friday, 12 September, 2008

A natural question to ask yourself is:- Was it really necessary to construct the Fredholm determinant by hand? In fact Boris did suggest something pretty close to that! Indeed the answer is no, it was not really necessary, but how else would you appreciate the following?

Recall the delooping sequence

$$G^{-\infty}_{\mathrm{sus}} \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus}} \xrightarrow{R} G^{-\infty}. \tag{7.1}$$

Now, I showed that the first odd Chern form, $\mathrm{Tr}(g^{-1}dg)$ can be 'lifted' to the central group – not by using R don't be confused on this – by using the evaluation map and push-forward. This defines, in general, the eta forms or in this case the eta invariant on the central group:-

$$\eta : \tilde{G}^{-\infty}_{\mathrm{sus}} \longrightarrow \mathbb{C},\ \eta(\tilde{g}) = \int_{\mathbb{R}} \mathrm{tr}\left(\tilde{g}^{-1}(t)\frac{d\tilde{g}(t)}{dt}\right) dt. \tag{7.2}$$

Is this really an eta invariant you ask? Well, in this context it is, it is not exactly *THE* eta invariant – that is essentially the same thing for the quantization sequence as we shall see later.

Exercise 1. Show that the eta invariant as defined in (7.2) is log-multiplicative:-

$$\eta(gh) = \eta(g) + \eta(h). \tag{7.3}$$

So, how does this construct the determinant? Well, I showed before that

$$d\eta = R^* \operatorname{Ch}_1. \tag{7.4}$$

In particular, combined with (7.3) this just means

$$\eta\big|_{G^{-\infty}_{\mathrm{sus}}} \text{ is locally constant.} \tag{7.5}$$

Exercise 2. Show by finite dimensional approximation (remember that $\mathcal{S}(\mathbb{R})$ behaves like $\dot{\mathcal{C}}^\infty[0,1])$) that

$$\eta : G^{-\infty}_{\mathrm{sus}} \longrightarrow 2\pi i\mathbb{Z}. \tag{7.6}$$

So we can define an integer-valued 'index' map (really it is the winding number of the Fredholm determinant)

$$\mathrm{ind} = \frac{\eta}{2\pi i} : G^{-\infty}_{\mathrm{sus}} \longrightarrow \mathbb{Z}. \tag{7.7}$$

This can be added to the exact sequence above to get a commutative diagram:

$$\begin{array}{ccccc} G^{-\infty}_{\mathrm{sus}} & \longrightarrow & \tilde{G}^{-\infty}_{\mathrm{sus}} & \xrightarrow{R} & G^{-\infty} \\ \downarrow{\scriptstyle \mathrm{ind}} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \det} \\ \mathbb{Z} & \xrightarrow{2\pi i\times} & \mathbb{C} & \xrightarrow{\exp} & \mathbb{C}^*. \end{array} \tag{7.8}$$

Now, what about

$$\begin{aligned} SG^{-\infty} &= \{g \in G^{-\infty}; \det(g) = 1\} \hookrightarrow G^{-\infty}, \\ \tilde{G}^{-\infty}_{\mathrm{sus},\eta=1} &= \{\tilde{g} \in \tilde{G}^{-\infty}_{\mathrm{sus}}; \eta(\tilde{g}) = 0\}, \\ G^{-\infty}_{\mathrm{sus,ind}=0} &= \{g \in G^{-\infty}_{\mathrm{sus}}; \eta(g) = 0\}. \end{aligned} \tag{7.9}$$

Exercise 3. Show that $\tilde{G}^{-\infty}_{\mathrm{sus},\eta=0}$ is contractible and that the combined diagram

(7.10)
$$\begin{tikzcd}
 & \{\mathrm{Id}\} \arrow[d] & \{\mathrm{Id}\} \arrow[d] & \{\mathrm{Id}\} \arrow[d] & \\
\{\mathrm{Id}\} \arrow[r] & G^{-\infty}_{\mathrm{sus,ind}=0} \arrow[r] \arrow[d, "\mathrm{ind}"] & \tilde{G}^{-\infty}_{\mathrm{sus},\eta=0} \arrow[r, "R"] \arrow[d, "\eta"] & SG^{-\infty} \arrow[r] \arrow[d, "\det"] & \{\mathrm{Id}\} \\
\{\mathrm{Id}\} \arrow[r] & G^{-\infty}_{\mathrm{sus}} \arrow[r] \arrow[d, "\mathrm{ind}"] & \tilde{G}^{-\infty}_{\mathrm{sus}} \arrow[r, "R"] \arrow[d, "\eta"] & G^{-\infty} \arrow[r] \arrow[d, "\det"] & \{\mathrm{Id}\} \\
\{0\} \arrow[r] & \mathbb{Z} \arrow[r, "2\pi i\times"] \arrow[d] & \mathbb{C} \arrow[r, "\exp"] \arrow[d] & \mathbb{C}^* \arrow[r] \arrow[d] & \{1\} \\
 & \{0\} & \{0\} & \{1\} &
\end{tikzcd}$$

is commutative and has exact rows and columns.

The first row is a reduced classifying sequence for K-theory.

Exercise 4. (Needs a bit more analysis) Define the unitary subgroup of $G^{-\infty}$, show that $G^{-\infty}$ retracts to it and construct a diagram as in (7.10) based on the unitary group and its loop groups and their subgroups.

8. Topic 2: Higher dimensional harmonic oscillator In place of lecture for Monday, 15 September

Carry through the discussion of the higher-dimensional isotropic smoothing operators, forming the algebra $\Psi^{-\infty}(\mathbb{R}^n)$, the associated group $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and corresponding loop groups. Similarly, for any compact manifold X, for the moment without boundary, discuss $\Psi^{-\infty}(X)$, $G^{-\infty}(X)$ and $\tilde{G}^{-\infty}_{\mathrm{sus}}(X)$ etc.

Here are some steps to help you along the way.

(1) Show that $\mathcal{S}(\mathbb{R}^{2n})$ becomes a non-commutative Fréchet algebra which will be denoted $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, with continuous product given by operator composition as in the 1-dimensional case

$$a \circ b(z,z') = \int_{\mathbb{R}^n} a(z,z'')b(z'',z')dz''. \tag{8.1}$$

(2) Discuss the higher dimensional harmonic oscillator using the n creation and annihilation operators

$$C_j = -\partial_{z_j} + z_j,\ A_j = C_j^* = \partial_{z_j} + z_j,$$
$$H = H(n) = \sum_{j=1}^n C_jA_j + n,\ [A_j, C_j] = 2,\ j = 1, \dots, n. \tag{8.2}$$

Show that H has eigenvalues $n + 2\mathbb{N}_0$ with the dimension of the eigenspace with eigenvalue $n + 2k$ equal to the dimension of the space of homogeneous polynomials of degree k in n variables.

(3) Compute the constants such that the functions

$$h_0 = c_0 \exp(-|z|^2/2),\ h_\alpha = c_\alpha C^\alpha h_0,\ \alpha \in \mathbb{N}_0^n \tag{8.3}$$

is orthonormal in $L^2(\mathbb{R}^n)$ and show that they form a complete orthonormal basis.

(4) Show that for any $u \in \mathcal{S}(\mathbb{R}^n)$ the Fourier-Bessel series

$$f = \sum_\alpha \langle f, h_\alpha\rangle h_\alpha \tag{8.4}$$

converges in $\mathcal{S}(\mathbb{R}^n)$ and that this gives an isomorphism

$$\mathcal{S}(\mathbb{R}^n) \longrightarrow \{\{c_\alpha\}; \sup_\alpha |\alpha|^N|c_\alpha| < \infty,\ \forall\ N \in \mathbb{N}\},\ |\alpha| = \sum_j \alpha_j. \tag{8.5}$$

(5) Show, either directly or by discussing the appropriate 'higher dimensional' versions of $\Psi^{-\infty}(\mathbb{N})$ based on sequences as in (8.5), that $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is topologically isomorphic to the algebra $\Psi^{-\infty}(\mathbb{N})$.

(6) Briefly describe and discuss the group $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

(7) Introduce the (higher, pointed, flat) loop groups of $G^{-\infty}_{\mathrm{sus}(k),\mathrm{iso}}(\mathbb{R}^n)$.

(8) Show that

$$\mathrm{tr}(a) = \int_{\mathbb{R}^n} a(z,z)dz \tag{8.6}$$

is the trace functional on $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

(9) Can you show that it is unique up to a constant multiple as a continuous linear functional which vanishes on commutators?

(10) See how everything else we have done so far looks in this setting!

(11) Extend these results further to any compact manifold, using the eigendecomposition for the Laplacian. I will come back to this and disuss it more seriously later.

9. Topic 3: Clifford algebras
In place of lecture for Wednesday, 17 September

Clifford algebra. If you know all about Clifford algebras, go to item 17 below.

(1) Let V be a real vector space, of dimension n, equipped with a Euclidean struture – that is a positive-definite symmetric bilinear form

$$h : V \times V \longrightarrow \mathbb{R}. \tag{9.1}$$

We will associate with V two algebras, the real and complex Clifford algebras. The latter can also be defined for a complex vector space with a positive definite hermitian bilinear form. In fact only non-degeneracy of the form is really needed but the Euclidean case is the one we want.

(2) Recall the full tensor algebra on V. This consists of the formal direct sum of the tensor products

$$\mathcal{T} = \sum_{k=0}^{\infty} (V^*)^{\otimes k}. \tag{9.2}$$

Thus an element of $\mathcal{T}$ is a sequence with kth term an element of $(V^*)^{\otimes k}$ and with all but finitely many terms zero. This is an algebra in the obvious way, with component-wise addition and tensor product. Of course the sequences should be thought of as finite sums terminating at some point.

(3) Note that I have not fixed the coefficients here. Thus (9.2) corresponds to real coefficients, with $(V^*)^0 = \mathbb{R}$. We are actually more interested in the complex version

$$\mathcal{T}_{\mathbb{C}} = \mathbb{C} \otimes \mathcal{T} \tag{9.3}$$

in which V^* is replaced throughout by its complexification.

(4) The tensor product of two copies $V^* \otimes V^*$ is the space of bilinear forms on V and so decomposes into the symmetric and antisymmetric parts, S^2V^* and Λ^2V^*. The former has dimension $\frac{1}{2}n(n+1)$ and the latter dimension $\frac{1}{2}n(n-1)$, with S^2V^* spanned by elements of the form

$$w_1 \otimes w_2 + w_2 \otimes w_1,\ w_1,\ w_2 \in V^*. \tag{9.4}$$

(5) If h^* is the dual metric, induces on V^* by h as a metric on V consider

$$J = \{w_1 \otimes w_2 + w_2 \otimes w_1 - 2h^*(w_1, w_2) \in \mathcal{T}. \tag{9.5}$$

This is a linear space, with complexification $J_{\mathbb{C}}$ given by the same terms with complex coefficients. Note that it is more conventional to replace the $-$ sign in (9.5) by a $+$. If that is the way you like it, bad luck.

(6) In $\mathcal{T}$ consider the ideal generated by J

$$\mathcal{J} = \mathcal{T} \otimes J \otimes \mathcal{T} \subset \mathcal{T},\ \mathcal{J}_{\mathbb{C}} = \mathbb{C} \otimes J \subset \mathcal{T}_{\mathbb{C}}. \tag{9.6}$$

(7) Finally then we have the Clifford algebras, real and complex:

$$\mathrm{Cl}(V) = \mathcal{T}/\mathcal{J},\ \mathbb{C}\mathrm{l}(V) = \mathcal{T}_{\mathbb{C}}/\mathcal{J}_{\mathbb{C}} = \mathbb{C} \otimes \mathrm{Cl}(V). \tag{9.7}$$

(8) Show by an inductive argument (or otherwise) that *as linear spaces* the Clifford algebras are isomorphic to the corresponding (real and complex)

exterior vector spaces

$$\Lambda^* V^* = \sum_{k=0}^{n} \Lambda^k V^* \tag{9.8}$$

of sums of totally antisymmetric k-linear forms on V.

(9) Show that the odd an even parts of the tensor product descend to the quotient so that the Cliffor algebras are $\mathbb{Z}_2$-graded

$$\mathrm{Cl}(V) = \mathrm{Cl}_{\mathrm{even}}(V) \oplus \mathrm{Cl}_{\mathrm{odd}}(V) \tag{9.9}$$

with the product graded in the sense that the product of two even, or two odd, elements is even and the product of an odd and an even element is odd.

(10) Show that the Clifford algebras are filtered by degree where an element is of degree k or less if it can be written as a sum of products each consisting of at most k elements of V^* :

$$\mathrm{Cl}^{(k)}(V) = \{u \in \mathrm{Cl}(V); u = \sum_{l \le k} c_\bullet w_1 \dots w_l, \tag{9.10}$$

$$\mathrm{Cl}^{(k)}(V)\,\mathrm{Cl}^{(l)}(V) \subset \mathrm{Cl}^{(k+l)}(V),\ \mathrm{Cl}(V) = \sum_{j=0}^{n} \mathrm{Cl}^{(n)}(V).$$

(11) Check that

$$\mathrm{Cl}^{(0)} = \mathbb{R},\ \mathrm{Cl}^{(1)} = V^* \oplus \mathbb{R} \Longrightarrow V^* \hookrightarrow \mathrm{Cl}(V). \tag{9.11}$$

(12) Show that an element of V^*, injected into $\mathrm{Cl}(V)$ has an inverse if and only if it is non-zero.

(13) Show that the associated graded algebra is canonically the exterior algebra

$$\sum_j \mathrm{Cl}^{(j)} \,/\, \mathrm{Cl}^{(j-1)} = \Lambda^* V \text{ as algebras.} \tag{9.12}$$

(14) Show that if V is given an orientation, so (using the metric as well) $\Lambda^n V^* = \mathbb{R}$ (or $\mathbb{C}$) this map defines the supertrace

$$\mathrm{str} : \mathrm{Ch}(V) \longrightarrow \mathrm{Cl}(V)/\,\mathrm{Cl}^{(n-1)}(V) = \Lambda^n V^* = \mathbb{R},\ \mathrm{str}(ab - (-1)^{\pm} ba) = 0 \tag{9.13}$$

where a and b are either even or odd and the sign is $+$ unless they are of opposite persuasions.

(15) Proceeding inductively (or otherwise) construct the fundamental spin representations, which are to say algebra isomorphisms to the matrix algebras

$$\begin{gathered} \mathbb{C}\mathrm{l}(\mathbb{R}) = \mathbb{C} \oplus \mathbb{C},\ \mathbb{C}\mathrm{l}(\mathbb{R}^2) = M(2, \mathbb{C}),\ \mathbb{C}\mathrm{l}(\mathbb{R}^3) = M(2, \mathbb{C}) \oplus M(2, \mathbb{C}), \\ \mathbb{C}\mathrm{l}(\mathbb{R}^{2k}) = M(2^k, \mathbb{C}),\ \mathbb{C}\mathrm{l}(\mathbb{R}^{2k+1}) = M(2^k, \mathbb{C}) \oplus M(2^k, \mathbb{C}). \end{gathered} \tag{9.14}$$

(16) Work out, if you have the time and energy, the 8-fold periodicity analogous to (9.14) for the real Clifford algebras!

(17) There is plenty more about Clifford algebras, but why are they useful here? In the even dimensional case, so $V = \mathbb{R}^{2p}$, the injection of V^* into $\mathrm{Cl}^{(1)}(V)$ leads to an embedding of the unit sphere

$$\mathbb{S}^{2k-1} = \{w \in \mathbb{R}^{2k}; |w|_{h^*} = 1\} \longrightarrow \mathbb{C}\mathrm{l}(\mathbb{R}^{2k}) = \mathrm{GL}(2^k, \mathbb{C}) \longrightarrow G^{-\infty}. \tag{9.15}$$

(18) We will show that (9.15) generates all the homotopy groups of $G^{-\infty}$.

10. Lecture 7: Semiclassical quantization Friday, 19 September, 2008

We will show that there is a way to directly 'pass from' $\mathcal{S}(\mathbb{R}^2)$ with its commutative product to $\Psi^{-\infty}_{\text{iso}}(\mathbb{R})$, which is the same space with the operator product. In fact this will 'work' much more generally, but it should be understood at the outset that this is not a map. It does define maps at various levels but 'semiclassical quantization' in this sense is not itself a map.

Going back to the definition of $\Psi^{-\infty}_{\text{iso}}(\mathbb{R})$ recall that I just defined this directly in terms of the operator product. We have already discussed smooth families of such operators. For one parameter families this just corresponds to $\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}))$. We will give this space another family of products in which the product depends on the parameter. Namely the 'semiclassical operator product' is initially only defined for $\epsilon > 0$ since the integrals look singular.

I will try to motivate it after writing it down. First let me change coordinates, to 'Weyl coordinates' on $\mathbb{R}^2$ which emphasize the diagonal

$$f(z,z') = F(\frac{1}{2}(z+z'), z-z'),\ F = Wf,\ F(t,s) = f(t+\frac{1}{2}s, t-\frac{1}{2}s). \tag{10.1}$$

Clearly, W is a linear isomorphism of $\mathcal{S}(\mathbb{R}^2)$.

Definition 3. A smooth family $A \in \mathcal{C}^\infty((0,1];\Psi^{-\infty}_{\text{iso}}(\mathbb{R}))$ is said to be a semiclassical family of smoothing operators if it is of the form

$$A_\epsilon u(z) = \epsilon^{-1}\int_{\mathbb{R}} F(\epsilon, \frac{\epsilon}{2}(z+z'), \frac{z-z'}{\epsilon})u(z')dz' \text{ with } F \in \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2)). \tag{10.2}$$

Note that as an operator A_ϵ only makes sense for $\epsilon > 0$ but the kernel function F is required to be smooth down to $\epsilon = 0$, thus the singularity at $\epsilon = 0$ is of a very particular kind. Since F is determined by its restriction to $\epsilon > 0$, it is actually determined by the family of operators A_ϵ.

Lemma 8. *If A and B are semiclassical families of smoothing operators then so is the composite $A_\epsilon \circ B_\epsilon$.*

Proof. Suppose A corresponds to the kernel function $F \in \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2))$ and B to G in the sense of the definition above. The composite operator, for each $\epsilon > 0$ has kernel in the ordinary sense

$$\begin{gathered}(A_\epsilon \circ B_\epsilon)u(z) = \int_{\mathbb{R}} c(z,z')u(z')dz', \\ c(\epsilon,z,z') = \epsilon^{-2}\int_{\mathbb{R}} F(\epsilon,\frac{\epsilon}{2}(z+z''),\frac{z-z''}{\epsilon})G(\epsilon,\frac{\epsilon}{2}(z''+z'),\frac{z''-z'}{\epsilon})dz''.\end{gathered} \tag{10.3}$$

Thus the kernel function defined, H, defined from (10.2) by c is

$$\begin{gathered}H(\epsilon,t,s) = \epsilon c(\epsilon,\epsilon^{-1}t+\frac{\epsilon}{2}s,\epsilon^{-1}t-\frac{\epsilon}{2}s) = \\ \epsilon^{-1}\int_{\mathbb{R}} F(\epsilon,\frac{t}{2}+\frac{\epsilon^2}{4}s+\frac{\epsilon}{2}z'',\epsilon^{-2}t+\frac{1}{2}s-\frac{z''}{\epsilon})G(\epsilon,\frac{\epsilon}{2}z''+\frac{1}{2}t-\epsilon^2\frac{s}{4},\frac{z''}{\epsilon}-\epsilon^{-2}t+\frac{1}{2}s)dz''.\end{gathered} \tag{10.4}$$

Changing variable of integration $z'' = \epsilon r + \epsilon^{-1}t$ this reduces to

$$H(\epsilon,t,s) = \int_{\mathbb{R}} F(\epsilon,t+\frac{\epsilon^2}{2}(r+\frac{1}{2}s),\frac{1}{2}s-r)G(\epsilon,t+\frac{\epsilon^2}{2}(r-\frac{1}{2}s),r+\frac{1}{2}s)dr. \tag{10.5}$$

The absolute convergence, and rapid decay of the result, is clear for $\epsilon > 0$ and for ϵ small follows uniformly from fact that the integrand is bounded by

$$\begin{aligned}(10.6)\quad & C_N(1+|t+\epsilon^2(r+s/2|)^{-N}(1+|r-s/2|)^{-N}(1+|t+\frac{\epsilon^2}{2}(r-s/2)|)^{-N} \\ & \times(1+|r+s/2|)^{-N} \\ & \leq C'(1+|t|)^{-N}(1+|r|)^{-N}(1+|s|)^{-N}.\end{aligned}$$

Derivatives can be estimated in the same way. Thus (10.6) defines a continuous bilinear map

$$\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2))\times\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2))\longrightarrow\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2)). \tag{10.7}$$

This shows that the semiclassical smoothing operators form an algebra. □

I will denote this algebra as $\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R})$, with the parameter suppressed into a suffix – remember it is by no means simply a smooth parameter as $\epsilon\downarrow 0$.

Notice what the product looks like at $\epsilon = 0$. The limiting rescaled kernel of the product is simply

$$H(0,t,s)=\int_{\mathbb{R}}F(0,t,\frac{1}{2}s-r)G(0,t,r+\frac{1}{2}s)dr. \tag{10.8}$$

This is a product on $\mathcal{S}(\mathbb{R}^2)$ so we have found another one! However, notice that it is commutative – changing variable from r to $-r$ effectively reverses the product. Not surprisingly this product can actually be reduced to the usual pointwise product on $\mathcal{S}(\mathbb{R}^2)$ by the simple expedient of taking the Fourier transform in s. For any semiclassical family, (10.2), we define the *semiclassical symbol* to be the Fourier transform of the limit at $\epsilon = 0$:

$$\sigma_{\mathrm{sl}}(A)(t,\tau)=\int_{\mathbb{R}}F(0,t,s)e^{-is\tau}ds. \tag{10.9}$$

Then for the product

$$\sigma_{\mathrm{sl}}(AB)(t,\tau)=\int_{\mathbb{R}}H(0,t,s)e^{-is\tau}ds \tag{10.10}$$

and then (10.8) becomes

$$\begin{aligned}(10.11)\quad & \sigma_{\mathrm{sl}}(AB)(t,\tau)= \\ & \int_{\mathbb{R}}\int_{\mathbb{R}}e^{-i(\frac{1}{2}s-r)\tau}F(0,t,\frac{1}{2}s-r)e^{-i(\frac{1}{2}s+r)\tau}G(0,t,r+\frac{1}{2}s)drds=\sigma_{\mathrm{sl}}(A)\sigma_{\mathrm{sl}}(B).\end{aligned}$$

Proposition 11. *The algebra of semiclassical smoothing operators with symbol homomorphism to the commutative algebra $\mathcal{S}(\mathbb{R}^2)$ gives a short exact (and multiplicative) sequence*

$$0\longrightarrow\epsilon\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R})\hookrightarrow\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R})\twoheadrightarrow\mathcal{S}(\mathbb{R}^2)\longrightarrow 0. \tag{10.12}$$

So, how do we get our (weak) homotopy equivalence? We simply 'turn on' the non-commutativity.

Exercise 5. (Will be done on Monday). Show that if $a\in\mathcal{S}(\mathbb{R}^2;M(N,\mathbb{C}))$ is such that $(\mathrm{Id}+a(t,\tau))^{-1}$ exists for all $(t,\tau)\in\mathbb{R}^2$ then if $A\in\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^N)$ is a (matrix-valued) semiclassical family with $\sigma_{\mathrm{sl}}(A)=a$ (which exists by (10.12)) then $\mathrm{Id}+A_\epsilon\in G^{-\infty}_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^N)$ for $\epsilon>0$ small. This works uniformly on compact sets, so if $f:X\longrightarrow$

$\mathcal{S}(\mathbb{R}^2;M(N,\mathbb{C}))\hookrightarrow G^{-\infty}_{\mathrm{sus}(2)}$ then the quantized map $f_\epsilon : X \longrightarrow G^{-\infty}$ (where these are different realizations of $G^{-\infty}$) for $\epsilon>0$ small, is well defined up to homotopy. This leads to the homotopy equivalence

$$Q_{\mathrm{sl}} : \pi_j(G^{-\infty}_{\mathrm{sus}(2)}) \longrightarrow \pi_j(G^{-\infty})\ \forall\ j, \tag{10.13}$$

which will prove Bott periodicity for us (with a bit more work).

Exercise 6. Consider the differential operators on $\mathbb{R}$ with polynomial coefficients

$$P=\sum_{k,j=0}^{N} c_{kj}x^kD_x^j,\ D_x=-i\frac{d}{dx}. \tag{10.14}$$

Give x and D_x 'homogeneity one' and so filter these operators by the combined order – this is the isotropic filtration.

Now, show that if A is a semiclassical family of smoothing operators then so is ϵ^NPA if P has total order N in this sense. Compute the semiclassical symbol of ϵ^NPA.

Exercise 7. Show that the definition of semiclassical families of smoothing operators extends directly to operators on $\mathbb{R}^n$ simply by reinterpretation of the formulæ.

Exercise 8. In preparation for what I will do on Monday, if A and B are semiclassical smoothing families as defined above, we have shown that the function $H\in\mathcal{C}^\infty([0,1]_\epsilon;\mathcal{S}(\mathbb{R}^2))$ fixing its kernel is determined by the corresponding functions F and G for A and B. Show that the Taylor series of H at $\epsilon=0$ is determined by the Taylor series of A and B and derive a formula for it – you will get a variant of the 'Moyal product' (although several differnt things go under this name). The most important thing for us is the second term in the expansion (well, given that we already know the first term!)

11. Lecture 8: Bott element
Monday, 22 September, 2008

The first thing I want to do today is to use the three Pauli matrices to construct the Bott element. Let's not worry about what this element is, or even where it is, for the moment. The initial objective is to find something non-trivial on $\mathbb{R}^2$.

The Pauli matrices, all elements of $M(2,\mathbb{C})$, I will denote

$$\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \gamma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \gamma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{11.1}$$

Clearly they are linearly independent over $\mathbb{C}$ and, together with $\mathrm{Id}_{2\times 2}$ span $M(2,\mathbb{C})$ as a linear space. Their products satisfy the cyclic conditions

$$\begin{gathered} \gamma_1\gamma_2 = i\gamma_3,\ \gamma_2\gamma_3 = i\gamma_1,\ \gamma_3\gamma_1 = i\gamma_2, \\ \gamma_2\gamma_1 = -i\gamma_3,\ \gamma_3\gamma_1 = -i\gamma_1,\ \gamma_1\gamma_3 = -i\gamma_2 \\ \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = \mathrm{Id},\ \gamma_1\gamma_2\gamma_3 = i\,\mathrm{Id}\,. \end{gathered} \tag{11.2}$$

and hence the Clifford identities

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij},\ i,j = 1,2,3. \tag{11.3}$$

This shows (see) that they give a representation, to wit the spin representation, of the complex Clifford algebra for $\mathbb{R}^2$. I will not use this explicitly here, but it is one way of getting a better understanding of what is going on. Each of the self-adjoint *involutions* γ_i has determinant -1 and so has two one-dimensional eigenspaces with eigenvalues ± 1.

Let us now choose a handy smooth function on $\mathbb{R}$ satisfying

$$\Theta : \mathbb{R} \longrightarrow \mathbb{R},\ \Theta(t) = 0,\ t < -2,\ \Theta(t) = \pi,\ t > -1,\ \Theta'(t) \geq 0. \tag{11.4}$$

So, $e^{i\Theta(t)}$ is a smooth flat pointed loop into the circle from the line.

Now consider the map from $\mathbb{R}^2$ into $M(2,\mathbb{C})$ given in terms of polar coordinates

$$\begin{gathered} (t,\tau) = r(\cos\theta, \sin\theta), \\ b(t,\tau) = \cos(\Theta(-r))\gamma_1 + \sin(\Theta(-r))\cos(\theta)\gamma_2 - \sin(\Theta(-r))\sin(\theta)\gamma_3, \\ b(t,\tau) = \begin{pmatrix} \cos(\Theta(-r)) & \sin(\Theta(-r))e^{i\theta} \\ \sin(\Theta(-r))e^{-i\theta} & -\cos(\Theta(-r)) \end{pmatrix} \end{gathered} \tag{11.5}$$

Let's hope I have made a sensible choice of signs!

The first thing to observe is that $-r$ is running from $-\infty$ to 0 as we come inwards from infinity, so

$$b(t,\tau) = \begin{cases} \gamma_1 & |(t,\tau)| > 2 \\ -\gamma_1 & |(t,\tau)| < 1. \end{cases} \tag{11.6}$$

It follows that $b : \mathbb{R}^2 \longrightarrow M(2,\mathbb{C})$ is smooth and is in fact a compactly-supported perturbation of γ_1 :

$$b - \gamma_1 \in \mathcal{C}_c^\infty(\mathbb{R}^2, M(2,\mathbb{C})). \tag{11.7}$$

Secondly, the Clifford identities show that

$$\begin{aligned} b^2 = {} & \cos^2(\Theta(-r))\gamma_1^2 + \sin^2(\Theta(-r))\cos^2(\theta)\gamma_2^2 + \sin^2(\Theta(-r))\sin^2(\theta)\gamma_3^2 \\ & + A(\gamma_1\gamma_2 + \gamma_2\gamma_1) + B(\gamma_2\gamma_3 + \gamma_3\gamma_2) + C(\gamma_3\gamma_1 + \gamma_1\gamma_3) = \mathrm{Id}\,. \end{aligned} \tag{11.8}$$

Thus, b is in fact a family of self-adjoint involutions. Moroever, it follows (without computation) that $\det(b(t,\tau)) = -1$, so again it has one-dimensional eigenspaces with eigenvalues ± 1. Let me denote the spectral decomposition by

$$\begin{gathered}b(t,\tau) = b_+(t,\tau) - b_-(t,\tau),\\ b_+ - \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \in \mathcal{C}^\infty_c(\mathbb{R}^2; M(2,\mathbb{C})),\ b_- - \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} \in \mathcal{C}^\infty_c(\mathbb{R}^2; M(2,\mathbb{C})).\end{gathered} \tag{11.9}$$

Here, $b_\pm$ are the orthogonal projections onto the ± 1 eigenspaces; probably b_+ best deserves to be called the Bott element. Its range is a 1-dimensional (over $\mathbb{C}$ of course) subbundle of $\mathbb{C}^2$ over $\mathbb{R}^2$ which is trivial (just the first component) near infinity but not globally trivial, as we shall see, if one only allows trivializations which are constant near infinity.

Exercise 9. Try to show directly that b_+ is not homotopically trivial, in the sense that there is no homotopy through families of projections to a constant projection where the constant projection has to be always constant at infinity (either fixed forever there, or just constant for each value of the parameter, it doesn't matter). I think the easiest way to do this is to find an homotopy invariant which shows that such a deformation is not possible. The obvious one is the total curvature of the line bundle. The curvature is a 2-form on $\mathbb{R}^2$ of compact support. I will compute it later, probably not today.

We can proceed with either the involution b or the projection b_+. Since it is more in the spirit of what we have done so far, consider the involution. It is definitely of the form

$$b(t,\tau) = \gamma_1 + \delta(t,\tau),\ \delta \in \mathcal{S}(\mathbb{R}^2; M(N,\mathbb{C})) \tag{11.10}$$

so we can apply 'semiclassical quantization' to the perturbation δ. Remember what this means: It just is the statement that

$$\exists\ D \in \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2) \text{ s.t. } \sigma_{\mathrm{sl}}(B) = \delta. \tag{11.11}$$

Here we need to work component by component in the 2×2 matrices. This we already know, from the surjectivity of the semiclassical symbol map, but we can do quite a lot more.

Lemma 9. *The semiclassical family D in* (11.11) *can be chosen so that (as operators on $\mathcal{S}(\mathbb{R};\mathbb{C}^2)$)*

$$(\gamma_1 + D_\epsilon)^2 = \mathrm{Id}\,. \tag{11.12}$$

That is, the quantization can also be chosen to be a family of involutions. Notice that we are 'quantizing' the constant matrix to the same constant matrix – really componentwise a multiple of the identity as an operator – which is not by any means a semiclassical family of smoothing operators. However, it is consistent with the way we (or rather you) showed that differential operators with polynomial coefficients compose with semiclassical families. This we are just demanding that the identity be quantized to the identity and this is consistent with the semiclassical symbol map, etc.

Exercise 10. Show (without doing any work) that in the same sense as in the Lemma, the projections $b_\pm$ can be quantized to commuting projections (also called idemptotents if we do not demand they be selfadjoint) $B_\pm$ with

$$B_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + D_+,\ D_+ \in \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2),\ \sigma_{\mathrm{sl}}(D_+) = b_\pm$$
$$B_- = \mathrm{Id} - B_+,\ B = B_+ - B_-,\ B_\pm^2 = B_\pm. \tag{11.13}$$

Exercise 11. Check, if only mentally, that it is consistent to extend the semiclassical symbol map to constant matrices, where the symbol is just the matrix itself, in the sense that this gives us a multiplicative map symbol

$$\sigma_{\mathrm{sl,iso}} : M(N;\mathbb{C}) + \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^N) \longrightarrow M(N;\mathbb{C}) + \mathcal{S}(\mathbb{R}^2; M(N;\mathbb{C})) \tag{11.14}$$

for this algebra into the algebra $\mathcal{S}(\mathbb{R}^2; M(N;\mathbb{C}))$ with constant multiples of the identity appended. We will do this more systematically later.

Proof. To do this I will need to check another couple of important facts about semiclassical quantization, but let's proceed anyway. For the first step we don't have much choice. Using the surjectivity of the symbol map, choose a D as in (11.11), but denoted D_0. The choice of symbol, together with the multipicativity and the exactness of the symbol sequence shows that

$$E_1 = (\gamma_1 + D_0)^2 - \mathrm{Id} = \gamma_1 D_0 + D_0\gamma_1 + D_0^2 \in \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2) \text{ satisfies}$$
$$\sigma_{\mathrm{sl}}(E_1) = 0 \Longrightarrow E_1 = \epsilon E_1' \in \epsilon\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2). \tag{11.15}$$

Thus our first choice 'works to first order'. We wish to modify D_0, the initial choice, by choosing $D_1 = \epsilon D_1' \in \epsilon\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2))$ to get the desired identity (11.12) to second order. Clearly adding $\epsilon D_1'$ changes the computation to

(11.16)
$$\begin{aligned} E_2 = (\gamma_1 + D_0 + \epsilon D_1')^2 - \mathrm{Id} &= E_1 + \epsilon(\gamma + D_0)D_1' + \epsilon D_1'(\gamma + D_0) + \epsilon^2(D_1')^2 \\ &= \epsilon(E_1' + (\gamma + D_0)D_1' + D_1'(\gamma + D_0)) + \epsilon^2(D_1')^2. \end{aligned}$$

Thus to ensure that $E_2 = \epsilon^2 E_2' \in \epsilon^2\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2)$ we wish to choose D_1' so that

$$\sigma_{\mathrm{sl}}(E_1' + (\gamma + D_0)D_1' + D_1'(\gamma + D_0)) = 0 \Longleftrightarrow$$
$$b\sigma_{\mathrm{sl}}(D_1') + \sigma_{\mathrm{sl}}(D_1')b = -\sigma_{\mathrm{sl}}(E_1'). \tag{11.17}$$

Here I have used the original choice of $\sigma_{\mathrm{sl}}(D_0) = \delta$. The symbols are still non-commutative, but only because they take values in 2×2 matrices; this is just matrix algebra. So why is such a choice possible? The matrix on the left in the last identity is not arbitrary. Indeed, recalling that $b = b_+ - b_-$, $\mathrm{Id} = b_+ + b_-$ it is necessarily diagonal with respect to this decomposition, since it is just

$$2b_+\sigma_{\mathrm{sl}}(D_1')b_+ - 2b_-\sigma_{\mathrm{sl}}(D_1')b_-. \tag{11.18}$$

Thus, (11.17) can only be solved if $\sigma_{\mathrm{sl}}(E_1')$ is also diagonal. Fortunately it is, because of the associativity of the (operator) product which shows that

$$B_0'E_1 = B_0'((B_0')^2 - \mathrm{Id}) = ((B_0')^2 - \mathrm{Id})B_0' = E_1B_0' \Longrightarrow$$
$$b\sigma_{\mathrm{sl}}(E_1) = \sigma_{\mathrm{sl}}(E_1)b \Longrightarrow \sigma_{\mathrm{sl}}(E_1) = b_+\sigma_{\mathrm{sl}}(E_1)b_+ + b_-\sigma_{\mathrm{sl}}(E_1)b_-. \tag{11.19}$$

Thus indeed we can choose D_1' to satisfy (11.17), for instance just require

$$\sigma_{\mathrm{sl}}(D_1') = -\frac{1}{2}b_+\sigma_{\mathrm{sl}}(E_1)b_+ + \frac{1}{2}b_-\sigma_{\mathrm{sl}}(E_1)b_-. \tag{11.20}$$

So, to complete the 'formal' part of the construction we just repeat this argument inducively. Suppose we have shown the existence of $D'_j \in \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2)$ for $0 \le j \le k$ so that (11.12) holds to order k (or is it $k+1$?)

$$(\gamma_1 + \sum_{0\le j\le k} \epsilon^j D'_j)^2 - \mathrm{Id} = E_{k+1} \in \epsilon^{k+1}\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2). \tag{11.21}$$

Then we want to choose D'_{k+1} to get to the next step. Adding $e^{k+1}D'_{k+1}$ changes the left side by

$$\epsilon^{k+1}\left((\gamma_1 + D_0)D'_{k+1} + D'_{k+1}(\gamma_1 + D_0)\right) \mod \epsilon^{k+1}\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2). \tag{11.22}$$

The choice (11.20), with 1 replaced by $k+1$ throughout works for the same reason. Thus, we can find a full formal solution.

Now, to proceed further we need first to pass from the formal series

$$\sum_{j=0}^{\infty} e^j D'_j \tag{11.23}$$

to an actual element of $\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2)$. This is Borel'e lemma.

Lemma 10. *[É. Borel] Given any sequence $D'_j \in \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^N))$ there exists an element $D' \in \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^N))$ such that*

$$D' - \sum_{j=0}^{k} \epsilon^j D'_j \in \epsilon^{k+1}\Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^N). \tag{11.24}$$

Proof. I will not do this, since it is just Borel's lemma when applied to the functions $F'_j(\epsilon;t,s)$ representing the opertors. Namely if $F'_j \in \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2))$ is such a sequence, with no constraints, then there exists $F \in \mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2))$ such that

$$F - \sum_{0\le j\le k} \epsilon^j F'_j \in \epsilon^{k+1}\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^2)) \ \forall\ k. \tag{11.25}$$

□

With this D' for our sequence, or series, as constructed above we conclude that

$$(\gamma + D')^2 - \mathrm{Id} \in \epsilon^\infty \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^N)) = \bigcap_{j=0}^{\infty} \epsilon^j \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2). \tag{11.26}$$

That is, this one element works to all orders.

Lemma 11. *The residual space*

$$\epsilon^\infty \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^N)) = \bigcap_{j=0}^{\infty} \epsilon^j \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R};\mathbb{C}^2) = \bigcap_{j=0}^{\infty} \epsilon^j \mathcal{C}^\infty([0,1];\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^2)) \tag{11.27}$$

is just the space of smooth families, in the ordinary sense, of Schwartz smoothing operators on $\mathbb{R}$ vanishing to infinite order at $\epsilon = 0$.

Exercise 12. I will almost certainly not have time to do this but it is straightforward. Think in terms of the kernel of the semiclassical family written out as

$$a(\epsilon, z, z') = \epsilon^{-1}F(\epsilon, \frac{\epsilon(z+z')}{2}, \frac{z-z'}{\epsilon}). \tag{11.28}$$

The function on the left is unque, it is the kernel of the operator and is certainly Schwartz for $\epsilon > 0$. The assumption is that the one function on the left can be written in the form

$$\epsilon^k \epsilon^{-1} F_k(\epsilon, \frac{\epsilon(z+z')}{2}, \frac{z-z'}{\epsilon}) \tag{11.29}$$

for each k where $F_k \in \mathcal{C}^\infty([0,1]; \mathcal{S}(\mathbb{R}^2))$. What we want are the estimates on a

$$\sup \epsilon^{-N} |D_\epsilon^p z^j (z')^k D_z^l D_{z'}^m a| < \infty \tag{11.30}$$

for all indices. Just check that any finite set of them follows from (11.29) by taking k large enough. The converse is easy.

So, now we know that our summed-up choice of quantization, D' satisfies

$$(\gamma_1 + D')^2 - \mathrm{Id} = E' \in \epsilon^\infty \mathcal{C}^\infty([0,1]; \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2)) \tag{11.31}$$

where the, perhaps improper, notation is too suggestive not to use. We still want to actually solve the problem, to get rid of the error term on the right. Just to keep you oriented, remember that $\sigma_{\mathrm{sl}}(D') = \delta$ and we are way beyond changing the leading term.

So finally the claim is that we can add an element of $\epsilon^\infty \mathcal{C}^\infty([0,1]; \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2))$ to D' to get rid of E'. This is now 'genuinely non-linear' where up to this point we have been linearizing.

So, first notice that if $d > 0$ and $z \in \mathbb{C}$ is such that $|z - \pm 1| \ge d$ then

$$\begin{gathered}(B'-z)^{-1} = \left((1-z)^{-1}\frac{1}{2}(B' + \mathrm{Id}) - (1+z)^{-1}\frac{1}{2}(B' - \mathrm{Id}) \right)(\mathrm{Id} + F),\\ F \in \epsilon^\infty \mathcal{C}^\infty([0,1]; \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2)),\ 0 < \epsilon < \epsilon_0(R) > 0.\end{gathered} \tag{11.32}$$

Here the operator on the left 'should have' eigenvalues $1 - z$ and $-(1 + z)$ on the positive and negative pieces of B' if this were an involution, so the first term on the right side is formally the inverse of this. To prove (11.32) compute the product

$$\begin{aligned}(B'-z)&\left((1-z)^{-1}\frac{1}{2}(B' + \mathrm{Id}) - (1+z)^{-1}\frac{1}{2}(\mathrm{Id} - B') \right)\\ &= (B'-z)\left(\frac{1}{1-z^2}B' + \frac{z}{1-z^2}\mathrm{Id}\right)\\ &= \frac{1}{1-z^2}(B')^2 - \frac{z^1}{1-z^2}\mathrm{Id} = \mathrm{Id} + \frac{1}{1-z^2}((B')^2 - \mathrm{Id}).\end{aligned} \tag{11.33}$$

As a result of our work so far, $(B')^2 - \mathrm{Id} = E' \in \epsilon^\infty \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2)$. Thus, by using Neumann series, the operator on the right in (11.33) is invertible, with inverse of the same form – at least for $0 \le \epsilon \le \epsilon_0$ for some $\epsilon_0 > 0$ depending on d and other constants. Thus (11.32) follows, with $F \in \epsilon^\infty \mathcal{C}^\infty([0,\epsilon_0]; \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2))$ and holomorphic in $|z - \pm 1| > d$.

Finally then we can use this to construct the quantization D, or $B = \gamma_1 + D$ we want. Just set

$$B_+ = \frac{i}{2\pi}\int_{|z-1|=\frac{1}{2}} (B'-z)^{-1} dz,\ B = 2B_+ - \mathrm{Id}. \tag{11.34}$$

I probably will not even get to this point today, but a few things remain. Namely we need to show that B_+ makes sense and is a projection, that

$$B - B_+ \in \epsilon^\infty \mathcal{C}^\infty([0,\epsilon_0]; \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}; \mathbb{C}^2))$$

and that B satisfies all our requirements. I leave this as an exercise in contour shifting – notice that the fact that $\epsilon_0 < 1$ is only a technical inconvenience, we can simply rescale the parameter (starting at $\frac{1}{2}\epsilon_0$) to make B exist out to $\epsilon = 1$. Mostly the semiclassical families are only of interest near $\epsilon = 0$. $\square$

12. Lecture 9: Adiabatic algebra and group Wednesday, 24 September

To carry through the argument for Bott periodicity that I have been edging towards, I have decided to take a slightly higher road than I initially intended. I hope this will actually be pretty clear but the first step is to throw together what we have done so far and work with an *adiabatic algebra* of smoothing operators. This is the same as the semiclassical algebra except that 'adiabatic' refers to a situation in which the 'semiclassical degeneration' occurs in only some of the variables. The name arises from Physics and refers to a formal motion which is so slow that the system remains in equilibrium. Here this just means that some of the variables become commutative. I will get to more geometric versions of this later in the semester. In fact we might as well jump into the higher dimensional case, which really makes very little difference.

Definition 4. A one-parameter family $A \in \mathcal{C}^\infty((0,1];\Psi^{-\infty}(\mathbb{R}^{d+k}))$ is *an adiabatic family* of smoothing operators, with respect to the the first d variables, if its Schwartz kernel is of the form

$$a_\epsilon(\epsilon,z,z',Z,Z') = \epsilon^{-d}F(\epsilon,\frac{\epsilon(z+z')}{2},\frac{z-z'}{\epsilon},Z,Z'),\ F\in\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2d+2k})). \tag{12.1}$$

So the case discussed up to this point corresponds to $d=1$ and $k=0$, although we allowed matrix values – which we could include here at only notational expense. If $k=0$ but $d>1$ we are in a higher dimensional semiclassical setting.

Proposition 12. *The adiabatic operators as in* (12.1) *form an algebra, denoted* $\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)$, *under operator compostion for* $\epsilon>0$.

Proof. The proof generalizes easily from the case above where $d=1$ and $k=0$. Let me give the defining isomorphism (12.1) a name:

$$\kappa:\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)\longrightarrow\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2d+2k})) \tag{12.2}$$

which we are pretty free to regard as an identification – indeed that is what I have been doing implicitly up to this point. What we showed when $d=1$ is that operator composition for $\epsilon>0$ induces a product which can be (corrected) generalized and written out explicitly:

$$\begin{gathered}H=\kappa(A\circ B)=\\ \int_{\mathbb{R}^d\times\mathbb{R}^k}F(\epsilon,t+\frac{\epsilon^2}{2}(r+\frac12 s),\frac12 s-r,Z,Z'')G(\epsilon,t+\frac{\epsilon^2}{2}(r-\frac12 s),r+\frac12 s,Z'',Z')dZ''\\ F=\kappa(A),\ G=\kappa(B).\end{gathered} \tag{12.3}$$

Recall that this just arises by noting the relationship of the Schwartz kernel, a, of A and $F=\kappa(A)$:

$$\begin{gathered}a(\epsilon,z,z',Z,Z')=\epsilon^{-d}F(\epsilon,\frac{\epsilon(z+z')}{2},\frac{z-z'}{\epsilon},Z,Z'),\\ F(\epsilon,t,s,Z,Z')=\epsilon^d a(\epsilon,\epsilon^{-1}t+\frac{\epsilon}{2}s,\epsilon^{-1}t-\frac{\epsilon}{2}s,Z,Z'),\end{gathered} \tag{12.4}$$

substituting into the formula for the product and changing variable. The same estimates as before show that this product is indeed a continuous bilinear map

$$\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2d+2k}))\times\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2d+2k}))\longrightarrow\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2d+2k})). \tag{12.5}$$

□

One thing I did not get to before is the extraction of the 'Moyel product' from the formula (12.4). Notice that (apart from the explicit dependence of $\kappa(A)$ on ϵ) ϵ only occurs through ϵ^2. Computing the Taylor series therefore gives

$$\hat{H}(\epsilon,t,\tau)\simeq\sum_{j=0}^{\infty}\epsilon^{2j}\sum_{|\alpha|+|\beta|=j}c_{\alpha,\beta}(\partial_t^{\alpha}\partial_{\tau}^{\beta}\hat{F}(\epsilon,t,\tau))\circ(\partial_t^{\beta}\partial_{\tau}^{\alpha}\hat{G}(t,\tau)). \tag{12.6}$$

where I have not (yet) computed the coefficients properly. Here the product is just the product in the suspended algebra $\Psi^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$.

A little later I will need (only in the 1-dimensional case in fact) the two leading terms. The first leads to the product law for the adiabatic symbol

$$\begin{gathered}\hat{H}(\epsilon,t,\tau)=\hat{F}(\epsilon,t,\tau)\circ\hat{G}(\epsilon,t,\tau)\\ +\epsilon^2\sum_{j=1}^{d}\big(\partial_{t_j}\hat{F}(\epsilon,t,\tau)\circ\partial_{\tau_j}\hat{G}(\epsilon,t,\tau)-\partial_{\tau_j}\hat{F}(\epsilon,t,\tau)\circ\partial_{t_j}\hat{G}(\epsilon,t,\tau)\big)+O(\epsilon^4)\\ \hat{H}(\epsilon,t,\tau,Z,Z')=\int_{\mathbb{R}^d}H(\epsilon,t,s,Z,Z')e^{-is\cdot\tau}ds.\end{gathered} \tag{12.7}$$

This implies the analogous symbolic property to 1-dimensional case:-

$$\begin{gathered}\sigma_{\mathrm{ad}}:\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)\longrightarrow\Psi^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k),\\ \sigma_{\mathrm{ad}}(A)(t,\tilde{a})=\int_{\mathbb{R}^d}\kappa(A)(0,t,s,Z,Z')e^{-is\cdot\tau}ds\end{gathered} \tag{12.8}$$

satisfies

$$\sigma_{\mathrm{ad}}(AB)=\sigma_{\mathrm{ad}}(A)\circ\sigma_{\mathrm{ad}}(B)\text{ in }\Psi^{-\infty}_{\mathrm{sus}(2),\mathrm{iso}}(\mathbb{R}^k). \tag{12.9}$$

This sum in (12.7) correspnd to the Poisson bracket, as it should! Of course I hardly need pause to say that σ_{ad} gives a short exact sequence

$$\epsilon\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)\hookrightarrow\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)\xrightarrow{\sigma_{\mathrm{ad}}}\Psi^{-\infty}_{\mathrm{sus}(2),\mathrm{iso}}(\mathbb{R}^k). \tag{12.10}$$

Now, one thing I have been pushing, rather relentlessly, in these lectures so far is that one should take these sorts of algebras 'seriously'. In particular look at the corresponding group and see what you get. Let me do again what we did earlier, perhaps with a little more care. Namely the algebra $\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)$ does not have a unit. So simply append a unit by taking the direct product and considering

$$\Psi^{-\infty,\dagger}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)=\mathbb{C}+\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k) \tag{12.11}$$

where the product is the obvious one and in particular, $\mathrm{Id}=1+0$ is the unit. Less abstractly one can consider $\mathbb{C}$ as being the complex multiples of the identity as operators on $\mathcal{S}(\mathbb{R}^{d+k})$ depending trivially on the parameter ϵ. Then one can consider the group

$$G^{-\infty,\dagger}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)=\{A\in\Psi^{-\infty,\dagger}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k);\exists\ B\in\Psi^{-\infty,\dagger}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k),\ AB=BA=\mathrm{Id}\}. \tag{12.12}$$

In fact it follows that if $A=z\,\mathrm{Id}+A'$ is invertible in this sense then $z\in\mathbb{C}^*$ and $\mathrm{Id}+z^{-1}A'$ is invertible. Thus we really do not lose anything by considering the group of the type we have been considering all along:-

$$G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)=\{\mathrm{Id}+A'\in G^{-\infty,\dagger}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)\}\hookrightarrow\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k). \tag{12.13}$$

This is all just formal. What is important, as I indicated earlier, is that this group happens to be open in terms of the inclusion (12.13) and hence is a nice topological (and of course smooth) group. We need to check this, but in fact lots of amusing things happen here so let me list this more formally.

Theorem 2. *The inclusion in* (12.13) *is open and the two maps, the adiabatic symbol* (12.8) *and the restriction map*

$$R : \Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) \xrightarrow{\big|_{\epsilon=1}} \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k}) \tag{12.14}$$

lead to a commutative diagram where the lower two maps are surjective, and admit compact lifting, and the upper two spaces are weakly contractible:

$$\begin{tikzcd}
\{A \in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k); \sigma_{ad}(A) = \mathrm{Id}\} \arrow[dr, hook] & & \{A \in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k); R(A) = \mathrm{Id}\} \arrow[dl, hook'] \\
& G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) \arrow[dl, "R"'] \arrow[dr, "\sigma_{\mathrm{ad}}"] & \\
G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k}) & & G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k).
\end{tikzcd} \tag{12.15}$$

Recall that weak contractibility here means that for any smooth map from a compact manifold into the space there is an homotopy to a constant map – in this case taking the value Id. The diagonal sequences are therefore exact. Note that compact lifting would usually be stated, at least in the topological literature, in the form that these sequences are 'Serre fibrations'. It means precisely that if $f : X \longrightarrow G$ is a smooth map into one of the bottom two spaces, then it can be lifted to $\tilde{f} : X \longrightarrow G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k)$ so that $R\tilde{f} = f$ or $\sigma_{\mathrm{ad}}\tilde{f} = f$ respectively. Of the five or so things to be proved here, three are reasonably straightforward and the remaining part, amounting to the (Serre) exactness of the 'R' sequence, depends heavily on the construction I did last time. I will postpone the proof, probably until next time.

Remark 1. (Frédéric Rochon) The two diagonal sequences in (12.15) are in fact fibrations, not just Serre fibrations. So, if you know a little topology, the Serre lifting condition does in fact follows from the surjectivity. I will prove it directly anyway but this observation makes it clear why the proof of the lifting condition is no harder than the proof of surjectivity!

So, suppose we have managed to prove the theorem, then what? Basically it amounts to a weak homotopy equivalence between the bottom two spaces. That is, the diagram induces a map, which is an isomorphism,

$$p_{\mathrm{ad}} : [X; G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)] \xrightarrow{\simeq} [X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})] \tag{12.16}$$

for any compact manifold X. Namely, take a smooth map $f : X \longrightarrow G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$. The 'Serre property' asserts that it can be lifted to $\tilde{f} : G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k)$ so the map in (12.16) is supposed to be induced by

$$[f] \longmapsto [R\tilde{f}]. \tag{12.17}$$

Of course, we need to check that this is well-defined. For fixed f two liftings $\tilde{f}(1)$ and $\tilde{f}(2)$ are such that $F = \tilde{f}(2)^{-1}\tilde{f}(1) : X \longrightarrow$ satisfies $\sigma_{\mathrm{ad}}(F) \equiv \mathrm{Id}$. So the stated weak contractibility in the Theorem implies that this is homotopic to the constant identiy map – hence $\tilde{f}(1)$ and $\tilde{f}(2)$ are homotopic. It follows that $R\tilde{f}(1)$ and $R\tilde{f}(2)$ are homotopic so the image class in (12.17) is well-defined given f. On the other hand if f_0 and f_1 are homotopic, so represent the same class $[f]$ on the left, then an homotopy $F : [0,1] \times X \longrightarrow G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$ can also be lifted and shows that the resulting image classes are the same. Thus (12.17) does lead to a well-defined map (12.16). Of course, the argument is reversible in the sense that there is a similar map defined the other way. These two maps are then inverses of each other. Recalling that we have defined

$$K^{-1-2d}(X) = [X; G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)] \ \forall\ d \geq 0, \tag{12.18}$$

with the result indepedent of the choice of k we conclude:

Corollary 1 (Bott periodicity). *For any compact manifold semiclassical quantization induces an isomorphism for any d :*

$$p_{\mathrm{ad}} : K^{-1-2d}(X) \longrightarrow K^{-1}(X). \tag{12.19}$$

In fact the Theorem and the isomorphism (12.19) extends to the case of non-compact manifolds X. We just need to consider the 'homotopy groups of maps with compact support'

$$\begin{aligned} \mathrm{K}_{\mathrm{c}}^{-1-j}(X) &= [X; G^{-\infty}_{\mathrm{sus}(j),\mathrm{iso}}(\mathbb{R})]_{\mathrm{c}} \\ &= \left\{ f : X \longrightarrow G^{-\infty}_{\mathrm{sus}(j),\mathrm{iso}}(\mathbb{R}); f(x) = \mathrm{Id},\ x \in X \setminus K,\ K \Subset X \right\} / \sim \end{aligned} \tag{12.20}$$

where the equivalence relation is through homotopies also reducing to the identity outside some compact subset. Then (12.19) extends to

$$p_{\mathrm{ad}} : \mathrm{K}_{\mathrm{c}}^{-1-2d}(X) \longrightarrow \mathrm{K}_{\mathrm{c}}^{-1}(X) \tag{12.21}$$

In fact,

Lemma 12.

$$\mathrm{K}_c^{-1-j}(X) \equiv \mathrm{K}_c^{-1}(X \times \mathbb{R}^j) \ \forall\ j \geq 0. \tag{12.22}$$

Proof. Left as an exercise, but said in brief as follows. Schwartz functions can always be approximated by functions of compact support. □

There are many ways to rewrite these isomorphism including the form of Bott periodicity mentioned earlier.

Corollary 2.

$$\pi_j(G^{-\infty}) = \begin{cases} \{0\} & j \ \mathit{even} \\ \mathbb{Z} & j \ \mathit{odd} \end{cases} \tag{12.23}$$

Proof. Assuming we know that $G^{-\infty}$ is connected and that $\pi_1(G^{-\infty}) = \mathbb{Z}$ then we just note that

$$\begin{aligned} \pi_{2j}(G^{-\infty}) &= K^{-1-2j}(\mathrm{pt}) = K^{-1}(\mathrm{pt}) = \{0\}, \\ \pi_{2j+1}(G^{-\infty}) &= K^{-1-2j-1}(\mathrm{pt}) = K^{-2}(\mathrm{pt}) = \pi_1(G^{-\infty}) = \mathbb{Z}. \end{aligned} \tag{12.24}$$

□

13. Lecture 10: Bott periodicity
Friday, 26 September

I will start the notes, if not the lecture, with an extended reply to a question from the end of the last lecture.

Question 1. (Jesse Gell-Redman) What has this got to do with index theory?

Answer 1. My first answer is that we need to develop K-theory in order to understand the index theorem, however I am trying to do more than that. This question is out or order of course, I don't mean parliamentary order here, just logical order. Still, let me run ahead a bit, taking this as an opportunity to indicate where I am trying to go – since for one thing you might not wish to come along!

The second answer is that 'this' meaning Theorem 2 really *is* an index theorem, or at least is closely related to one. Let me try to describe this relationship, even though I will use some as-yet-undefined objects. The index theorem most closely related to Theorem 2 is Fedosov's index theorem on isotropic operators. Well, the original theorem was about the numerical index but let me jazz it up to the families theorem. First, an isotropic pseudodifferential operator corresponds to equal scaling for z and D_z on $\mathbb{R}^n$, as I have indicated earlier in relation to isotropic smoothing operators. So, whatever they are, isotropic pseudodifferential operators of order 0 are bounded operators on $L^2(\mathbb{R}^n)$ and they have symbols. Since they are 'isotropic' the symbol is just a homogeneous function on $\mathbb{R}^{2n}\setminus\{0\}$, or equivalently on $\mathbb{S}^{2n-1}$, with values in $M(N;\mathbb{C})$. The operator is Fredholm if and only if the symbol is invertible. Let X be a parameter space, then the symbol of an elliptic family of such operators, parameterized by X, is a smooth map

$$a : X\times\mathbb{S}^{2n-1}\longrightarrow \mathrm{GL}(N;\mathbb{C})\hookrightarrow G^{-\infty}. \tag{13.1}$$

This defines a K-class, $[a]\in K^{-1}(X\times\mathbb{S}^{2n-1})$. If we had the Künneth formula at our disposal, which we do not, we would know that $K^{-1}(X\times\mathbb{S}^{2n-1})\equiv K^{-1}(X)\otimes K^0(\mathbb{S}^{2n-1})\oplus K^0(X)\otimes K^{-1}(\mathbb{S}^{2n-1})$, where $K^0(X)$ is the soon-to-be-introduced group based on vector bundles, or projections. Now, both K-groups of the sphere are $\mathbb{Z}$ so this means that $K^{-1}(X\times\mathbb{S}^{2n-1})\equiv K^{-1}(X)\oplus K^0(X)$. This can be understood more directly here in terms of two maps

$$\mathrm{K}^{-1}(X)\xleftarrow{S^*}\mathrm{K}^{-1}(X\times\mathbb{S}^{2n-1})\xrightarrow{\mathrm{cl}}\mathrm{K}^0_{\mathrm{c}}(X\times\mathbb{R}^{2n}). \tag{13.2}$$

The map on the left is just pull-back by choosing a point, say the South Pole, on the sphere. The map on the right is a version of the 'clutching construction' which in this context just means a map made explicitly with matrices which turns an isomorphism into a bundle. The maps in (13.2) are each isomorphisms when restricted to the null space of the other, so the K-space splits as indicated.

Now, the elliptic family of symbols can be quantized to a family of operators which are not only Fredholm but have constant rank null spaces. The null spaces then form a bundle over X as do the null spaces of the adjoints and the formal difference of these (we will get to this next week) define an element of $K^0(X)$; this is the index (in K-theory) and it only depends on the class of the symbol $[a]$. This gives us the little diagram

$$
\begin{tikzcd}
K^{-1}(X) \\
K^{-1}(X \times \mathbb{S}^{2n-1}) \arrow[u, "S^*"'] \arrow[d, "\mathrm{ind}_{\mathrm{iso}}"] \\
K^0(X).
\end{tikzcd}
\tag{13.3}
$$

where I put in the upward map because one consequence of the discussion below is that the null space of this isotropic index map is the same as the map on the right in (13.2). Now we can add the clutching construction above and another variant of the clutching construction both above and below to get a bigger diagram

$$
\begin{tikzcd}
K^{-1}(X) & & \\
K^{-1}(X \times \mathbb{S}^{2n-1}) \arrow[u, "S^*"] \arrow[r, "\mathrm{cl}"] \arrow[d, "\mathrm{ind}_{\mathrm{iso}}"'] & \mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}^{2n}) \arrow[r, "\mathrm{cl}", "\simeq"'] \arrow[dl, "p_{\mathrm{sl}}^{\mathrm{even}}"', "\simeq"] & \mathrm{K}^1_{\mathrm{c}}(X \times \mathbb{R}^{2n+1}) \arrow[dl, "p_{\mathrm{sl}}^{\mathrm{odd}}"', "\simeq"] \\
K^0(X) \arrow[r, "\mathrm{cl}", "\simeq"'] & \mathrm{K}^{-1}_{\mathrm{c}}(X \times \mathbb{R}) &
\end{tikzcd}
\tag{13.4}
$$

where I have added in two more maps. Namely, the 'odd' semiclassical Bott periodicity map – on the far right – that we are currently discussing and its even brother in the middle that we will soon get to. The $\simeq$'s indicate isomorphisms.

So, there is your index theorem. The main claim is that this diagram commutes, so the index for isotropic operators is equal to the product going around the right. In this context the Bott periodicity maps are 'topological' and the index map is 'analytic'. Of course the semiclassical definition makes the periodicity maps rather analytic too, but that is one thing I am trying to get at! So, how to prove it? The Atiyah-Singer approach was to give enough properties of these maps that they forced into uniqueness, the general principle being that if you have a natural construction – so it is universal in X – and it is non-trivial and has a few more properties then there is only one possibility. The proof I will give later is more analytic, as you might guess. Basically we can deform the isotropic pseudodifferential operators, following the clutching construction, into families of projections valued in smoothing operators and then into the group of invertible perturbations by smoothing operators – and this corresponds precisely to the three maps along the top.

To make the picture more symmetric I can add in the odd version of the isotropic index theorem, for elliptic self-adjoint operators or suspended operators, and get a

bigger commutative diagram:
(13.5)

$$\begin{tikzcd}
\mathrm{K}^{-1}(X) & & & \mathrm{K}_{\mathrm{c}}^{-1}(X\times\mathbb{R}) \\
\mathrm{K}^{-1}(X\times\mathbb{S}^{2n-1}) \arrow[u, "S^*"] \arrow[r, "\mathrm{cl}"] \arrow[dr, "\mathrm{ind}_{\mathrm{iso}}^{\mathrm{even}}"'] & \mathrm{K}_{\mathrm{c}}^{0}(X\times\mathbb{R}^{2n}) \arrow[r, "\mathrm{cl}", "\simeq"'] \arrow[d, "p_{\mathrm{sl}}^{\mathrm{even}}", "\simeq"'] & \mathrm{K}_{\mathrm{c}}^{1}(X\times\mathbb{R}^{2n+1}) \arrow[d, "\simeq", "p_{\mathrm{sl}}^{\mathrm{odd}}"'] & \mathrm{K}_{\mathrm{c}}^{-1}(X\times\mathbb{R}\times\mathbb{S}^{2n-1}) \arrow[u, "S^*"'] \arrow[l, "\mathrm{cl}"'] \arrow[dl, "\mathrm{ind}_{\mathrm{iso}}^{\mathrm{odd}}"] \\
& \mathrm{K}^{0}(X) \arrow[r, "\mathrm{cl}", "\simeq"'] & \mathrm{K}_{\mathrm{c}}^{-1}(X\times\mathbb{R}) &
\end{tikzcd}$$

What I am really after in the course is not only to do these things, and of course the geometric versions of them which include the Atiyah-Singer theorem, but also to do it in such a way as to carry the Chern charcter along. Diagrams such as (13.5) need to be 'subsumed' into a smooth K-theory.

Jesse, does this start to answer your question?

Just so that we don't get too lost, let me very briefly outline the proof of Theorem 2 – which I should finish on Monday.

(1) The group $G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)$ is open in $\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d;\mathbb{R}^k)$: – I have not given the appropriate product estimates in this algebra. Instead I show for perturbations near 0 in the algebra the operators on L^2 for each $\epsilon\in(0,1]$ are invertible and then show that the inverse is in the group.
(2) The map $\sigma_{\mathrm{ad}}:G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)\longrightarrow G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$ is surjective and any compact map into the image lifts:- Invertibility of the adiabatic symbol implies invertibility of the operator for small $\epsilon>0$ and uniformly on compact sets. Modify the family in $\epsilon>0$ to get the lifting property.
(3) The subgroup $\{A\in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k);\sigma_{\mathrm{ad}}(A)=0\}$ is weakly contractible:- Show that on compact sets one can 'cut the family off' near $\epsilon=0$ preserving invertibility. Then we are reduced to contractibility of the half-free loop group shown earlier.
(4) The map $R:G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)\longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})$, given by restriction to $\epsilon=1$, is surjective and any compact map into the image lifts:- This is the most involved part. The main thing to show is that the semiclassical quantization of the Bott element, introduced last week, is a rank one perturbation of the matrix projection at infinity, and so can be deformed to

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}+\Pi_1\begin{pmatrix}0&0\\0&1\end{pmatrix}\tag{13.6}$$

where P_1 is the projection onto the group state of the harmonic oscillator. Then it follows that the element

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}+g(x)\Pi_1\begin{pmatrix}0&0\\0&1\end{pmatrix}+(1-\Pi_1)\begin{pmatrix}0&0\\0&1\end{pmatrix}\tag{13.7}$$

is in the image, where everything has been tensored by $M(N,\mathbb{C})$ and g is an arbitrary map $X\longrightarrow\mathrm{GL}(N,\mathbb{C})$ of compact support. However, any element in the image space is homotopy to one of these, so we have the lifting property.

(5) The subgroup $\{A \in G^{-\infty}_{\text{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k); R(A) = \text{Id}\}$ is weakly contractible:- This is where we use Atiyah's clever rotation, and this follows rather miraculously from the previous step it.

So, to work. Let me go through the simpler parts of the proof of Theorem 2 first, probably leaving the last step until Monday. For convenience I will break the result up into pieces and I will likely not go through the 'easier' part in as much detail in the lecture as in the notes.

Lemma 13. *The group $G^{-\infty}_{\text{ad,iso}}(\mathbb{R}^d; \mathbb{R}^k)$ is open in $\Psi^{-\infty}_{\text{ad,iso}}(\mathbb{R}^d; \mathbb{R}^k)$.*

Proof. There may be a more direct proof than the one I will give here – if you find one please let me know! Since we are in a group we know that the issue is only the invertibility of $\text{Id} + A$ where A lies in some small metric ball around the origin. As we know this just means that for one of the norms on $\mathcal{C}^\infty([0,1]; \mathcal{S}(\mathbb{R}^{2d+2k})$, and for some $\epsilon > 0$, $\|A\|_{(N)} < \epsilon$ implies the existence of $B \in \Psi^{-\infty}_{\text{ad,sl}}(\mathbb{R}^d; \mathbb{R}^k)$ such that $(\text{Id} + B) = (\text{Id} + A)^{-1}$. We will get this by using the 'old-fashioned' method of invertibility acting on $L^2(\mathbb{R}^{k+d})$.

Thus, we first need to show that each element of $\Psi^{-\infty}_{\text{ad,sl}}(\mathbb{R}^d; \mathbb{R}^k)$ defines a uniformly bounded operator on L^2 for $\epsilon \in (0,1]$. Note that ϵ is a parameter so the only problem is at $\epsilon = 0$ were the operator blows up. Just to make sure there is no confusion, we are considering A_ϵ as an operator on say $L^2(\mathbb{R}^{d+k})$ through the usual integral formula

$$(A_\epsilon u)(z, Z) = \epsilon^{-d} \int_{\mathbb{R}^d \times \mathbb{R}^k} A(\epsilon, \frac{\epsilon(z+z')}{2}, \frac{z-z'}{\epsilon}, Z, Z') u(\epsilon, z', Z') dz' dZ' \tag{13.8}$$

and we want to get a uniform estimate on the L^2 norm as $\epsilon \downarrow 0$. This follows from Schur's lemma (not the one in representation theory of course) which says that the norm satisfies

(13.9)
$$\|A_\epsilon\|^2_{L^2} \le \sup_{z,Z} \int_{\mathbb{R}^{d+k}} |a(\epsilon, z, z', Z, Z')| dz' dZ' \times \sup_{z',Z'} \int_{\mathbb{R}^{d+k}} |a(\epsilon, z, z', Z, Z')| dz dZ,$$

assuming I have not missed out a constant. So, we just need to show that the right side is small if some norm on A is small. There is symmetry between the two terms so it suffices to consider the first and to see that

$$\begin{aligned} &\sup_{z,Z} \int_{\mathbb{R}^{d+k}} |a(\epsilon, z, z', Z, Z')| dz' dZ' \\ &= \sup_{z,Z} \int_{\mathbb{R}^{d+k}} \epsilon^{-d} |A(\epsilon, \frac{\epsilon(z+z')}{2}, \frac{z-z'}{\epsilon}, Z, Z')| dz' dZ' \\ &\sup_{z,Z} \int_{\mathbb{R}^{d+k}} |A(\epsilon, -\epsilon^2 s/2 + \epsilon z, s, Z, Z')| ds dZ' \le C \|A\|_{(N)} \end{aligned} \tag{13.10}$$

where we have just made the change of variable of integration from z' to $s = (z - z')/\epsilon$. Here N is just large enough to ensure convergence of the integrals.

So, from this it follows that if $\|A\|_{(N)} < \epsilon$ for some $\epsilon > 0$ then the family $\text{Id} + A_\epsilon$ has an inverse as operators uniformly on L^2, $\text{Id} + B_\epsilon$, where B_ϵ is small with A. So, it remains to show that this inverse actually comes from an element

$B \in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k)$. To do this we just need to construct the inverse near $\epsilon = 0$ since we already know what happens for $\epsilon > \epsilon_0 > 0$. The adiabatic symbol,

$$\sigma_{\mathrm{ad,iso}}(A) \in \Psi^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^d) \tag{13.11}$$

is small with $\|A\|_{(N)}$ and hence

$$(\mathrm{Id} + \sigma_{\mathrm{ad,iso}}(A))^{-1} = \mathrm{Id} + \sigma_{\mathrm{ad,iso}}(B) \text{ in } G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^d). \tag{13.12}$$

Now we can choose $B_0 \in \Psi^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^d)$ with this symbol, as usual, and we only need to invert

$$(\mathrm{Id} + B_0)(\mathrm{Id} + A) = \mathrm{Id} + \epsilon A'. \tag{13.13}$$

To do this we can use Neumann series to remove the Taylor series at $\epsilon = 0$:

$$\sum_j (-1)^j \epsilon^j (A')^j. \tag{13.14}$$

Sum this series using Borel's lemma and then we are back to a trivial case $\mathrm{Id} + A''$ where $A'' \in \epsilon^\infty \mathcal{C}^\infty([0,1]; \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})$, which is automatically invertible for small ϵ with inverse of the same type. Of course, we could have done this from the start. However, summing the Taylor series invovles norms of all orders and the problem is uniformity. However, we already know the existence of the L^2 inverse uniformly down to $\epsilon = 0$. Here we have shown that this inverse, being unique, is in fact an element of $G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k)$ without directly getting a bound on B.

This might make one wonder about the continuity of the inverse map, $A \longrightarrow (\mathrm{Id} + A)^{-1} - \mathrm{Id}$. However, the construction above works uniformly on compact sets and so the sequential continuity of the map follows – and we are in a complete metric space so all is well. ☐

The last part of this proof is very close to proving the properties of the 'adiabatic' sequence.

Lemma 14. *The adiabatic symbol gives a surjective map*

$$\sigma_{\mathrm{ad,iso}} : G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) \longrightarrow G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k), \tag{13.15}$$

and any smooth map $X \longrightarrow G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$ on a manifold, reducing to the identity outside a compact set, can be lifted under (13.15) *and the elements mapping to* Id *under* (13.15) *form a weakly contractible subgroup.*

Proof. The argument in the proof of the Lemma above shows that if

$$b \in G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$$

then any element $B \in \mathrm{Id} + \Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k)$ which has $\sigma_{\mathrm{ad,iso}}(B) = b$ is invertible on some smaller interval $[0, \epsilon_0]$ which depends on the choice of B. However, ϵ is just a parameter so we can 'expand' it by choosing a diffeomorphism $[0, \epsilon_0] \longrightarrow [0, 1]$ which is the identity near 0. Thus in fact the adiabatic symbol map is surjective. The same argument works uniformly on compact sets gives the lifting property.

The uniquess of the lift, up to homotopy, is the weak contractibility of the kernel-group. That is, we need to show that a smooth map

$$f : X \longrightarrow \left\{ A \in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k); \sigma_{\mathrm{ad,iso}}(A) = \mathrm{Id} \right\} \tag{13.16}$$

is smoothly homotopic to the identity, where if X is not compact both the map and the homotopy are required to restrict to the identity outside some compact subset of X.

This looks very like the contractibility of the half-open loop group and it may be that there is a global retraction of a similar sort to used there. At the moment I do not know it, so we actually reduce to that case using the compactness of the supports. So, given a map as in (13.16) we can insert a cutoff, choosing

$$\rho \in \mathcal{C}^\infty([0,1]),\ \rho(\epsilon) = 1 \text{ in } \rho < \frac{1}{4},\ \rho(\epsilon) = 0 \text{ in } \epsilon > \frac{1}{2}, 0 \le \rho(\epsilon) \le 1 \tag{13.17}$$

and consider the family

$$f_t(x) = \operatorname{Id} + \epsilon t \rho(\epsilon/\delta) + \epsilon(1 - \rho(\epsilon/\delta))A(\epsilon, x),\ f(x) = \operatorname{Id} + \epsilon A(\epsilon, x). \tag{13.18}$$

The uniformity in the construction of inverses above (and the factor of ϵ) shows that $\delta > 0$ is chosen small enough then this is an homotopy in the group in (13.16). At $t = 1$ it is f and at $f = 0$ it is in the flat loop group, since it reduces to the identity near $\epsilon = 0$. The earlier contraction argument therefore allows it to be retracted to the identity. □

14. Lecture 11: Adiabatic periodicity map Monday, 29 September

Last time I started the 'easier', or perhaps better to say 'routine', part of the proof of Theorem 2 – giving the adiabatic diagonal sequence, from top left to bottom right in

(14.1)

$$\begin{tikzcd}
\{A \in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k); \sigma_{ad}(A) = \mathrm{Id}\} \arrow[dr, hook] & & \{A \in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k); R(A) = \mathrm{Id}\} \arrow[dl, hook'] \\
& G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d : \mathbb{R}^k) \arrow[dl, "R"'] \arrow[dr, "\sigma_{\mathrm{ad}}"] & \\
G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k}) & & G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k).
\end{tikzcd}$$

Observe that this already gives us a map – following the arguments of last lecture –

$$[X; G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)]_{\mathrm{c}} \longrightarrow [X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})]_{\mathrm{c}}. \tag{14.2}$$

So once we see the same properties for the other sequence we conclude that this must be an isomorphism. In fact for the moment I will only do this for $d = 1$.

Addendum to Lecture 11: $\mathrm{GL}(N, \mathbb{C})$ **and the Bott element From Paul Loya**

Here we present the proof on Sept. 26-th that the restriction map to $\varepsilon = 1$:

$$R : G^{-\infty}_{ad,iso}(\mathbb{R}, \mathbb{R}^k) \to G^{-\infty}_{iso}(\mathbb{R}^{k+1}),$$

is surjective at the level of homotopies using the Bott element.

Preparing for the Bott element: Lemmas from Lecture 14

The following lemma is Lemma 16 in Lecture 14.

Lemma 15 (**Finite Rank Approximation**)**.** *Let* Π *be the orthogonal projection onto an* N*-dimensional subspace of* $\mathcal{S}(\mathbb{R}^d)$ *and choose an identification of linear maps on the range of* Π *with* $M(N, \mathbb{C})$*, and consider the map*

$$M(N, \mathbb{C}) \ni A \longmapsto \mathrm{Id} - \Pi + A\Pi \in \mathrm{Id} + \Psi^{-\infty}_{iso}(\mathbb{R}^d),$$

where $A\Pi$ *on the right is the matrix* A *acting on the range of* Π *through the chosen identification of linear maps on the range of* Π *with* $M(N, \mathbb{C})$*. This map restricts to a map*

$$\mathrm{GL}(N, \mathbb{C}) \ni A \longmapsto \mathrm{Id} - \Pi + A\Pi \in G^{-\infty}_{iso}(\mathbb{R}^d),$$

and for any topological space X*, induces a map*

$$[X, \mathrm{GL}(N, \mathbb{C})]_c \to [X, G^{-\infty}_{iso}(\mathbb{R}^d)]_c$$

that is defined independent of the choice of the N*-dimensional subspace of* $\mathcal{S}(\mathbb{R}^d)$ *chosen and the choice of identification of linear maps on the range of* Π *with* $M(N, \mathbb{C})$*. Moreover, any element of* $[X, G^{-\infty}_{iso}(\mathbb{R}^d)]_c$ *is in the image of this map for a sufficiently large* N*.*

The following lemma is Lemma 17 in Lecture 14.

Lemma 16. *If Π_1 is the orthogonal projection onto a 1-dimensional subspace of $\mathcal{S}(\mathbb{R}^d)$, then the map*[1]

$$G^{-\infty}_{iso}(\mathbb{R}^k) \ni h \longmapsto \mathrm{Id} - \Pi_1 + h\Pi_1 \in G^{-\infty}_{iso}(\mathbb{R}^{k+d})$$

induces an isomorphism

$$[X, G^{-\infty}_{iso}(\mathbb{R}^k)]_c \to [X, G^{-\infty}_{iso}(\mathbb{R}^{k+d})]_c$$

that is defined independent of the choice of the 1-dimensional subspace.

Proof of the theorem: The Magical Bott element

Recall that the Bott element is an operator $B \in \mathcal{H}^{-\infty}_{ad}(\mathbb{R}) = \{\text{involutions in}^2\ \gamma_1 + \Psi^{-\infty}_{ad}(\mathbb{R}, \mathbb{C}^2)\}$, which has the property that

$$B_+|_{\varepsilon=1} \sim \begin{pmatrix} 1 & 0 \\ 0 & \Pi_1 \end{pmatrix}$$

and

$$B_-|_{\varepsilon=1} \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 - \Pi_1 \end{pmatrix}$$

where $\Pi_1 \in \Psi^{-\infty}_{iso}(\mathbb{R})$ is the projection onto a one-dimensional subspace of $\mathcal{S}(\mathbb{R}^1)$ (say the ground state of the harmonic oscillator).

Theorem 3. *The restriction map to $\varepsilon = 1$:*

$$R : G^{-\infty}_{ad,iso}(\mathbb{R}, \mathbb{R}^k) \to G^{-\infty}_{iso}(\mathbb{R}^{k+1}),$$

is surjective at the level of homotopies. That is, for any topological space X and any element $[g] \in [X, G^{-\infty}_{iso}(\mathbb{R}^{k+1})]_c$ there is an element $[\tilde{g}] \in [X, G^{-\infty}_{ad,iso}(\mathbb{R}, \mathbb{R}^k)]_c$ such that $[R\tilde{g}] = [g]$.

Proof. By finite rank approximation, any element of $[X, G^{-\infty}_{iso}(\mathbb{R}^{k+1})]_c$ is homotopic to an invertible matrix through Lemma 15 and by further stabilization we may assume that

$$g = F_0 \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix}, \quad \text{where} \quad g_0 : X \to \mathrm{GL}(N, \mathbb{C}),$$

Id_N is the $N \times N$ identity matrix, and

$$F_0 : \mathrm{GL}(2N, \mathbb{C}) \to G^{-\infty}_{iso}(\mathbb{R}^{k+1})$$

is the map in Lemma 15 defined by some choice of $2N$-dimensional subspace of $\mathcal{S}(\mathbb{R}^{k+1})$ — it's not important now what subspace we choose although at the end of this proof we'll take a subspace in $\mathcal{S}(\mathbb{R}^{k+1}) = \mathcal{S}(\mathbb{R}^k \times \mathbb{R}^1)$ spanned by $2N$ independent functions in $\mathcal{S}(\mathbb{R}^k)$ times a function in $\mathcal{S}(\mathbb{R}^1)$. The reason we take g in terms of a 2×2 matrix (of $N \times N$ matrices) is because the Bott element is given in terms of 2×2 matrices. We shall find a map

$$\tilde{g} : X \to G^{-\infty}_{ad,iso}(\mathbb{R}, \mathbb{R}^k)$$

such that $[R\tilde{g}] = [g]$. To define $\tilde{g}$, let

$$F_1 : \mathrm{GL}(2N, \mathbb{C}) \to G^{-\infty}_{iso}(\mathbb{R}^k),$$

---

[1] On the right-hand side, as operators on $\mathcal{S}(\mathbb{R}^{k+d}) = \mathcal{S}(\mathbb{R}^k \times \mathbb{R}^d)$, Π_1 only acts on the $\mathbb{R}^d$ factor and h on the $\mathbb{R}^k$ factor.

[2] Recall that $\gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and note that $\Psi^{-\infty}_{ad}(\mathbb{R}, \mathbb{C}^2)$ consists of 2×2 matrices of operators in $\Psi^{-\infty}_{ad}(\mathbb{R})$.

be the map in Lemma 15 induced by some choice of $2N$-dimensional subspace of $\mathcal{S}(\mathbb{R}^k)$, and then define

$$\boxed{\tilde{g} = F_1 g_1,}$$

where

$$g_1 = \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} (\mathrm{Id}_N \otimes B_+) + \mathrm{Id}_N \otimes B_-.$$

Let's pause to think about this! Note that $B_\pm$ are 2×2 matrices whose entries are operators in $\mathrm{Id} + \Psi_{ad}^{-\infty}(\mathbb{R})$, so $\mathrm{Id}_N \otimes B_\pm$ are $2N \times 2N$ matrices of the same sort. Using that $B = \gamma_1$ modulo a 2×2 matrix of operators in $\Psi_{ad}^{-\infty}(\mathbb{R})$ one can check that $g_1 = \mathrm{Id}_{2N} + R$ where R is a $2N \times 2N$ matrix of operators in $\Psi_{ad}^{-\infty}(\mathbb{R})$. Moreover, g_1 is invertible with inverse

$$\begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0^{-1} \end{pmatrix} (\mathrm{Id}_N \otimes B_+) + \mathrm{Id}_N \otimes B_-.$$

It follows that

$$\tilde{g} = F_1 g_1 \in G^{\infty}_{ad,iso}(\mathbb{R}, \mathbb{R}^k).$$

Now we claim that

$$\tilde{g}|_{\varepsilon=1} \sim g = F_0 \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix}.$$

To see this, recall that

$$B_+|_{\varepsilon=1} \sim \begin{pmatrix} 1 & 0 \\ 0 & \Pi_1 \end{pmatrix} \quad \text{and} \quad B_-|_{\varepsilon=1} \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 - \Pi_1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} g_1|_{\varepsilon=1} &= \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} (\mathrm{Id}_N \otimes B_+)|_{\varepsilon=1} + \mathrm{Id}_N \otimes B_-|_{\varepsilon=1} \\ &\sim \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & \mathrm{Id}_N \Pi_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_N - \mathrm{Id}_N \Pi_1 \end{pmatrix} \\ &= \mathrm{Id}_{2N} - \mathrm{Id}_{2N} \Pi_1 + \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} \Pi_1. \end{aligned}$$

Hence,

$$\tilde{g}|_{\varepsilon=1} = F_1 g_1|_{\varepsilon=1} \sim \mathrm{Id} - \Pi_1 + \left(F_1 \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} \right) \Pi_1.$$

In other words, if

$$F_2 : [X, G^{-\infty}_{iso}(\mathbb{R}^k)]_c \to [X, G^{-\infty}_{iso}(\mathbb{R}^{k+1})]$$

is the isomorphism induced by the map, which we also denote by F_2,

$$G^{-\infty}_{iso}(\mathbb{R}^k) \ni h \longmapsto \mathrm{Id} - \Pi_1 + h\Pi_1 \in G^{-\infty}_{iso}(\mathbb{R}^{k+1}) \tag{14.3}$$

found in Lemma 16 with $d = 1$, then we see that

$$\tilde{g}|_{\varepsilon=1} = F_2 \left(F_1 \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} \right).$$

To summarize, we are left to show that

$$\left[F_2 \left(F_1 \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} \right) \right] = \left[F_0 \begin{pmatrix} \mathrm{Id}_N & 0 \\ 0 & g_0 \end{pmatrix} \right].$$

Thus, our theorem is finished off by proving that the composition

$$[X, \mathrm{GL}(2N,\mathbb{C})] \overset{F_1}{\to} [X, G^{-\infty}_{iso}(\mathbb{R}^k)] \overset{F_2}{\to} [X, G^{-\infty}_{iso}(\mathbb{R}^{k+1})]$$

is exactly the same as the map

$$[X, \mathrm{GL}(2N,\mathbb{C})] \overset{F_0}{\to} [X, G^{-\infty}_{iso}(\mathbb{R}^{k+1})];$$

in other words, we need to prove that the following diagram commutes:

$$\begin{tikzcd} {[X, \mathrm{GL}(2N,\mathbb{C})]} \arrow[r, "F_1"] \arrow[rr, bend right, "F_0"'] & {[X, G^{-\infty}_{iso}(\mathbb{R}^k)]} \arrow[r, "F_2"] & {[X, G^{-\infty}_{iso}(\mathbb{R}^{k+1})]} \end{tikzcd} .$$

To prove this we just have to look at what the maps F_0, F_1, and F_2. Let Π be the orthogonal projection onto any $2N$-dimensional subspace of $\mathcal{S}(\mathbb{R}^k)$. Then F_1 is the map induced by

$$\mathrm{GL}(2N,\mathbb{C}) \ni A \longmapsto \mathrm{Id} - \Pi + A\Pi \in G^{-\infty}_{iso}(\mathbb{R}^k).$$

Therefore, at the homotopy group level, F_2F_1 is the map (see (14.3) for F_2) induced by

$$\begin{aligned} \mathrm{GL}(2N,\mathbb{C}) \ni A \longmapsto \mathrm{Id} - \Pi_1 + \big(\mathrm{Id} - \Pi + A\Pi\big)\Pi_1 &\in G^{-\infty}_{iso}(\mathbb{R}^{k+1}) \\ = \mathrm{Id} - \Pi\,\Pi_1 + A\Pi\,\Pi_1 &\in G^{-\infty}_{iso}(\mathbb{R}^{k+1}) \end{aligned}$$

On the other hand, $\Pi\,\Pi_1$ is the orthogonal projection onto a $2N$-dimensional subspace of $\mathcal{S}(\mathbb{R}^{k+1})$ (namely, the space of functions of the form $f(x)\,g(y)$ where $(x,y) \in \mathbb{R}^k \times \mathbb{R}^1$ with f and g in the range of Π and Π_1, respectively). Hence, we can take the map F_0 to be induced by

$$\mathrm{GL}(2N,\mathbb{C}) \ni A \longmapsto \mathrm{Id} - \Pi\,\Pi_1 + A\Pi\,\Pi_1 \in G^{-\infty}_{iso}(\mathbb{R}^{k+1}),$$

which is exactly F_2F_1. This completes the proof. □

Extra: Bott periodicity for the general linear group

This section is not needed for the Bott element but might be useful to be written down. In this section we prove that $\pi_0(\mathrm{GL}(N,\mathbb{C})) = \{0\}$ and $\pi_1(\mathrm{GL}(N,\mathbb{C})) = \mathbb{Z}$.

Lemma 17. *There is a smooth map*

$$T : \mathbb{S}^{2N-1} \setminus \{-e_1\} \to \mathrm{GL}(N,\mathbb{C}) \quad , \quad v \mapsto T_v$$

such that $T_{e_1} = \mathrm{Id}$ *and for all* $v \in \mathbb{S}^{2N-1} \setminus \{-e_1\}$, $T_v v = e_1$.

Proof. Given $v \in \mathbb{S}^{2N-1}$, for all $x \in \mathbb{C}^N$, define

$$T_v(x) = x + (x\cdot v)(e_1 - v) + \frac{(e_1\cdot v)(x\cdot v) - x\cdot e_1}{1 + e_1\cdot v}(e_1 + v).$$

Here "$\cdot$" denotes the usual Hermitian inner product on $\mathbb{C}^N$ (linear in the first slot and conjugate linear in the second slot). Using this formula, it's easy to show that T_v depends smoothly on $v \in \mathbb{S}^{2N-1} \setminus \{-e_1\}$, $T_{e_1} = \mathrm{Id}$, and $T_v v = e_1$. □

Using this smooth map T_v we prove prove the following theorem. (I haven't seen a proof of this theorem that uses the linear map T_v. Has anyone?)

Theorem 4. *If* $0 < k < N$*, then the inclusion map*

$$\mathrm{GL}(k,\mathbb{C}) \to \mathrm{GL}(N,\mathbb{C}) \ ; \quad A \mapsto \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & A \end{pmatrix}$$

induces an isomorphism between homotopy spaces

$$[\mathbb{S}^1, \mathrm{GL}(k,\mathbb{C})] \to [\mathbb{S}^1, \mathrm{GL}(N,\mathbb{C})].$$

Proof. By iteration we may assume that $k = N-1$. Our theorem follows immediately from the following two claims, which we'll prove using the lemma: For any $N > 1$,

(1) Any element of $[\mathbb{S}^1, \mathrm{GL}(N,\mathbb{C})]$ has a representative of the form $\begin{pmatrix} 1 & 0 \\ 0 & g(x) \end{pmatrix}$, where $g : \mathbb{S}^1 \to \mathrm{GL}(N-1,\mathbb{C})$.

(2) Two maps $g_0, g_1 : \mathbb{S}^1 \to \mathrm{GL}(N-1,\mathbb{C})$ are homotopic if and only if the maps $\begin{pmatrix} 1 & 0 \\ 0 & g_0(x) \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & g_1(x) \end{pmatrix}$ are homotopic as maps into $\mathrm{GL}(N,\mathbb{C})$.

Let $f : \mathbb{S}^1 \to \mathrm{GL}(N,\mathbb{C})$ be a continuous map. Since $f(\mathbb{S}^1)$ is a compact subset of $\mathrm{GL}(N,\mathbb{C})$, an open set in the set of all $N \times N$ matrices, it follows that any $N \times N$ matrix sufficiently close to the image $f(\mathbb{S}^1)$ must lie in $\mathrm{GL}(N,\mathbb{C})$. Using this fact plus a standard compactness argument, it is straightforward to show[3] that f is homotopic to a map (again denoted by f) such that the first column $w_1(x)$ of f is not a positive real multiple of $-e_1$. Hence, $v(x) = w_1(x)/\|w_1(x)\|$ is never equal to $-e_1$. By Lemma 17 we have $T_{v(x)} w_1(x) = \|w_1(t)\|$, so for all $x \in \mathbb{S}^1$,

$$T_{v(x)} f(x) = \begin{pmatrix} \|w_1(x)\| & * \\ 0 & g(x) \end{pmatrix},$$

where $*$ is unimportant components and $g : \mathbb{S}^1 \to \mathrm{GL}(N-1,\mathbb{C})$. We may homotopy the first column to e_1, so

$$f(x) \sim T_{v(x)}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & g(x) \end{pmatrix}$$

Now $v(x) \in \mathbb{S}^{2n-1} \setminus \{-e_1\} \cong \mathbb{R}^{2n-1}$ so we can homotopy $v(x)$ to the constant vector e_1 within $\mathbb{S}^{2n-1} \setminus \{-e_1\}$. Since $T_{e_1} = \mathrm{Id}$ it follows that $T_{v(x)}^{-1} \sim \mathrm{Id}$. This proves Claim 1.

We now prove 2. Certainly the "only if" part holds, so assume that $\begin{pmatrix} 1 & 0 \\ 0 & g_0(x) \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & g_1(x) \end{pmatrix}$ are homotopic as maps into $\mathrm{GL}(N,\mathbb{C})$, which means there is a continuous map $F : \mathbb{S}^1 \times [0,1] \to \mathrm{GL}(N,\mathbb{C})$ such that

$$F(x,0) = f_0(x) = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x) \end{pmatrix} \quad \text{and} \quad F(x,1) = f_1(x) = \begin{pmatrix} 1 & 0 \\ 0 & g_1(x) \end{pmatrix}. \tag{14.4}$$

By a similar argument as we stated in the previous paragraph we may assume that $w_1(x,t)$, the first column of $F(x,t)$, is never a positive multiple of $-e_1$. Hence, $v(x,t) = w_1(x,t)/\|w_1(x,t)\|$ is never equal to $-e_1$. By Lemma 17, we have

$$T_{v(x,t)} F(x,t) = \begin{pmatrix} 1 & * \\ 0 & g(x,t) \end{pmatrix},$$

[3]If $f_{N1}(x)$ denotes the N-th row, 1-st column element of $f(x)$, all you have to do is replace this function by a new function such that $f_{N1}(x) \neq 0$ for $x \neq 1$.

where $g(x,t) \in \mathrm{GL}(N-1,\mathbb{C})$. Since $v(x,0) = e_1 = v(x,1)$ and $T_{e_1} = \mathrm{Id}$, it follows that $g(x,0) = g_0(x)$ and $g(x,1) = g_1(x)$. Thus, $g : \mathbb{S}^1 \times [0,1] \to \mathrm{GL}(N-1,\mathbb{C})$ provides a homotopy between g_0 and g_1. $\square$

Corollary 3. $\pi_0(\mathrm{GL}(N,\mathbb{C})) = \{0\}$ *and* $\pi_1(\mathrm{GL}(N,\mathbb{C})) = \mathbb{Z}$.

Proof. The second statement follows from Theorem 4 (with $k = 1$) and the fact that $\pi_1(\mathrm{GL}(1,\mathbb{C})) = [\mathbb{S}^1, \mathrm{GL}(1,\mathbb{C})] = \mathbb{Z}$ (which can be proved using for example the winding number). The proof of Theorem 4 also works if we replace $\mathbb{S}^1$ with a point, so the first statement follows from the fact that $\pi_0(\mathrm{GL}(1,\mathbb{C})) = \{0\}$. $\square$

15. Lecture 12: Atiyah's rotation Wednesday, 1 October

Let me try to clarify what we have done so far as regards the proof of Bott periodicity. Last time we proved that there is a map

$$p_{\mathrm{ad}} : [X; G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)]_{\mathrm{c}} \longrightarrow [X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})]_{\mathrm{c}} \tag{15.1}$$

obtained by semiclassical quantization. Indeed let me quickly recall how this map is defined – I have generalized from homotopy classes of smooth maps from a compact manifold to homotopy classes of compactly-supported smooth map from a general manifold, but this makes very little difference to the argument. To define (15.1) take a representative, $f : X \longrightarrow G^{-\infty}_{\mathrm{sus}(2d),\mathrm{iso}}(\mathbb{R}^k)$ and choose, as we can, a family $\tilde{f} : X \longrightarrow \mathrm{Id} + \Psi^{-\infty}_{\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d : \mathbb{R}^k)$ which has adiabatic symbol f. Last time I showed that in fact $\tilde{f}$ is invertible for $\epsilon \in (0, \epsilon_0]$, $\epsilon_0 > 0$, with inverse given by a similar family. We can expand the parameter so that $\epsilon_0 = 1$ and then (15.1) is obtained by restriction to $\epsilon = 1$. It doesn't matter where we restrict, provided $\tilde{f}_\epsilon$ is invertible for that ϵ and smaller. So, we showed that this induces a map (15.1).

Now, we also showed that the map (15.1) is surjective for $d = 1$. To do this we proved that for a given map with compact support $g : X \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})$ we can first make a smooth homotopy, and then 'lift' it to a family which takes the form

$$\tilde{g} = \mathrm{Id} + A(t,\tau)g^{-1}(x) + A'(t,\tau)g(x) + A''(t,\tau) \in G^{-\infty}_{\mathrm{sus}(2),\mathrm{iso}}(\mathbb{R}^k) \tag{15.2}$$

and where the perturbation is compactly supported on $\mathbb{R}^2$. Note that this family has the nice property that $g = \mathrm{Id}$ implies $\tilde{g} = \mathrm{Id}$ which is important now that we want to treat compactly supported families. Thus we proved (modulo properties of the Bott element which have not yet been checked) that

$$p_{\mathrm{ad}}(\tilde{g}) = [g], \tag{15.3}$$

thus establishing surjectivity.

Now we want to prove injectivity of p_{ad}, which reduces to the weak contractibility of a certain subgroup of $G^{-\infty}_{\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d : \mathbb{R}^k)$. This can be done by rather tedious, if imaginative, computation but Atiyah realized that it is a consequence of the multiplicativity of K-theory. Now, I have not discussed this multiplicativity but we can just proceed directly and then sort out what we have done afterwards.

Stripped down in this way, Atiyah's idea goes as follows. Look at $\tilde{g}$ in (15.2). It is itself a compactly supported smooth map

$$\tilde{g} : X \times \mathbb{R}^2 \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k). \tag{15.4}$$

So, we can (after homotopy to finite rank) apply the same construction to it. Let me call the result

$$G(t', \tau', t, \tau, x) = \mathrm{Id} + A(t', \tau')\tilde{g}^{-1}(x, t, \tau) + A'(t', \tau')\tilde{g}(x, t, \tau) + A''(t', \tau'). \tag{15.5}$$

So this is a function on $\mathbb{R}^4 \times X$ with compact support. We recover $\tilde{g}$ (if you like up to homotopy) by quantizing it in the variables (t', τ'). However, let us make a rotation between (t',τ') and (t,τ), substituting in (15.5)

$$\begin{aligned} t' &\longmapsto t'\cos\theta + t\sin\theta, \\ \tau' &\longmapsto \tau'\cos\theta + \tau\sin\theta, \\ t &\longmapsto -t'\sin\theta + t\cos\theta, \\ \tau &\longmapsto -\tau'\sin\theta + \tau\cos\theta. \end{aligned} \tag{15.6}$$

This linear change of variables leads to a smooth map

$$\tilde G : [0,\pi/2]\times\mathbb{R}^4\times X \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k). \tag{15.7}$$

So, we can think of this as an homotopy in the variable θ and in particular, we can quantize it in (t',τ') uniformly in θ, $\mathbb{R}^2$ and X. At $\theta=\pi/2$ the (t',τ') variables are replaced by (t,τ) in the sense that

$$\begin{aligned} &\tilde G(\pi/2,t,\tau,t',\tau',x) = \\ &\qquad \mathrm{Id} + A(-t,-\tau)\tilde g^{-1}(x,t',\tau') + A'(-t,-\tau)\tilde g(x,t',\tau') + A''(-t,-\tau). \end{aligned} \tag{15.8}$$

Thus, at $\theta=\pi/2$ we are simply quantizing $\tilde g$ and $\tilde g^{-1}$. However, by construction the quantization (which we may choose) of $\tilde g$ is g and hence that of $\tilde g^{-1}$ is g^{-1}.

Finally then see what happens. If $\tilde g$ is such that $p_{\mathrm{ad}}\tilde g = \mathrm{Id}$ then we can find $\tilde G(\theta)$ which at $\theta=0$ quantizes to $\tilde g$ and at $\theta=\pi/2$ quantizes to Id. Thus in fact $\tilde g$ is homotopic to the identity and we have proved that (15.1) is an isomorphism, well for $d=1$ and modulo the discussion of projections – which I will proceed to do.

Maybe it is worthwhile going back and checking, modulo the same issues of course, that I have now proved Theorem 2 since that *looks* more substantial. So, I claim that, for $d=1$, I can pretend to have done everything except the weak contractibility of the subgroup

$$\mathcal{N} = \{A\in G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}:\mathbb{R}^k); R(A)=\mathrm{Id}\}. \tag{15.9}$$

That is, consider a compactly supported smooth map into it. We need to show that it can be deformed to the constant-at-the-identity map. This is almost what we have shown. Namely we have shown that if $\gamma : X\longrightarrow\mathcal{N}$ is such that $R(\gamma)=\mathrm{Id}$ then $\sigma_{\mathrm{ad}}(\gamma)$, which is what we discussed above, is homotopic to the identity in $G^{-\infty}_{\mathrm{sus(2),iso}}(\mathbb{R}^k)$. Now, quantizing this family of symbols (including the homotopy) gives us an homotopy in $G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}:\mathbb{R}^k)$ which I can call $\Gamma(t,x)$. At $t=0$ it is γ and at $t=1$ it is Id. We are almost there, but it is *not* (necessarily) an homotopy in $\mathcal{N}$. It starts there and finishes there but we have done nothing to control $R(\Gamma(t,x))$ for $t\in(0,1)$. Fortunately we have the lifting map. That is we can lift $R(\Gamma(t,x))$ to a family $\Gamma' : [0,1]\times X\longrightarrow G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}:\mathbb{R}^k)$ so that $R(\Gamma'(t,x)) = R(\Gamma(t,x))$ *and* so that $\Gamma'(0,x)=\Gamma'(1,x)=\mathrm{Id}$ since $R(\Gamma(t,x)=\mathrm{Id}$ there. Then $(\Gamma'(t,x))^{-1}\Gamma(t,x)$ is a new homotopy from γ to Id which *is* in $\mathcal{N}$.

Okay, so it is on to projections to check the little facts about the Bott projection and its quantization.

Let me formalize what we were doing earlier as regards projections and set

$$\mathcal{H}^{-\infty}(\mathbb{R}^k) = \left\{a\in\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k;\mathbb{C}^2); I_\infty + a \text{ is an involution}\right\},\ I_\infty = \begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{15.10}$$

This is not a group, but it has certain features making it similar to $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)$. Notably it is an infinite dimensional manifold 'modelled on (two copies of) $\mathcal{S}(\mathbb{R}^{2k})$. This I will discuss next time.

Definition 5. The K-groups (not quite immediately obvious that they are groups) associated to a smooth manifold X are the compactly-supported homotopy classes of smooth maps

$$\mathrm{K}^0_{\mathrm{c}}(X) = [X; \mathcal{H}^{-\infty}(\mathbb{R})]_{\mathrm{c}}. \tag{15.11}$$

Of course we will quickly show that one could just as well take

$$\mathrm{K}^0_{\mathrm{c}}(X) = [X; \mathcal{H}^{-\infty}(\mathbb{R}^k)]_{\mathrm{c}} \tag{15.12}$$

for any k and the results are naturally isomorphic. If X is compact one can drop the 'c' suffix – and historically it is even dropped in the general case, meaning that in the literature $\mathrm{K}^0(X)$ denotes what I am calling $\mathrm{K}^0_{\mathrm{c}}(X)$.

The work I did on the Bott element now extends directly.

Proposition 13. *There is a well-defined (adiabatic) periodicity map*

$$p_{\mathrm{ad}} : \mathrm{K}^0_c(X \times \mathbb{R}^{2d}) \longrightarrow \mathrm{K}^0_c(X). \tag{15.13}$$

Proof. A class in $\mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}^{2d})$ is represented by a compactly-supported map $X \times \mathbb{R}^{2d} \longrightarrow \mathcal{H}^{-\infty}(\mathbb{R})$. The discussion in Lecture 8 shows that this can be quantized to an adiabatic family of involutions. We need to check homotopy invariance of the result but this follows the same lines. ☐

Now we have lots of groups and lots of identifications between them:-

$$\begin{gathered} p_{\mathrm{ad}} : \mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}^{2d}) = \mathrm{K}^0_{\mathrm{c}}(X),\ p_{\mathrm{ad}} : \mathrm{K}^1_{\mathrm{c}}(X \times \mathbb{R}^{2d}) = \mathrm{K}^1_{\mathrm{c}}(X), \\ \mathrm{cl} : \mathrm{K}^0_{\mathrm{c}}(X) \longrightarrow \mathrm{K}^1_{\mathrm{c}}(X \times \mathbb{R}). \, \mathrm{cl} : \mathrm{K}^1_{\mathrm{c}}(X) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}) \end{gathered} \tag{15.14}$$

We do need to make sure that these maps are consistent under compostion – so we can regard them as identifications (with some care!). Typically the top row are regarded really as identifications – this is Bott periodicity – and the bottom row as maps.

16. Lecture 13: Involutions and K^0 Friday, 3 October

Last time I introduced the space of smooth involutions $\mathcal{H}^{-\infty}(\mathbb{R})$, let me immediately note some properties of it.

Proposition 14. *There is a surjective index, or relative dimension, map*

$$\operatorname{ind} : \mathcal{H}^{-\infty}(\mathbb{R}^k) \longrightarrow \mathbb{Z},\ \operatorname{ind}(I_\infty + a) = \frac{1}{2}\operatorname{tr}(a) \tag{16.1}$$

which labels the components, $\mathcal{H}^{-\infty}_k(\mathbb{R}^k)$, of $\mathcal{H}^{-\infty}(\mathbb{R}^k)$. The base component, where the index vanishes, is a homogeneous space

$$\mathcal{H}^{-\infty}_0(\mathbb{R}^k) = G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k;\mathbb{C}^2)/\left(G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)\oplus G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)\right) \tag{16.2}$$

through conjugation and the other components are isomorphic to the base component – but not naturally so.

Proof. We use finite rank approximation to prove this. In the construction of the quantized Bott element I used the idea which lies behind:

Lemma 18. *For each $I \in \mathcal{H}^{-\infty}(\mathbb{R}^k)$ there is a neighbourhood*

$$0 \in B \subset \Psi^{-\infty}(\mathbb{R}^k;\mathbb{C}^2)$$

such that if $b \in B$ then the complex integral

$$J(b) = -\operatorname{Id} - \frac{\pi i}{\int}_{|z-1|=\frac{1}{2}} \left(\frac{1}{2}(I+b) - (z-\frac{1}{2})\operatorname{Id}\right)^{-1} dz \tag{16.3}$$

is an element of $\mathcal{H}^{-\infty}(\mathbb{R}^k)$.

Proof. As Boris said: Just use the functional calculus!

If $b = 0$ in (16.3), then the inverse of $\frac{1}{2}I - (z-\frac{1}{2})\operatorname{Id} = (1-z)B_+ - zB_-$ is $(1-z)^{-1}B_+ - z^{-1}B_-$ where $I = B_+ - B_-$ is the decomposition into projections. The inverse is uniformly bounded on $|z-1| = \frac{1}{2}$ so remains invertible there if perturbed by $b/2$ in a small ball around the origin. Thus the integrand in (16.3) does exist and is of the form

$$\left(\frac{1}{2}(I+b) - (z-\frac{1}{2})\operatorname{Id}\right)^{-1} = (1-z)^{-1}B_+ - z^{-1}B_- + \gamma(z;b) \tag{16.4}$$

where $\gamma(z;b)$ is holomorphic near $|z-1| = \frac{1}{2}$ and valued in smoothing operators. The integral of the first term on the right in (16.4) is $-B_+$ so $J(b) = I + b'$ with $b' \in \Psi^{-\infty}(\mathbb{R}^k;\mathbb{C}^2)$. Moreover, b' is small with b and depends continuously on it. It remains to check that $J(b)$ is an involution. The square can be written

$$\begin{aligned} J(b)^2 = \operatorname{Id} + 2\frac{1}{\pi i}\int_{|z-1|=\frac{1}{2}} \left(\frac{1}{2}(I+b) - (z-\frac{1}{2})\operatorname{Id}\right)^{-1} dz \\ + \frac{1}{(\pi i)^2}\int_{|z-1|=\frac{1}{2}}\int_{|t-1|=\frac{1}{2}+\delta} \left(\frac{1}{2}(I+b) - (z-\frac{1}{2})\operatorname{Id}\right)^{-1} \\ \left(\frac{1}{2}(I+b) - (t-\frac{1}{2})\operatorname{Id}\right)^{-1} dzdt \end{aligned} \tag{16.5}$$

where the t contour has been moved slightly where $\delta > 0$. Applying the resolvent identity

$$\left(\frac{1}{2}(I+b)-(z-\frac{1}{2})\operatorname{Id}\right)^{-1}\left(\frac{1}{2}(I+b)-(t-\frac{1}{2})\operatorname{Id}\right)^{-1}=$$
$$(z-t)^{-1}\left(\frac{1}{2}(I+b)-(z-\frac{1}{2})\operatorname{Id}\right)^{-1}$$
$$-(z-t)^{-1}\left(\frac{1}{2}(I+b)-(t-\frac{1}{2})\operatorname{Id}\right)^{-1}$$

and inserting this into the the last term allows it to be evaluated by residues as

$$\begin{aligned}(16.6)\quad \frac{1}{(\pi i)^2}\int_{|z-1|=\frac{1}{2}}\int_{|t-1|=\frac{1}{2}+\delta}&\left(\frac{1}{2}(I+b)-(z-\frac{1}{2})\operatorname{Id}\right)^{-1}\\ &\times\left(\frac{1}{2}(I+b)-(t-\frac{1}{2})\operatorname{Id}\right)^{-1}dzdt\\ &=-2\frac{1}{\pi i}\int_{|z-1|=\frac{1}{2}}\left(\frac{1}{2}(I+b)-(z-\frac{1}{2})\operatorname{Id}\right)^{-1}dz.\end{aligned}$$

Thus indeed, $J(b)^2=\operatorname{Id}$. □

This 'retraction onto $\mathcal{H}^{-\infty}(\mathbb{R}^k)$' allows any element $I_\infty+a$ to be connected to a finite rank perturbation of I_∞. Namely, if k is large enough, depending on a, then

$$(16.7)\qquad I_\infty+(1-t)a+t\Pi_k a\Pi_k$$

is sufficiently close to $I_\infty+a$, for $t\in[0,1]$, for the Lemma to apply. Moreover it follows directly from the formula for $J(b)$ that

$$(16.8)\qquad J(\Pi_k a\Pi_k)=I+\Pi_k a'\Pi_k$$

is indeed a finite rank perturbation. Thus, as an involution it is equal to

$$(16.9)\qquad I_\infty(\operatorname{Id}-\Pi_k)+\Pi_k A\Pi_k$$

where the second term is an involution in $M(2,\mathbb{C})\otimes M(k,\mathbb{C})$, the latter being matrices acting on the range of Π_k in $\mathcal{S}(\mathbb{R}^k)$.

For finite rank involutions the first statements in the Proposition become obvious. In a given vector space they correspond to a decomposition as a direct sum, of the 1 and -1 eigenspaces, of dimensions d_+ and d_-, $d_++d_-=N$ being the dimension of the space on which the involution acts. Moreover, for fixed N any two such decompositions are linearly equivalent if and only the positive eigenspaces have the same dimension, d_+. The trace of the involution, $d_+-d_-=-2N+2d_-$, is an even integer which determines the involution up to linear equivalence. It follows that for the decomposition (16.9), in which Π_k acts as a multiple of the identity on the $\mathbb{C}^2$ factor,

$$(16.10)\qquad \operatorname{tr}(J(\Pi_k a\Pi_k)-I_\infty)=\operatorname{tr}(\Pi_k A\Pi_k)-\operatorname{tr}(I_\infty\Pi_k)=2p\in 2\mathbb{Z}$$

determines the linear equivalence class.

So, it remains to show that $\frac{1}{2}\operatorname{Tr}(a)$ is locally constant. However differentiating the identity $I_t^2 = \operatorname{Id}$ shows that

$$I_t I_t' + I_t' I_t = 0 \Longrightarrow \operatorname{tr}(I_t') = 0,\quad \text{hence } \frac{d}{dt}\operatorname{tr}(I_t - I) = 0. \tag{16.11}$$

since I_t' is off-diagonal with respect to I_t.

This proves (16.1) and that the 'index' map is constant on the components of $\mathcal{H}^{-\infty}(\mathbb{R}^k)$.

In the case that $\operatorname{ind}(I_\infty + a) = 0$ it follows from the discussion above that $I + a$ is connected by a smooth path $I_\infty + a(t)$, $t \in [0,1]$, in $\mathcal{H}^{-\infty}(\mathbb{R}^k)$ to I_∞ itself, so $a(1) = a$, $a(0) = 0$. For each $I \in \mathcal{H}^{-\infty}(\mathbb{R}^k)$, if b is small enough and $I+b \in \mathcal{H}^{-\infty}(\mathbb{R}^k)$ then

$$T = (I+b)_+ I_+ + (I+b)_- I_- \in G^{-\infty}(\mathbb{R}^k;\mathbb{C}^2) \tag{16.12}$$

where $I + b = (I_b)_+ - (I+b)_-$ is the decomposition into projections. Moreover,

$$TI = (I+b)T \Longrightarrow (I+b) = T^{-1}IT.$$

Thus, nearby involutions are conjugate under the action of $G^{-\infty}(\mathbb{R}^k;\mathbb{C}^2)$.

Apply this at each point $t \in [0,1]$ it follows that there is a finite decomposition of the interval such that $I + a(t)$ at each lower end-point is so conjugate to the upper end-point. Composing the action shows that $I + a$ is conjugate to I_∞.

Thus we see that the action by conjugation of $G^{-\infty}(\mathbb{R}^k;\mathbb{C}^2)$, is transitive on $\mathcal{H}_0^{-\infty}(\mathbb{R}^k)$. It is clear that the subgroup fixing I_∞ is the diagonal group $G^{-\infty}(\mathbb{R}^k) \oplus G^{-\infty}(\mathbb{R}^k)$ which is (16.2).

In each $\mathcal{H}_k^{-\infty}(\mathbb{R}^k)$ there is a 'base point'

$$\begin{cases} I_\infty + (\operatorname{Id} - I_\infty)\Pi_k & \in \mathcal{H}_k^{-\infty} \\ I_\infty - (\operatorname{Id} + I_\infty)\Pi_k & \in \mathcal{H}_{-k}^{-\infty} \end{cases}, k > 0 \tag{16.13}$$

Thus it suffices to show that these are conjugate to I_∞. This can be done by renumbering the bases – of course these conjugating operators are not in $G^{-\infty}(\mathbb{R}^k;\mathbb{C}^2)$. $\square$

This result has quite a few consequences for our defintion of $\mathrm{K}_c^0(X)$. However, the first thing I need to do – to finish the proof of Bott periodicity – is to go back and look at the quantized Bott involution constructed in Lemma 9. What we want to do is to compute $\frac{1}{2}\operatorname{tr}(D_\epsilon)$, which we now know to be constant as a function of $\epsilon > 0$. Of course we must somehow compute it in terms of the semiclassical limit as $\epsilon \downarrow 0$. By construction D_ϵ comes from a semiclassical family, with kernels

$$D_\epsilon = \epsilon^{-1} D(\epsilon, \frac{\epsilon(t+t')}{2}, \frac{t-t'}{\epsilon}) \tag{16.14}$$

valued in 2×2 matrices. So, for $\epsilon > 0$

$$\operatorname{tr}(D_\epsilon) = \epsilon^{-1}\int_{\mathbb{R}} \operatorname{tr} D(\epsilon, \epsilon t, 0)dt = \epsilon^{-2}\int_{\mathbb{R}} D(\epsilon, T, 0)dT = \frac{1}{2\pi\epsilon^2}\int_{\mathbb{R}^2} \operatorname{tr}\hat{D}(\epsilon, t, \tau)dt d\tau. \tag{16.15}$$

So, what we know is $\hat{D}(0,t,\tau) = \delta(t,\tau)$ and what we need to compute is the (integral of the trace of) the coefficient of ϵ^2 in the Taylor series expansion of $\hat{D}(\epsilon,\dots)$. Fortunately, the ϵ^2 term is the next after the leading term.

In fact if you recall the construction of D what we did was start with D_0 which is a quantization of δ; we can take it not to depend explicitly on ϵ. Then we need to compute the semiclassical symbol of the error term

$$(I_\infty + D_0)^2 - \mathrm{Id} = \epsilon^2 E_1,\ \sigma_{\mathrm{sl}}(E_1) = \frac{1}{2i}\left(\partial_t\delta\partial_\tau\delta - \partial_\tau\delta\partial_t\delta\right). \tag{16.16}$$

Now, the *correction term* is $\epsilon^2 D_1$ where $\sigma_{\mathrm{sl}}(D_1)$ has to satisfy

$$b\sigma_{\mathrm{sl}}(D_1) + \sigma_{\mathrm{sl}}(D_1)b = \sigma_{\mathrm{sl}}(E_1) \tag{16.17}$$

which we did by noting that the right side satisfies

$$b\sigma_{\mathrm{sl}}(E_1) = \sigma_{\mathrm{sl}}(E_1)b \text{ so } \sigma_{\mathrm{sl}}(D_1) = \frac{1}{2}b\sigma_{\mathrm{sl}}(E_1)$$

works. Thus combining these formulæ we need to compute

$$-\frac{1}{8\pi}\int_{\mathbb{R}^2} \mathrm{tr}\left(b\left(\partial_t\delta\partial_\tau\delta - \partial_\tau\delta\partial_t\delta\right)\right) dtt\tau. \tag{16.18}$$

Since $\partial_t\delta$ and $\partial_\tau\delta$ are derivatives of $b = I_\infty + \delta$ we know that $b(\partial_t\delta) = -(\partial_t\delta)b$, etc, anticommute. So in fact the two terms in (16.18) are the same. Since δ is written in terms of polar coordinates, it is natural to change variable and use a similar rearrangement to reduce to the integral

$$-\frac{1}{4\pi}\int_0^\infty\int_0^{2\pi} \mathrm{tr}\left(b(\partial_r\delta)(\partial_\theta\delta)\right) drd\theta. \tag{16.19}$$

Now, recall what $b = I_\infty + \delta$ is! It was defined in terms of Pauli matrices

$$b(t,\tau) = \cos(\Theta(-r))\gamma_1 + \sin(\Theta(-r))\cos(\theta)\gamma_2 - \sin(\Theta(-r))\sin(\theta)\gamma_3. \tag{16.20}$$

There are three constant matrices in (16.20). Each of them has trace zero and the product of any two of them (which is $\pm i$ times the other one) has trace zero. The product of all three $\gamma_1\gamma_2\gamma_3 = -\mathrm{Id}_{2\times 2}$ has trace -2. Thus there are four terms which can contribute. Namely the product of $\Theta'(-r)$ and

$$\begin{aligned}\sin^3(\Theta)\sin^2\theta\gamma_3\gamma_1\gamma_2 - \sin(\Theta)\cos^2(\Theta)\sin^2\theta\gamma_1\gamma_3\gamma_2\\ - \sin^3(\Theta)\cos^2\theta\gamma_2\gamma_1\gamma_3 + \sin(\Theta)\cos^2(\Theta)\cos^2\theta\gamma_1\gamma_2\gamma_3\\ = -\sin(\Theta)\,\mathrm{Id},\end{aligned} \tag{16.21}$$

where $\Theta = \Theta(-r)$. The integral is therefore

$$-\int_0^{2\pi}\int_0^{\pi} \sin^2\theta\sin(\Theta)d\theta d\Theta = 8\pi. \tag{16.22}$$

Combining all this we conlude that

$$\mathrm{ind}(B) = \frac{1}{2}\,\mathrm{tr}(D) = 1. \tag{16.23}$$

Phew, that proves Bott periodicity.

17. Lecture 14: Even periodicity map
Monday, 6 October

Question 2. (Jesse Gell-Redman) The construction of the (odd) periodicity map looks a bit fishy, would you care to clarify it?

Answer 2. Well, he was politer than that. Let me put things together, maybe a little more carefully than I did before. First let me try to clarify a couple of things – the sense in which we are free to switch the number of variables in which our smoothing operators act and are also free to work with finite dimensional matrices.

Lemma 19. *For any N and for any selection of N of the elements of the standard basis of eigenfunctions of the harmonic oscillator, $e_{i_1}, \dots, e_{i_N}$ the inclusion*

$$\mathrm{GL}(N;\mathbb{C}) \ni g_{ij} \longmapsto \hat{g} \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}),\ \hat{g}e_{i_l} = \sum_{p=1}^{N} g_{pl}e_{i_p},\ \hat{g}e_j = e_j,\ j \neq i_l, \tag{17.1}$$

is a group homomorphism which induces a map on homotopy classes

$$[X; \mathrm{GL}(N,\mathbb{C})]_c \longrightarrow [X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R})]_c \tag{17.2}$$

which is independent of the choice of basis and such that every element in the target group is in the image for N sufficiently large.

Proof. That the stabilizing map (17.1) is a group homorphism is clear enough and so it induces a map (17.2). Any two choices of basis are conjugate. To see this it suffices to change one element at a time to another eigenfunction which is not in the current set, and then finally relabel. The relabelling is given by an element of $\mathrm{GL}(N,\mathbb{C})$ acting by conjugation and the switching is given by a rotation between the two elements. In either case on the image space this is conjugation by a fixed (in terms of X) element of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R})$. Since this group is connected, the conjugation can be removed by a homotopy, constant in X. This proves that the induced maps (17.2) are all the same. □

One can easily go further somewhat further, as we will later, and conclude that as long as $\mathrm{GL}(N,\mathbb{C})$ is made to act on an N-dimensional subspace of the range of Π_K, for some k, and the identity on a complementary space, and on the range of $\mathrm{Id} - \Pi_k$, the same map (17.2), at the level of homotopy, results.

Lemma 20. *If $0 \neq e \in \mathcal{S}(\mathbb{R}^l)$ and $\pi_e = e \otimes \bar{e} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^l)$ is the orthogonal projection onto e then the group homomorphism*

$$G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^d) \ni \mathrm{Id} + a \longmapsto \mathrm{Id} + \pi_e \otimes a \in G^{-\infty}(\mathbb{R}^{d+k}) \tag{17.3}$$

is a weak homotopy equivalence (Is it an homotopy equivalence?) which induces an isomorphism, for any manifold X,

$$[X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^d)]_c = [X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k})]_c \tag{17.4}$$

which is independent of the choice of e.

Note that e is fixed but arbitrary. As in the proof above, we can deform any of the maps (17.3) to any other by rotating e.

Note that the embeddings above are constant, i.e. we are not permitting twisting which depends on the point in X. It is also worth re-emphasizing that in both these maps the identity is 'increased in size', I have been regarding these maps as inclusions but one does need to be a little careful about this.

So, what is the inverse of the adiabatic periodicity map exactly? It is based on the lifting statement for restriction to $\epsilon = 1$

$$R : G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R} : \mathbb{R}^k) \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+k}). \tag{17.5}$$

Namely, take a smooth map $h : X \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+k})$ which is constant at the identity outside a compact set. We can 'lift' this back into one of the groups (17.1), that is there is a homotopy h_t where h_0 is in the image of the group and $h_1 = h$. So, consider h_0 instead.

Now, what do we know about $b = \gamma_1 + \delta$, $\delta \in \mathcal{C}^\infty_c(\mathbb{R}^2; M(2,\mathbb{C}))$, the Bott element. We showed that this involution is the semiclassical symbol of an involution $B = \gamma_1 + D$, $D \in \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R}; \mathbb{C}^2)$. More precisely, $D_\epsilon \in \mathcal{C}^\infty((0,1]; \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}; \mathbb{C}^2))$ is a smooth family of 2×2 matrices of isotropic smoothing operators on $\mathbb{R}$, forming a semiclassical family with symbol δ and such that $\gamma_1 \operatorname{Id} + D_\epsilon$ is an involution for all $\epsilon \in (0,1]$. Moreover, last time I finally computed the relative index of this family, showing it was 1, and hence that say $\gamma \operatorname{Id}_1 + D_{\frac{1}{2}}$ can be deformed to have

$$R(B) = \gamma \operatorname{Id}_1 + 2\Pi_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{17.6}$$

(which has the same relative index) through involutions – here Π_1 is projection onto the ground state of the harmonic oscillator (or onto some other element of $\mathcal{S}(\mathbb{R})$ of your choice). Thus, modifying the parameter in $\epsilon > \frac{1}{4}$ a bit we have a semiclassical family, B, of 2×2 matrices, in 1 dimension, which has semiclassical symbol b and takes the value (17.6) at $\epsilon = 1$.

Now, much as above we want to turn this into an adiabatic family. First we can undo B into the projections onto its positive and negative partis, $B_\pm$ and then consider the semiclassical family of $2N \times 2N$ matrices

$$g' = (h_0^{-1}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})(h_0(x) \otimes B_+ + B_-) \in G^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R}; \mathbb{C}^{2N}). \tag{17.7}$$

The first factor makes the semiclassical family have 'unital part' – the leading matrix multiple of the identity – be Id_{2N}, giving (17.7) and from (17.6)

$$R(g') = \operatorname{Id} + \Pi_1 \otimes (h_0 - \operatorname{Id}_N) \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{17.8}$$

This is just $h_0(x)$ acting as an $N \times N$ matrix on $\mathbb{C}^N$, extended to the second part of the $\mathbb{C}^2$, plus the identity on everything else.

Now, we are free to embed $M(2N,\mathbb{C})$ to act on a finite span of the harmonic oscillator eigenfunctions in $\mathbb{S}(\mathbb{R}^k)$ however we want, and we can do this so that the $h_0(x)$ in (17.8) is the h_0 we started with. Moreover this embedding corresponds to the same sort of map as (17.3) but now giving

$$G^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R}; \mathbb{C}^{2N}) \longrightarrow G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R} : \mathbb{R}^k; \mathbb{C}^2). \tag{17.9}$$

So, combining these steps the image, g, of g' under (17.9) has $R(g) = h_0$. Modifying the family in $\epsilon > 0$ we can insert the extra homotopy to arrange that $R(g) = h$ as desired.

Let us note some stabilization results of the same type as discussed above, but for the classifying spaces $\mathcal{H}^{-\infty}(\mathbb{R}^d)$. As for the corresponding groups there are isomorphisms – for the moment fixed, but at some point I will have to discuss the possible choices –

$$\begin{aligned}\mathcal{H}^{-\infty}(\mathbb{R}) &\longrightarrow \mathcal{H}^{-\infty}(\mathbb{R}^d)\\ \mathcal{H}^{-\infty}(\mathbb{R}^d;\mathbb{C}^N) &\longrightarrow \mathcal{H}^{-\infty}(\mathbb{R}^d).\end{aligned}\tag{17.10}$$

They are obtained by relabelling basis elements or by extending the perturbation to have rank 1 in the other 'variables'. One can also think about this as going back to our original sequential group and considering the corresponding

$$\mathcal{H}^{-\infty}(\mathbb{N}) = \left\{a\in\Psi^{-\infty}(\mathbb{N};\mathbb{C}^2);\begin{pmatrix}1&0\\0&-1\end{pmatrix}+a\text{ is an involution}\right\}.\tag{17.11}$$

Now, we also have adiabatic and suspended versions of these spaces:-

$$\begin{gathered}\mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^k)=\\ \left\{a\in\Psi^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^k;\mathbb{C}^2);\begin{pmatrix}1&0\\0&-1\end{pmatrix}+a(T)\in\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)\ \forall\ T\in\mathbb{R}^p\right\}\\ \mathcal{H}^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)=\\ \left\{a\in\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k;\mathbb{C}^2);\begin{pmatrix}1&0\\0&-1\end{pmatrix}+a(\epsilon)\in\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{d+k}),\ \epsilon>0\right\},\end{gathered}\tag{17.12}$$

including of course those acting on $\mathbb{C}^2\otimes\mathbb{C}^N$ instead of just $\mathbb{C}^2$.

At this stage it probably has already occurred to you that we can 'put things together'. That is, we can define a combined adiabatic-suspended-isotropic space of involutions

$$\begin{aligned}\mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)=\Big\{a\in\mathcal{S}(\mathbb{R}^p;\Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k;\mathbb{C}^2\otimes\mathbb{C}^N));&\\ \begin{pmatrix}1&0\\0&-1\end{pmatrix}+a(\epsilon)\in\mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^{d+k};\mathbb{C}^N),\ \forall\ \epsilon>0\Big\}.&\end{aligned}\tag{17.13}$$

Here it is understood that if $d=0$ or $p=0$ one is reduced to the earlier cases.

Proposition 15. *The space in* (17.13) *is classifying for K-theory of the parity of p (provided $k>0$), the base component (which is the whole space if p is odd) is homogeneous*

$$\begin{gathered}\mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)=\\ G^{-\infty}_{\mathrm{sus}(p),\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k;\mathbb{C}^2)\Big/\Big(G^{-\infty}_{\mathrm{sus}(p),\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k)\oplus G^{-\infty}_{\mathrm{sus}(p),\mathrm{ad,iso}}(\mathbb{R}^d:\mathbb{R}^k;\mathbb{C}^2)\Big)\end{gathered}\tag{17.14}$$

and for $d=1$ (also for $d>1$ shown later) the lower maps in the following diagram are surjective, with the lifting property for compactly-supported families, and with

the upper spaces weakly contractible:-
(17.15)

$$
\begin{tikzcd}
\left\{\sigma_{\mathrm{ad}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \arrow[dr] & & \left\{R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \arrow[dl] \\
& \mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{ad},\mathrm{iso}}(\mathbb{R}^d : \mathbb{R}^k) \arrow[dl, "R"'] \arrow[dr, "\sigma_{\mathrm{ad}}"] & \\
\mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^{d+k}) & & \mathcal{H}^{-\infty}_{\mathrm{sus}(p+2d),\mathrm{iso}}(\mathbb{R}^k).
\end{tikzcd}
$$

Proof. Everything here, except the lifting property for R is fairly straightforward, meaning it is much the same as before. For the momemnt I omit proofs from the notes for these parts. To prove the properties of R we need some more properties of involutions $\square$

18. Lecture 15: Vector bundles and $\mathrm{K}^0_{\mathrm{c}}(X)$ Wednesday, 8 October

We start with an involution which is a finite rank perturbation of γ_1, $\gamma_1 + a$, $\Pi_k a = a\Pi_k = a$. Thus, restricting to $\mathbb{C}^2\otimes\Pi_k$ which we can identify with any other $2k$-dimensional vector space we have an involution

$$I = I_+ - I_- \text{ acting on } \mathbb{C}^2\otimes\operatorname{Ran}(\Pi_k)\equiv\mathbb{C}^{2k}. \tag{18.1}$$

Then consider a further slice $\mathbb{C}^2\otimes(\Pi_{3k}-\Pi_k)$. Here we can identify $\operatorname{Ran}(\Pi_{3k}-\Pi_k)$ with $\mathbb{C}^{2k}$ and so write the restriction of $\gamma_1\otimes\operatorname{Id}$ as

$$\gamma_1\otimes(I_+ + I_-). \tag{18.2}$$

So the part of the involution in $\mathbb{C}^2\otimes\operatorname{Ran}(\Pi_{3k})$ is

$$\begin{gathered}(I_+(x)-I_-(x))\oplus E_+\otimes(I_+ + I_-)\oplus -E_-\otimes(\Pi_{3k}-\Pi_k),\\ E_+ = \begin{pmatrix}1&0\\0&0\end{pmatrix},\ E_- = \begin{pmatrix}0&0\\0&1\end{pmatrix}.\end{gathered} \tag{18.3}$$

Now, the I_- part of the first block can be rotated rotated with the I_- part of the second block and thus there is an homotopy leading from (18.2) to

$$\begin{aligned}&(I_+(x)+I_-(x))\oplus E_+\otimes(I_+ - I_-)\oplus -E_-\otimes(\Pi_{3k}-\Pi_k)\\ &= (E_+ + E_-)\oplus\Pi_k + E_+\otimes(I_+ - I_-)\oplus -E_-\otimes(\Pi_{3k}-\Pi_k).\end{aligned} \tag{18.4}$$

This computation proves:-

Lemma 21. *Any $f\in\mathcal{C}^\infty_c(X;\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^d)$ is homotopic through such maps to one of the form*

$$\tilde f(x) = \gamma_1\otimes(\operatorname{Id}-\Pi_{3k}) + \begin{pmatrix}I(x)&0\\0&-\operatorname{Id}\end{pmatrix}\otimes(\Pi_{3k}-\Pi_k) + \begin{pmatrix}\operatorname{Id}&0\\0&\operatorname{Id}\end{pmatrix}\otimes\Pi_k \tag{18.5}$$

where $I(x)$ is a smooth family of involutions acting on the $2k$-dimensional space which is the range of $\Pi_{3k}-\Pi_k$.

In consequence $\tilde f$ commutes with γ_1 and has positive and negative projections of the form

$$\begin{gathered}\tilde f_+(x) = E_+\otimes(\operatorname{Id}-\Pi_{3k}+I_+ + \Pi_k) + E_-\otimes\Pi_k\\ \tilde f_-(x) = E_-\otimes(\operatorname{Id}-\Pi_k) + E_+\otimes I_-,\end{gathered} \tag{18.6}$$

which therefore commute (with each other of course and) with E_+ and E_-. One really might as well write this in the more symmetric form

$$\begin{gathered}\tilde f_+(x) = E_+\otimes(\operatorname{Id}-P^-(x)) + E_-\otimes(P^+(x)),\\ \tilde f_-(x) = E_-\otimes(\operatorname{Id}-P^+(x)) + E_+\otimes(P^-(x)),\\ \Pi_l P^\pm = P^\pm\Pi_l = (P^\pm)^2 \text{ and } P^+P^- = P^-P^+ = 0\end{gathered} \tag{18.7}$$

where $l = 3k$. Then (18.6) shows that we can take $P^+ = \Pi_k$, $k\le l$; by considering $-I$ it follows similarly that one can arrange by homotopy that $P^- = \Pi_k$ instead. Note that it follows from (18.7) that

$$\begin{gathered}\operatorname{ind}(\tilde f(x)) =\\ \frac12\operatorname{tr}\big((E_- + E_+)\otimes P^+(x) - (E_+ + E_-)\otimes P^-(x)\big) = \operatorname{rank}(P^+) - \operatorname{rank}(P^-).\end{gathered} \tag{18.8}$$

This gives us the basic relationship between vector bundles and smooth families of involutions, namely $P^+ \ominus P^-$ is a 'superbundle' – the formal difference of two bundles – which also determines the element of $\mathrm{K}^0_{\mathrm{c}}(X)$ fixed by $\tilde{f}$.

Said a different way, the space $\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)$ of involutions itself has an involution acting on it, namely

$$\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k) \ni \gamma_1 + a \longmapsto \gamma_1(\gamma_1 + a)\gamma_1 \in \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k). \tag{18.9}$$

This is however 'trivial' as far as homotopy is concerned. Namely

Lemma 22. *Any map $f \in \mathcal{C}^\infty_c(X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k))$ is homotopic to some*

$$\begin{gathered} \tilde{f} \in \mathcal{C}^\infty_c(X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)) \textit{ satisfying} \\ \gamma_1 \tilde{f}(x) = \tilde{f}(x)\gamma_1 \textit{ and } a = \Pi_k a = a\Pi_k \end{gathered} \tag{18.10}$$

for some k.

Proof by Jesse and Paul, not proofread yet. First suppose that

$$f_i : X \to \mathcal{H}^{-\infty}_{iso}\left(\mathbb{R}^d\right),\ i = 0, 1,$$

are maps with $f_1 \sim f_2$. Then there is a map

$$\begin{array}{rcl} F : [0,1] \times X & \to & \mathcal{H}^{-\infty}_{iso}\left(\mathbb{R}^d\right) \\ F(0,x) & = & f_0(x) \\ F(1,x) & = & f_1(x) \end{array}$$

f By the above lemma, there is a homotopy from F to a map $\widetilde{F}$ so that $\widetilde{F}$ has a decomposition

$$\widetilde{F}(t,x) \quad = \quad E_+ \otimes (Id - 2P_-(t,x)) - E_- \otimes (Id - 2P_+(t,x)),$$

and that furthermore P_- can be chosen so that $P_- \equiv \pi_k$ for some big k, so in particular $P_-(0,x) = P_-(1,x)$. It follows that the $P_+(t,\cdot)$ define isomorphic bundles for all t by an open and closed argument (openness is always true, and the closed part follows from the constancy of the rank.)

For the converse, suppose we have an equivalence of bundles

$$P^0_- \oplus P^1_+ \oplus S = P^1_- \oplus P^0_+ \oplus S = \mathbb{C}^l, \tag{18.11}$$

over a space X. Then we choose an identification of $\mathbb{C}^l$ with a subspace of $\mathcal{S}(\mathbb{R}^d)$ so that π_l is projection thereon, and define

$$f^i \quad = \quad E_+ \otimes \left(Id - 2P^i_-\right) - E_- \otimes \left(Id - 2P^i_+\right),$$

for $i = 0, 1$. The lemma then follows by using (18.11) and rotating blocks as follows.

$$\begin{array}{rcl} f^0 & = & E_+ \otimes \left(Id - 2P^0_-\right) - E_- \otimes \left(Id - 2P^0_+\right) \\ & = & E_+ \otimes \left((Id - 2P^0_-)\pi_l\right) - E_- \otimes \left((Id - 2P^0_+)\pi_l\right) \\ & & +E_+ \otimes \left((Id - 2P^0_-)(Id - \pi_l)\right) - E_- \otimes \left((Id - 2P^0_+)(Id - \pi_l)\right), \end{array}$$

so just deal with the middle line, so that we only consider $f^0\left(E_+ \otimes \pi_k + E_- \otimes \pi_k\right)$, which is

$$\begin{array}{rl} = & E_+ \otimes \left(Id_{\mathbb{C}^l} - 2P^0_-\right) - E_- \otimes \left(id_{\mathbb{C}^l} - 2P^0_+\right) \\ = & E_+ \otimes \left(P^1_+ + S - P^0_-\right) - E_- \otimes \left(P^1_- + S - P^0_+\right) \end{array}$$

Everything here is in blocks, so you can rotate the two S's into one another, which switches their signs. This and another substitution gives

$$\begin{aligned} &= E_+ \otimes \left(P^1_+ - S - P^0_-\right) - E_- \otimes \left(P^1_- - S - P^0_+\right) \\ &= E_+ \otimes \left(2P^1_+ - Id_{\mathbb{C}^l}\right) - E_- \otimes \left(2P^1_- - Id_{\mathbb{C}^l}\right) \\ &= E_+ \otimes \left(Id_{\mathbb{C}^l} - 2P^1_-\right) - E_- \otimes \left(Id_{\mathbb{C}^l} - 2P^1_+\right) \end{aligned}$$

Adding this back to the part we ignored gives the homotopy we wanted. □

This is a direct consequence of Lemma 21.

Proposition 16. *Any map $\tilde{f} \in \mathcal{C}^\infty_c(X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k))$ satisfying* (18.10) *is of the form* (18.7) *and two such maps $\tilde{f}_i$ are homotopy if and only if there is a vector bundle S over X which is identified with $\mathbb{C}^p$ outside a compact set and a bundle isomorphism*

$$\mathrm{Ran}(P^+_1) \oplus \mathrm{Ran}(P^-_2) \oplus S \longrightarrow \mathrm{Ran}(P^+_2) \oplus \mathrm{Ran}(P^-_1) \oplus S \tag{18.12}$$

which is the natural identification outside a compact set; here the ranges of the projections are considered as vector bundles over X.

Proof. □

The adiabatic Bott element constructed ealier

$$B = \gamma_1 \otimes \mathrm{Id} + D,\ D \in \Psi^{-\infty}_{\mathrm{sl,iso}}(\mathbb{R}; \mathbb{C}^2) \tag{18.13}$$

is an involution, $B^2 = \mathrm{Id}$, and satisfies

$$\begin{aligned} \sigma_{\mathrm{sl}}(B) &= \gamma_1 \otimes \mathrm{Id} + \delta(t,\tau) = b(t,\tau) \\ R(B) &= \gamma_1 \otimes (\mathrm{Id} - \Pi_1) + \mathrm{Id} \otimes \Pi_1 \in M(2,\mathbb{C}) + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}; \mathbb{C}^2) \end{aligned} \tag{18.14}$$

which is (18.7) with $P^- = 0$, $P^+ = \Pi_1$, $l = 1$.

Completion of proof of Proposition 15. To prove the even semiclassical lifting property we can take an element in the form (18.7). Consider

$$\begin{aligned} \tilde{B} = \gamma_1 \otimes (\mathrm{Id} - P^+(x) - P^-(x)) + B \otimes P^+(x) - B \otimes P^-(x) \\ \in \mathcal{C}^\infty_c(X; M(2,\mathbb{C}) + \Psi^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R} : \mathbb{R}^k). \end{aligned} \tag{18.15}$$

I think this quantizes to the right thing and so proves the surjectivity of R in (17.15). Injectivity follows using Atiyah's rotation again. □

Now, let me consider the clutching constructions. Perhaps I will take the time to do this carefully, for the moment I have just written these down and am hoping for the best!

First, from even to odd. There is an actual map

$$\begin{aligned} \mathrm{cl}_{\mathrm{eo}} : \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k) \ni I = \gamma_1 + a \longmapsto \\ \big(\cos(\Theta(t)) - i\sin(\Theta(t))\gamma_1\big)\big(\cos(\Theta(t)) + i\sin(\Theta(t))I\big) \\ = \mathrm{Id} + i\sin(\Theta(t))\big(\cos(\Theta(t)) - i\sin(\Theta(t))\gamma_1\big)a \in G^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^k; \mathbb{C}^2). \end{aligned} \tag{18.16}$$

Here $\Theta \in \mathcal{C}^\infty(\mathbb{R})$ is non-decreasing, vanishes for t sufficiently negative and is equal to π for t positive. Similarly, from odd to even

$$
\begin{aligned}
(18.17) \quad & \mathrm{cl}_{\mathrm{oe}} : G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k) \ni g \longmapsto \\
& I(t) = \begin{cases} \begin{pmatrix} \cos(\Theta(t)) & \sin(\Theta(t))g \\ \sin(\Theta(t))g^{-1} & -\cos(\Theta(t)) \end{pmatrix} & t \le 0 \\ \begin{pmatrix} \cos(2\pi - \Theta(-t)) & \sin(2\pi - \Theta(-t)) \\ \sin(2\pi - \Theta(-t)) & -\cos(2\pi - \Theta(-t)) \end{pmatrix} & t > 0 \end{cases} \in \mathcal{H}^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^k).
\end{aligned}
$$

Proposition 17. *The clutching maps in* (18.16) *and* (18.17) *induce isomorphisms in K-theory giving commutative diagrams for any manifold X :*
(18.18)

$$
\begin{tikzcd}
{[X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)]_c} \arrow[r, "\mathrm{cl}_{\mathrm{eo}}"] \arrow[d, equal] & {[X; G^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^k; \mathbb{C}^2)]_c} \arrow[r, "\mathrm{cl}_{\mathrm{oe}}"] \arrow[d, equal] & {[X; \mathcal{H}^{-\infty}_{\mathrm{sus(2),iso}}(\mathbb{R}^k; \mathbb{C}^2)]_c} \arrow[d, equal] \arrow[ll, bend right, "\mathrm{p}_{\mathrm{ad}}"'] \\
\mathrm{K}^0_c(X) \arrow[r, "\mathrm{cl}_{\mathrm{eo}}"] & \mathrm{K}^1_c(\mathbb{R} \times X) \arrow[r, "\mathrm{cl}_{\mathrm{oe}}"] & \mathrm{K}^0_c(\mathbb{R}^2 \times X) \arrow[ll, bend left, "\mathrm{p}_{\mathrm{ad}}"]
\end{tikzcd}
$$

and

$$
(18.19) \quad
\begin{tikzcd}
{[X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)]_c} \arrow[r, "\mathrm{cl}_{\mathrm{oe}}"] \arrow[d, equal] & {[X; \mathcal{H}^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^k)]_c} \arrow[r, "\mathrm{cl}_{\mathrm{oe}}"] \arrow[d, equal] & {[X; G^{-\infty}_{\mathrm{sus(2)\ iso}}(\mathbb{R}^k; \mathbb{C}^2)]_c} \arrow[d, equal] \arrow[ll, bend right, "\mathrm{p}_{\mathrm{ad}}"'] \\
\mathrm{K}^{-1}_c(X) \arrow[r, "\mathrm{cl}_{\mathrm{eo}}"] & \mathrm{K}^0_c(\mathbb{R} \times X) \arrow[r, "\mathrm{cl}_{\mathrm{oe}}"] & \mathrm{K}^{-1}_c(\mathbb{R}^2 \times X) \arrow[ll, bend left, "\mathrm{p}_{\mathrm{ad}}"]
\end{tikzcd}
$$

19. Lecture 16: The Chern character Friday, 10 October

We now have two 'series' of classifying spaces for K-theory. The (loop) groups based on $G^{-\infty}$ and the spaces of involutions $\mathcal{H}^{-\infty}$ and their looped (or suspended) versions. I have previously introduced the Chern forms in the first case. Today I want to introduce the basic Chern forms in the second and discuss their properties – leading to the definition of the Chern character. You might want to recall that $\mathcal{H}^{-\infty}$ is a Fredholm manifold.

Now, the basis 'zeroth' Chern form is the index, the relative dimension invariant used to analyze the components of $\mathcal{H}^{-\infty}$:

$$\mathrm{Ch}_0^{\mathcal{H}} = \mathrm{ind} : \mathcal{H}^{-\infty} \ni I \longmapsto \frac{1}{2}\operatorname{tr}(I - \gamma_1) \in \mathbb{Z}. \tag{19.1}$$

The higher forms do not require 'regularization' – the subtraction of γ_1 – because they involve derivatives, so we can think of (19.1) as a regularized version of $\frac{1}{2}\operatorname{tr}(I)$. Thus, consider the forms on $\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)$

$$\mathrm{Ch}_{2k}^{\mathcal{H}} = 2^{-2k-1}\operatorname{tr}(I(dI)^{2k}) = 2^{-2k-1}\operatorname{tr}(IdI \wedge dI \cdots \wedge dI). \tag{19.2}$$

Recall that $I^2 = \mathrm{Id}$, so $I(dI) = -(dI)I$ shows that I is an additional anticommuting factor. Antisymmetry shows that an odd number of factors of dI would lead to zero, so we only consider the even cases. Notice that $dI = da$, $I = \gamma + a$, is a 1-form valued in the underlying algebra $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k; \mathbb{C}^2)$ so the trace does exist. Moreover it follows directly that

$$d\,\mathrm{Ch}_{2k}^{\mathcal{H}} = 2^{-2k-1}\operatorname{tr}(dI(dI)^{2k}) = 0 \tag{19.3}$$

since all the terms dI are closed and the involution I and dI anticommute, so

$$\begin{gathered}(dI)^{2k+1} = I^2(dI)^{2k+1} = -I(dI)^{2k+1}I \\ \Longrightarrow \operatorname{tr}((dI)^{2k+1}) = -\operatorname{tr}(I(dI)^{2k+1}I) = -\operatorname{tr}(I^2(dI)^{2k+1}) = -\operatorname{tr}((dI)^{2k+1})\end{gathered} \tag{19.4}$$

using the trace identity.

Why the factors of 2 in (19.2)? Consider what happens when we pull back these forms under a map $f \in \mathcal{C}^\infty_c(X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k))$. Since $\mathrm{Ch}_{2k}^{\mathcal{H}}$ is closed, it follows that under homotopy it changes by an exact form:

Exercise 13. Observe, following the discussion in Lecture 4 that the under for an homotopy f_t the parameter derivative $\frac{d}{dt}f_t^*\,\mathrm{Ch}^{\mathcal{H}}$ is exact and hence so is the differnece $f_1^*\,\mathrm{Ch}^{\mathcal{H}} - f_0^*\,\mathrm{Ch}^{\mathcal{H}}$.

So, in view of Lemma 21 we can assume that f is replaced by $\tilde f$ in (18.7). Then,

$$\begin{aligned}&\tilde f^* dI = \\ &d(E_+ \otimes (\mathrm{Id} - P^-(x)) + E_- \otimes (P^+(x))) - d(E_- \otimes (\mathrm{Id} - P^+(x)) + E_+ \otimes (P^-(x))) \\ &\qquad = 2\big(E_- \otimes dP^+(x) - E_+ \otimes dP^-(x)\big).\end{aligned} \tag{19.5}$$

Since $E_+E_- = 0$, the big wedge product in (19.2) decomposes into two pieces:

$$\Big(2\big(E_- \otimes dP^+(x) - E_+ \otimes dP^-(x)\big)\Big)^{2k} = 2^{2k}\big(E_-(dP^+)^{2k} - E_+(dP^-)^{2k}\big). \tag{19.6}$$

Now, recall that the differential of a projection satisfies

$$PdP + (dP)P = dP \Longrightarrow PdP = dP(\mathrm{Id} - P),\ (\mathrm{Id} - P)dP = dP(P). \tag{19.7}$$

Thus, inserting the identity as $P^{\pm} + (\operatorname{Id} - P^{\pm})$ into the corresponding term and expanding out we find

$$
(19.8)\quad E_-(dP^+)^{2k} = \\
E_-\big(P^+(dP^+)(\operatorname{Id} - P^+)(dP^+)P^+)^k + (\operatorname{Id} - P^+)(dP^+)P^+(dP^+)(\operatorname{Id} - P^+)\big).
$$

Inserting this (for both signs) back into (19.2), note that the factor of I swithces half the sings, then the trace identity shows that the two terms in (19.8) give the same contribution. Thus in fact

$$
(19.9)\quad \tilde{f}^* \operatorname{Ch}^{\mathcal{H}}_{2k} = \\
\operatorname{tr}\big((P^+(x)(dP^+(x))(dP^+(x))P^+(x))^k - (P^-(x)(dP^-(x))(dP^-(x))P^-(x))^k\big)
$$

with the constants cancelling, which is why they were included in the first place.

Exercise 14. Show that if $P(x)$ is a smooth family of projections, valued in $M(N, \mathbb{C})$ for some N then the curvature of the connection defined on the bundle $E_P = \operatorname{Ran}(P)$ by $\nabla u = P(x)(du)$ for any section, is precisely $P(x)(dP(x))(dP(x))P(x)$.

Thus in fact, (19.9) shows that $\operatorname{Ch}^{\mathcal{H}}_{2k}$ represents the difference of the trace of the kth powers of the curvature of the two bundles. The standard definition of the Chern character is then

$$
(19.10)\qquad \operatorname{Ch}^{\mathcal{H}} = \sum_{k=0}^{\infty} \frac{1}{(2\pi i)^k k!} \operatorname{Ch}^{\mathcal{H}}_{2k} = \frac{1}{2} \operatorname{tr}\big(\exp(\frac{I(dI)^2 I}{4\pi i})\big).
$$

The $2\pi i$'s are included to make the first Chern class, here $\operatorname{Ch}^{\mathcal{H}}_2$, integral – this is the usual normalization for the curvature of a line bundle. It is sometimes omitted, but will then crop up somewhere else. This normalization corresponds to the multiplicativity.

For pairs of vector bundles, or 'superbundles', $V_+ \ominus V_-$, on a compact manifold the super tensor product is

$$
(19.11)\quad (V_+^{(1)} \ominus V_-^{(1)}) \otimes (V_+^{(2)} \ominus V_-^{(2)}) = \\
\Big((V_+^{(1)} \otimes V_+^{(2)}) \oplus (V_-^{(1)} \otimes V_-^{(2)})\Big) \ominus \Big((V_+^{(1)} \otimes V_1^{(2)}) \oplus (V_-^{(1)} \otimes V_+^{(2)})\Big).
$$

Proposition 18. *The universal Chern character (in deRham cohomology) pulls back to define a homomorphism of Abelian groups*

$$
(19.12)\qquad \operatorname{Ch}^{\mathcal{H}} : \mathrm{K}^0_c(X) \longrightarrow \mathrm{H}^{even}_c(X)
$$

which is multiplicative under the super tensor product

$$
(19.13)\qquad \operatorname{Ch}^{\mathcal{H}}([f][g]) = \operatorname{Ch}^{\mathcal{H}}([f]) \wedge \operatorname{Ch}^{\mathcal{H}}([g]).
$$

Proof. Compute after arranging that the projections all commute. □

We also want to understand the behaviour of the Chern character under the clutching and periodicity maps. To do the latter, for instance to see what happens under pull-back for (18.17) we need first consider the 'suspended' versions of the

Chern character. These are just obtained as case of the loop groups of $G^{-\infty}$ using the evaluation maps

$$\begin{gathered}\mathrm{ev}_p : \mathbb{R}^p \times \mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^k) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k),\\ \mathrm{Ch}^{\mathcal{H}(sus(p))}_{2k-p} = \int_{\mathbb{R}^p} \mathrm{ev}_p^* \mathrm{Ch}^{\mathcal{H}}_{2k},\ \mathrm{Ch}^{\mathcal{H}(\mathrm{sus}(p))} = \int_{\mathbb{R}^p} \mathrm{ev}_p^* \mathrm{Ch}^{\mathcal{H}},\end{gathered} \tag{19.14}$$

giving a $2k-p$ form, or sum of such, on $\mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^k)$. The pull-back of this form under a smooth map into $\mathcal{H}^{-\infty}_{\mathrm{sus}(p),\mathrm{iso}}(\mathbb{R}^k)$ is the same as interpreting this as a map from $\mathbb{R}^p \times X$ into $\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)$, pulling back $\mathrm{Ch}^{\mathcal{H}}_{2k}$ and then pushing forward to X, i.e. integrating over $\mathbb{R}^p$.

Thus, under the map (18.17) we can pull back the whole once-suspended Chern character to $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k)$.

Lemma 23. *The pull back of the once-suspended Chern character*

$$\begin{aligned}\mathrm{cl_{oe}}^* \mathrm{Ch}^{\mathcal{H}(\mathrm{sus})} &= \sum_k \frac{1}{2^{2k+2}\pi i k!} \int_{\mathbb{R}} \mathrm{tr}(I(dI(t))^{2k})\\ &= \frac{1}{2\pi i}\int_0^1 (g^{-1}dg)\exp\big(s(1-s)\frac{(g^{-1}dg)^2}{2\pi i}\big)ds.\end{aligned} \tag{19.15}$$

Proof. From (18.17),

$$\begin{aligned} dI(t) = &\\ &\begin{pmatrix} -\sin(\Theta(t)) & \cos(\Theta(t))g\\ \cos(\Theta(t))g^{-1} & \sin(\Theta(t))\end{pmatrix}\Theta'(t)dt\\ &\quad + \begin{pmatrix} 0 & \sin(\Theta(t))dg\\ -\sin(\Theta(t))g^{-1}(dg)g^{-1} & 0\end{pmatrix}, t \le 0,\end{aligned} \tag{19.16}$$

where we can ignore the part in $t > 0$ since it is only a function of t. Proceeding term by term, only one factor of dt can occur so there are $2k$ terms, depending on which factor of dt is selected. If dt is taken from the p slot then then there are $p-1$ factor of the second term in (19.16) before it and $2k-p-1$ after. These anticommute with I and using the trace identity and antisymmetry it follows that these terms are all the same. Thus, after taking the trace, we are reduced to computing

$$\begin{gathered}\frac{2k}{2^{2k+2}\pi i k!}\int_{-\infty}^0 \mathrm{tr}\left(I(t)\frac{dI}{dt}\Theta'(t)dt\right.\\ \left.\begin{pmatrix} 0 & -\sin^{2k-1}(\Theta(t))(dg(g^{-1})^{2k-2}dg)\\ \sin^{2k-1}(\Theta(t))g^{-1}(g^{-1}dg)^{2k-1}g^{-1} & 0\end{pmatrix}\right)\end{gathered} \tag{19.17}$$

Computing directly,

$$I(t)\frac{dI}{dt} = \begin{pmatrix} 0 & -g\\ g^{-1} & 0\end{pmatrix} \tag{19.18}$$

so again the two terms from (19.17) are equal and reduce to

$$\begin{aligned} &-\frac{1}{2^{2k+2}\pi i(k-1)!}\int_0^\pi \sin^{2k-1}(\Theta)d\Theta\,\mathrm{tr}((g^{-1}dg)^{2k-1}\\ &\qquad = \frac{1}{2^{2k+2}\pi i(k-1)!}\int_0^1 (1-t^2)^{k-1}dt\,\mathrm{tr}((g^{-1}dg)^{2k-1}\end{aligned} \tag{19.19}$$

Thus, the pull-back of the Chern character can be written

$$\int_0^1 (\frac{g^{-1}dg}{4\pi i}) \exp\big((1-t^2)(\frac{g^{-1}dg}{4\pi i})^2\big)dt. \tag{19.20}$$

Here the variable of integration has been changed to $\cos t$ and the integral divided into two equal parts, by symmetry. It is more conventional, to replace t by say $s=(1-t)/2$ and hence arrive at (19.15). □

To compute the pull-back of the odd Chern character under the clutching map (18.16) we can start by noting that the image $g(t)=\mathrm{cl}_{\mathrm{eo}}(I)$ is the product of two invertibles, $g(t)=\mathcal{U}_0(t)\mathcal{U}(t)$ where $\mathcal{U}_0$ does not depend on I – it is just there to make the leading part the identity. Moreover

$$\begin{gathered} g^{-1}(t)dg(t) = i\mathcal{U}^{-1}\sin(\Theta(t))dI + \mathcal{U}^{-1}\mathcal{U}_0^{-1}\frac{d\mathcal{U}_0}{dt}\mathcal{U} + \mathcal{U}^{-1}\frac{d\mathcal{U}}{dt} \\ i\mathcal{U}^{-1}\sin(\Theta(t))dI + i\mathcal{U}^{-1}\big(\cos(2\Theta(t)) - i\sin(2\Theta(t))\gamma_1\big)(I-\gamma)a\Theta'(t)dt. \end{gathered} \tag{19.21}$$

Expanding out $\operatorname{tr}((g^{-1}dg)^{2k+1})$ one again gets a contribution of one dt from each factor and by virtue of the trace identity these are all the same. Thus

$$\begin{aligned} &\operatorname{tr}((g^{-1}dg)^{2k+1}) \\ &= (2k+1)i^{2k}\sin^{2k}(\Theta(t))\operatorname{tr}\left(\big(\mathcal{U}^{-1}\mathcal{U}_0^{-1}\frac{d\mathcal{U}_0}{dt}\mathcal{U} + \mathcal{U}^{-1}\frac{d\mathcal{U}}{dt}\big)dt(\mathcal{U}^{-1}dI)^{2k}\right). \end{aligned} \tag{19.22}$$

Consider the last $2k$-fold wedge product as the k-fold product of

$$\mathcal{U}^{-1}dI\wedge\mathcal{U}^{-1}dI = dI\wedge dI \tag{19.23}$$

since $\mathcal{U}^{-1}dI=(dI)\mathcal{U}$. Using the trace identity to move the first factor of $\mathcal{U}^{-1}$ and the same identity again we arrive at:-

(19.24)

$$\begin{aligned} \operatorname{tr}((g^{-1}dg)^{2k+1}) &= (2k+1)i^{2k}\sin^{2k}(\Theta(t))\operatorname{tr}\left(\big(\mathcal{U}_0^{-1}\frac{d\mathcal{U}_0}{dt} + \mathcal{U}^{-1}\frac{d\mathcal{U}}{dt}\big)dt(dI)^{2k}\right) \\ &= -2(2k+1)i^{2k+1}\sin^{2k}(\Theta(t))\cos(\Theta(t))\Theta'(t)dt\operatorname{tr}\big(I(dI)^{2k}\big) + d_I\mathcal{T}_{2k}(I). \end{aligned}$$

(I think!) Here the exact form comes from the terms with no 'I' factor. It looks as though I have not made a good choice of normalization here, as regards the i's for a start.

Exercise 15. So, it remains to work out the constants here after integration, i.e.

$$C_{2k} = -2(2k+1)i^{2k+1}\int_0^\pi \sin^{2k}(\Theta)\cos(\Theta)d\Theta \tag{19.25}$$

and to insert these into the formula for the odd Chern character to see whether we do indeed recover the even Chern character!

Then look at the diagrams (18.18) and (18.19) and conclude what happens to the Chern character under the periodicity maps.

20. Lecture 17: Isotropic calculus and looping sequence Monday, 13 October

I am including here more detail (and may add even more later) than I will give in the lecture where I will assume some familiarity with pseudodifferential operators. In fact, in the lecture, I started at (20.27) and tried to explain the nature of the space in the middle that we need to construct – and then talked a little about the isotropic calculus (the case $\epsilon = 1$ of what follows) and the corresponding group.

Earlier, I worked out the product formula for semiclassical families of smoothing operators, in terms of their 'renormalized' kernels, $a_\epsilon(t,t') = \epsilon^{-n}F(\epsilon, \frac{1}{2}\epsilon(t+t'), (t-t')/\epsilon)$ where F is Schwartz on $\mathbb{R}^{2n}$ and smooth in ϵ. From (10.5) the product is

$$(20.1)\quad H(\epsilon,t,s) = \int_{\mathbb{R}^n} F(\epsilon, t+\frac{\epsilon^2}{2}(r+\frac{1}{2}s), \frac{1}{2}s - r)G(\epsilon, t+\frac{\epsilon^2}{2}(r-\frac{1}{2}s), r+\frac{1}{2}s)dr.$$

The 'full symbol', or Weyl form of this product is obtained by taking the Fourier transform in s and using the Fourier inversion formula:

$$(20.2)\quad \hat H(\epsilon,t,\tau) = (2\pi)^{-2n}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \hat F(\epsilon, t+\frac{\epsilon^2}{2}(r+\frac{1}{2}s),\tau_1)e^{i(\frac{1}{2}s-r)\tau_1} \\ \times \hat G(\epsilon, t+\frac{\epsilon^2}{2}(r-\frac{1}{2}s),\tau_2)e^{i(r+\frac{1}{2}s)\tau_2}e^{-i\tau s}drdsd\tau_1 d\tau_2.$$

Introducing $t_1 = t+\frac{\epsilon^2}{2}(r+\frac{1}{2}s)$ and $t_2 = t+\frac{\epsilon^2}{2}(r-\frac{1}{2}s)$ in place of r and s as variables of integration, so $r+\frac{1}{2}s = 2(t_1-t)/\epsilon^2$ and $r-\frac{1}{2}s = 2(t_2-t)/\epsilon^2$ and $drds = dt_1dt_2$, gives

$$(20.3)\quad \hat H(\epsilon,t,\tau) = (2\pi)^{-2n}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} \hat F(\epsilon,t_1,\tau_1)\hat G(\epsilon,t_2,\tau_2) \\ \times \exp\left(\frac{2i}{\epsilon^2}\big(\omega(t_1-t,\tau_1-\tau,t_2-t,\tau_2-\tau)\big)\right)dw_1dw_2, \\ \omega(t_1,t_2,\tau_1,\tau_2) = t_1\cdot\tau_2 - t_2\cdot\tau_1,\ dw = dtd\tau.$$

Here $\omega : \mathbb{R}^{2n}\times\mathbb{R}^{2n} \longrightarrow \mathbb{R}$ is the standard (antisymmetric) symplectic form on $\mathbb{R}^{2n}$ and dw is the corresponding (Lebesgue) volume form on $\mathbb{R}^{2n}$. In fact the formula makes sense for an arbitrary symplectic vector space, W, i.e. is invariant under the application of the same symplectic transformation in all three copies of $\mathbb{R}^{2n}$. Thus it can be written

$$(20.4)\quad h(\epsilon,w) = M(f,g)(\epsilon,w) = (2\pi)^{-\dim W}\int_W\int_W f(\epsilon,w_1)g(\epsilon,w_2) \\ \times\exp\left(\frac{2i}{\epsilon^2}\big(\omega(w_1-w,w_2-w)\big)\right)dw_1dw_2, \\ M : \mathcal{C}^\infty([0,1];\mathcal{S}(W))\times\mathcal{C}^\infty([0,1];\mathcal{S}(W)) \longrightarrow \mathcal{C}^\infty([0,1];\mathcal{S}(W)).$$

Consider various 'symbol spaces' associated to $\mathbb{R}^p$, and ultimately any vector space. First the Fréchet topologies on 'symbols with bounds' on $\mathbb{R}^p$, namely

$$(20.5)\qquad ||a||_{m,k} = \sup_{(t,\tau)\in\mathbb{R}^p, |\alpha|\le k} ||(1+|z|)^{-m+|\alpha|}|D_z^\alpha a|$$

is a sequence of norms. Denote by $S^m_\infty(\mathbb{R}^p)$ the subspace of $\mathcal{C}^\infty(\mathbb{R}^p)$ on which all these norms are bounded then

(1) For each m, $S^m_\infty(\mathbb{R}^p)$ is a Fréchet space, increasing with m. In particular these are complete metric spaces.
(2) $\mathcal{S}(\mathbb{R}^p)$ is dense in $S^m_\infty(\mathbb{R}^p)$ with respect to the topology of $S^{m'}_\infty(\mathbb{R}^p)$ for any $m' > m$.
(3) Pull back gives an action of $\mathrm{GL}(p,\mathbb{R})$ on these spaces, which therefore make sense on any finite dimensional vector space.
(4) Consider the quadratic compactification ${}^q\overline{\mathbb{R}^p}$ of $\mathbb{R}^p$, with quadratic boundary defining function ρ_q^2 (e.g. $\rho_q^2 = (|z|^2+1)^{-1}$. This is a compact manifold with boundary which is diffeomorphic to a ball and has interior canonically diffeomorphic to $\mathbb{R}^p$,

$$Q : \mathbb{R}^p \longrightarrow {}^q\overline{\mathbb{R}^p}. \tag{20.6}$$

It is defined to have the property

(20.7)
$$Q^*\mathcal{C}^\infty({}^q\overline{\mathbb{R}^p}) = \big\{u \in \mathcal{C}^\infty(\mathbb{R}^p);\ u = \tilde u(\frac{1}{|z|^2}, \frac{z}{|z|}) \text{ in } z \neq 0,\ \tilde u \in \mathcal{C}^\infty([0,\infty)\times\mathbb{S}^{p-1})\big\}.$$

The quadratic compactification is invariant under linear isomorphisms, i.e. the action of $\mathrm{GL}(p,\mathbb{R})$ on $\mathbb{R}^p$ extends to act as diffeomorphisms on ${}^q\overline{\mathbb{R}^p}$.
(5) Similarly the radial compactification, $\overline{\mathbb{R}^p}$, with boundary defining function ρ (e.g. $\rho = \rho_q$) is a compact manifold with boundary, again diffeomorphic to a ball, with compactifying map giving a commutative diagram of smooth maps

(20.8)
$$\begin{tikzcd} \mathbb{R}^p \arrow[rr, "R"] \arrow[dr, "Q"'] & & \overline{\mathbb{R}^p} \arrow[dl, "\beta"] \\ & {}^q\overline{\mathbb{R}^p} & \end{tikzcd}$$

with β a parabolic blow-down map for the boundary. The analogue of (20.7) is

(20.9)
$$R^*\mathcal{C}^\infty(\overline{\mathbb{R}^p}) = \{u \in \mathcal{C}^\infty(\mathbb{R}^p);\ u = u'(\frac{1}{|z|}, \frac{z}{|z|}) \text{ in } z \neq 0,\ u' \in \mathcal{C}^\infty([0,\infty)\times\mathbb{S}^{p-1})\}.$$

The radial compactification is again invariant under invertible linear transformations and in addition translations on $\mathbb{R}^p$ lift to be smooth on it.
(6) Then (with the pull-back maps suppressed)

$$\rho_q^{-m}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^p}) \subset \rho^{-m}\mathcal{C}^\infty(\overline{\mathbb{R}^p}) \subset S^m(\mathbb{R}^p) \tag{20.10}$$

are linearly invariant.
(7)

Theorem 5. *The bilinear form M defines continuous bilinear maps, consistent under the natural inclusions,*

$$\begin{gathered} M:\mathcal{C}^\infty([0,1];\rho_q^{-m}\mathcal{C}^\infty({}^q\overline{W}))\times\mathcal{C}^\infty([0,1];\rho_q^{-m'}\mathcal{C}^\infty({}^q\overline{W}))\\ \longrightarrow\mathcal{C}^\infty([0,1];\rho_q^{-m-m'}\mathcal{C}^\infty({}^q\overline{W}))\\ M:\mathcal{C}^\infty([0,1];\rho^{-m}\mathcal{C}^\infty(\overline{W}))\times\mathcal{C}^\infty([0,1];\rho^{-m'}\mathcal{C}^\infty(\overline{W}))\\ \longrightarrow\mathcal{C}^\infty([0,1];\rho^{-m-m'}\mathcal{C}^\infty(\overline{W}))\\ M:\mathcal{C}^\infty([0,1];S^m_\infty(W))\times\mathcal{C}^\infty([0,1];S^{m'}_\infty(W))\longrightarrow\mathcal{C}^\infty([0,1];S^{m+m'}_\infty(W)). \end{gathered} \tag{20.11}$$

Note that this 'consistency' is the reason for introducing the spaces S^m. The map in the last line is defined by density from $\mathcal{C}^\infty([0,1];\mathcal{S}(\mathbb{R}^{2n})$, when m and m' are both increased by $\epsilon >$. Then the map itself follows by restriction (and of course has to be shown to be continuous). Then the other two maps are by restriction – when f and g are in the appropriate space from (20.10) then so is $M(f,g)$ and it depends continuosly on them in the stronger topology.

Proof. Not included for the moment – ultimately it can be proved by some form of the lemma of stationary phase. Much more is proved in the paper of Hörmander [4]. There are other sources which are maybe a bit more accessible. □

(8) The corresponding associative filtered algebras will be denoted $\Psi^m_{\mathrm{qisy}}(W)$, $\Psi^m_{\mathrm{isy}}(W)$ and $\Psi^m_{\infty\text{-isy}}(W)$. Note that for a symplectic vector space these are not naturally algebras of operators, just algebras. However, in the case of $W=\mathbb{R}^{2n}$ they are all operators on $\mathcal{S}(\mathbb{R}^n)$ and then $\Psi^m_{\mathrm{qiso}}(\mathbb{R}^n)=\Psi^m_{\mathrm{qisy}}(\mathbb{R}^{2n})$ etc. Moreover the action on $\mathcal{S}(\mathbb{R}^n)$ extends and restricts, much as for the product itself to give

$$\begin{gathered} \Psi^m_{\mathrm{qiso}}(\mathbb{R}^n)\times\rho_q^{-k}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^n})\longrightarrow\rho_q^{-k-m}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^n}),\\ \Psi^m_{\mathrm{iso}}(\mathbb{R}^n)\times\rho^{-k}\mathcal{C}^\infty(\overline{\mathbb{R}^n})\longrightarrow\rho^{-k-m}\mathcal{C}^\infty(\overline{\mathbb{R}^n}),\\ \Psi^m_{\infty\text{-iso}}(\mathbb{R}^n)\times S^k_\infty(\mathbb{R}^n)\longrightarrow S^{k+m}_\infty(\mathbb{R}^n). \end{gathered} \tag{20.12}$$

(9) The various algebras of isotropic pseudodifferential operators are what we get by setting $\epsilon=1$ (or up to invertible linear change of variable, any other $\epsilon>0$). The 'classical' space of isotropic pseudodifferential operators have a leading symbol map

$$\sigma_{m,\mathrm{iso}}:\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)\longrightarrow\mathcal{C}^\infty(\mathbb{S}^{2n-1};(d\rho)^{-m}) \tag{20.13}$$

which should be thought of as a section of a certain trivial line bundle over the sphere at infinity – namely the products $\rho^{-m}f$, $f\in\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}}$ modulo the $\rho^{-m+1}f$). This symbol is multiplicative in the obvious sense

$$\sigma_{m+m',\mathrm{iso}}(AB)=\sigma_{m,\mathrm{iso}}(A)\sigma_{m',\mathrm{iso}}(B),\ A\in\Psi^m_{\mathrm{iso}}(\mathbb{R}^n),B\in\Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \tag{20.14}$$

and gives a short exact sequence

$$\Psi^{m-1}_{\mathrm{iso}}(\mathbb{R}^n)\hookrightarrow\Psi^m_{\mathrm{iso}}(\mathbb{R}^n)\xrightarrow{\sigma_{m,\mathrm{iso}}}\mathcal{C}^\infty(\mathbb{S}^{2n-1};(d\rho)^{-m}). \tag{20.15}$$

(10) For the quadratic isotropic algebra we get the same thing, with an improved 'error estimate'

$$\Psi^{m-2}_{\mathrm{qiso}}(\mathbb{R}^n) \hookrightarrow \Psi^m_{\mathrm{qiso}}(\mathbb{R}^n) \xrightarrow{\sigma_{m,\mathrm{qiso}}} \mathcal{C}^\infty(\mathbb{S}^{2n-1}; (d\rho_q)^{-m}). \tag{20.16}$$

For the corresponding algebras on a symplectic vector space the same properties hold, now with $\mathbb{R}^{2n}$ replaced by W and the sphere $\mathbb{S}^{2n-1}$ replace by the 'sphere of W' which is $\mathbb{S}W = (W \setminus 0)/\mathbb{R}^+$, so for instance (20.16) becomes the exact sequence

$$\Psi^{m-2}_{\mathrm{qisy}}(W) \hookrightarrow \Psi^m_{\mathrm{qisy}}(W) \xrightarrow{\sigma_{m,\mathrm{qiso}}} \mathcal{C}^\infty(\mathbb{S}W; (d\rho_q)^{-m}). \tag{20.17}$$

(11) The adjoint (or transpose) is an involution on each of the algebras described above and it follows by duality that $\Psi^m_{\infty\text{-iso}}(\mathbb{R}^n)$ acts on $\mathcal{S}'(\mathbb{R}^n)$.

(12) We are mostly interested below in the algebras of operators of order 0. For these the symbol can be recovered in part by noting that

$$\Psi^0_{\mathrm{iso}}(\mathbb{R}^n) : \mathcal{C}^\infty(\overline{\mathbb{R}^n}) \longrightarrow \mathcal{C}^\infty(\overline{\mathbb{R}^n}) \tag{20.18}$$

where $\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \subset \mathcal{S}'(\mathbb{R}^n)$. Then restriction to the sphere at infinity gives

$$u \in \mathcal{C}^\infty(\overline{\mathbb{R}^n}),\ A \in \Psi^0_{\mathrm{iso}}(\mathbb{R}^n),\ (Au)\big|_{\mathbb{S}^{n-1}} = \sigma_{\mathrm{iso},0}(A)\big|_{[(\mathbb{R}^n,0)]} u\big|_{\mathbb{S}^{n-1}} \tag{20.19}$$

which allows the symbol on the equatorial sphere, $\tau = 0$, to be recovered. To get the symbol at all other points of the sphere, except at the 'vertical subsphere' $t = 0$, one can take a real quadratic (homogeneous) polynomial q. Then $q(t) - q(t') = (t - t') \cdot L(\frac{t+t'}{2})$ where L is a linear map. There is a mapping property extending (20.18):

$$\Psi^0_{\mathrm{iso}}(\mathbb{R}^n) : e^{iq}\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \longrightarrow e^{iq}\mathcal{C}^\infty(\overline{\mathbb{R}^n}) \tag{20.20}$$

and then as in (20.19)

$$(e^{-iq}Ae^{iq}u)\big|_{\mathbb{S}^{n-1}} = \sigma_{\mathrm{iso},0}(A)\big|_{[(\mathbb{R}^n,L\mathbb{R}^n)]} u\big|_{\mathbb{S}^{n-1}}\ \forall\ u \in \mathcal{C}^\infty(\overline{\mathbb{R}^n}). \tag{20.21}$$

From this one can recover the symbol everywhere on the sphere at infinity by continuity.

(13) The Fourier transform is also an isomorphism on the space of isotropic operators, thus

$$\begin{gathered} A_{\mathcal{F}}\hat{v} = \hat{f} \text{ if } Av = f,\ v \in \mathcal{S}(\mathbb{R}^n),\ A \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n) \\ \Longrightarrow A_{\mathcal{F}} \in \Psi^m_{\mathrm{iso}}(\mathbb{R}^n),\ \sigma_{m,\mathrm{iso}}(A_{\mathcal{F}}(t,\tau) = \sigma_{m,\mathrm{iso}}(A)(\tau, -t). \end{gathered} \tag{20.22}$$

(14) For the full semiclassical (and 'classical') there is both a conventional symbol as described above and a semiclassical symbol reduced to the previous case for smoothing operators – I will discuss these more later.

(15) There is yet another generalization of the isotropic algebras that we need to consider. Namely we want to allow them to 'take values in (isotropic) smoothing operators. This is not so hard. I will denote the corresponding algebras of operators in the form $\Psi^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R}^n;\mathbb{R}^p)$. These are smoothing in the last variables. The kernels can be thought of as just Schwartz maps

$$k \in \mathcal{S}(\mathbb{R}^{2p}; \Psi^0_{\mathrm{qiso}}(\mathbb{R}^n). \tag{20.23}$$

The composition is then given by composing in the isotropic algebra and then in the usual way as smoothing operators

$$k \circ k'(\bullet; z, z') = \int_{\mathbb{R}^p} k(\bullet; z, z'')k'(\bullet; z'', z')dz''. \tag{20.24}$$

Make sure you have a picture of how these isotropic operators, especially the ones of order zero 'work'. For the moment look at (20.24) and take $n = 1$, and for the picture $p = 1$. Then the kernels can be consider as distributions on $\mathbb{R}^2 \times \mathbb{R}^2$ where everything is Schwartz in the last two variables. Recall that we are considering the partial Fourier transform of the Schwartz kernels, so $k = k(t, \tau; z, z')$ where the product is given by (20.4) (or (20.3)) in the t, τ variables with $\epsilon = 1$. So the function k is $\mathcal{C}^\infty$ on the product of two disks and vanishes to infinite order at the boundary of the second (with the z, z' variables).

Picture: Product of two disks.

The operator product takes two such functions and composes then – the composition in the second disk(s) is usual composition of Schwartz-smoothing operators. The composition in the first disk(s) is really the same, but we have taken a partial Fourier transform of everything and then this same product extends to $\mathcal{C}^\infty$ functions up to the boundary. Both parts of the product are non-commutative of course, but at a point approaching the boundary in the first fact the product becomes more and more commutative and, as I will discuss later, the Taylor series at the boundary of the product only depends on the Taylor series of the factors. So the principal symbol – on in t, τ – is a function on the circle with values in the smoothing operators on $\mathbb{R}$ (or just as well $\mathbb{R}^p$) and composes as loops:-

$$\begin{gathered}\sigma_{0,\mathrm{qiso}}(AB) = \sigma_{0,\mathrm{qiso}}(A)\sigma_{0,\mathrm{qiso}}(B) \\ \text{in } \mathcal{C}^\infty(\mathbb{S}; \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p), \text{ for } A, B \in \Psi^{0;-\infty}_{\mathrm{qiso}}(\mathbb{R}; \mathbb{R}^p).\end{gathered} \tag{20.25}$$

So, let me identify the looping, or quantization sequence in terms of these algebras. This involves three groups, two of which we are already familiar with:-

$$G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}) \longrightarrow \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R}; \mathbb{R}^p) \xrightarrow{\sigma_{0,\mathrm{iso}}} G^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^p). \tag{20.26}$$

In this form it is *not quite* exact. What precisely is the central group? It is made up from (20.25). First consider the subalgebra of $\Psi^{0;-\infty}_{\mathrm{qiso}}(\mathbb{R}; \mathbb{R}^p)$ obtained by denanding that the (partially-Fourier-transformed) kernel $k \in \mathcal{C}^\infty({}^q\overline{\mathbb{R}^2} \times \overline{\mathbb{R}^2})$ – which by assumption vanishes to infinite order at the boundary in the second variable – also vanishes to infinite order at one point, $N \in {}^q\overline{\mathbb{R}^2}$, on the boundary of the first disk, say the North Pole (i.e. nothing interesting happens at the North Pole). As I say, this is a subalgebra because of the Taylor-series-locality at the boundary. Then the group $\dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R}; \mathbb{R}^p)$ is the operators (on $\mathcal{S}(\mathbb{R}^{1+p})$ or instance) of the form $\mathrm{Id} + A$ with A of this form and invertible, with inverse of the same form.

The first map in (20.26) is then inclusion. The Schwartz-smoothing operators correspond to those kernels (before and after Fourier transform) which vanish to infinite order at the whole boundary of the first disk as well as the second. The second map is just the principal symbol – given by the restriction to the boundary in the first variable (but not in the second set of variables). The identity appears here either formally, or as it turns out corresponding to the function with is constant in the first variable (if you like 1 from the Fourier transform of a delta function) and actually the identity, i.e. $\delta(z - z')$, in the second variable. Anyway, it is just the identity in the second variables. Thus the image of an element $\operatorname{Id} + A$ of the central group is $\operatorname{Id} + s_{0,\mathrm{iso}}(A)$ which is the identity plus a function on the circle, flat at N, with values in the Schwartz-smoothing operators on $\mathbb{R}^p$ and as such invertible! This gives the sequence (20.26).

Now, why go to all the gymnastics of the flatness at N? Well, otherwise we would not get the loop group out on the right for one thing. More seriously

Theorem 6. *The central group in* (20.26) *is weakly contractible and the range is precisley the component of the identity in the loop group.*

Even when we adjust the targent group the resulting sequence is not exact, but only for a silly reason. Namely we have not taken into account the higher terms in the Taylor series at the boundary. When we do this we get the *looping sequence* which is an exact sequence of groups:-

$$\{\mathrm{Id}\} \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}) \longrightarrow \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p) \xrightarrow{\sigma_{0,\mathrm{iso}}} G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso}}(\mathbb{R}^p) \xrightarrow{\mathrm{ind}} \mathbb{Z} \longrightarrow \{0\}. \tag{20.27}$$

Here $G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso}}(\mathbb{R}^p)$ is a 'star product extension' or formal quantization of the original group $G^{-\infty}_{\mathrm{sus},\mathrm{iso}}(\mathbb{R}^p)$. Namely it consists of formal power series in a formal variable ρ (which can be identified with the defining function for the boundary of ${}^q\overline{\mathbb{R}^2}$) where the leading term is an element of the suspended group:-

$$G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso}}(\mathbb{R}^p) \ni a = \sum_{j\ge 0}\rho^j a_j,\ a_0 \in G^{-\infty}_{\mathrm{sus},\mathrm{iso}}(\mathbb{R}^p),\ a_j \in \Psi^{-\infty}_{\mathrm{sus},\mathrm{iso}}(\mathbb{R}^p),\ j \ge 1 \tag{20.28}$$

and the product is given by differential operators – more about this later! However, it is important to note that invertibility of such a formal power series is just invertibility of the leading term and the lower order terms are just 'affine junk' from a topoligical point of view – they can be deformed away. However, as we shall see, from an analytic viewpoint they turn out to be important.

21. Lecture 18: The determinant bundle Wednesday, 15 October

Even though I have not carefully explained everything that goes into the looping sequence let me proceed to use it to define the determinant bundle – and then see what more we need to do. Here I will proceed in a way that is closely parallel to the content of Section 7. There I used the delooping sequence to (re-)construct the determinant on $G^{-\infty}$. Let me recall that construction now tying in some of the things that have come up in the meantime. The delooping sequence is

$$G^{-\infty}_{\mathrm{sus}}(\mathbb{R}^k) \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{R}^k) \xrightarrow{R} G^{-\infty}(\mathbb{R}^k). \tag{21.1}$$

The clutching map in (18.16) has the property

$$\mathrm{cl_{eo}}^*\left(\frac{1}{2\pi i}\int_{\mathbb{R}}\mathrm{tr}(g^{-1}\frac{dg}{dt})dt\right) = \frac{1}{2}\,\mathrm{tr}(I-\gamma_1) = \mathrm{ind}\,. \tag{21.2}$$

To see this, just compute away:-

$$\begin{aligned}&\mathrm{cl_{eo}}^*\left(\int_{\mathbb{R}}\mathrm{tr}(g^{-1}\frac{dg}{dt})dt\right)\\ &= \int_0^\pi \mathrm{tr}((\cos\Theta - i\sin\Theta I)(-\sin\Theta + i\cos\Theta I)\\ &\qquad - (\cos\Theta - i\sin\Theta\gamma_1)(-\sin\Theta + i\cos\Theta\gamma_1)d\Theta\\ &= i\int_0^\pi \mathrm{tr}(I-\gamma_1)d\Theta = (2\pi i)\,\mathrm{ind}\end{aligned} \tag{21.3}$$

where all the non-trace class terms consisently cancel out. Hence we can say that this is the same 'index functional' on $G^{-\infty}_{\mathrm{sus}}(\mathbb{R}^k)$ as is represented by ind on $\mathcal{H}^{-\infty}(\mathbb{R}^k)$. This will be more fully justified by Fedosov's index theorem a bit later.

So we have defined the first vertical map in

$$\begin{tikzcd} G^{-\infty}_{\mathrm{sus}}(\mathbb{R}^k) \arrow[r] \arrow[d, "\mathrm{ind}"] & \tilde{G}^{-\infty}_{\mathrm{sus}}(\mathbb{R}^k) \arrow[r, "R"] \arrow[d, "\tilde{\eta}"] & G^{-\infty}(\mathbb{R}^k) \arrow[d, "\det"] \\ \mathbb{Z} \arrow[r] & \mathbb{C} \arrow[r, "\exp(2\pi i\cdot)"] & \mathbb{C}^* \end{tikzcd} \tag{21.4}$$

and – we know that it maps into $\mathbb{Z}$. The second vertical map, $\tilde{\eta}$ we defined by *regularization* – or extension – of the index functional. Namely we just used the 'same' definition but now on the 'one-end-open' loops

$$\tilde{\eta}(\tilde{g}) = \frac{1}{2\pi i}\int_{\mathbb{R}}\mathrm{tr}(\tilde{g}^{-1}(t)\frac{d\tilde{g}(t)}{dt})dt. \tag{21.5}$$

This still makes sense since $d\tilde{g}(t)/dt \in \mathcal{S}(\mathbb{R};\Psi^{-\infty}(\mathbb{R}^k))$ because of the flatness condition on these half-open loops. Now, we can no longer see that this extended functional takes integral values – indeed it doesn't – but we checked directly that $\exp(2\pi i\tilde{\eta})$ descends to the quotient and there it satisfies all the properties we want of the determinant, and reduces to it under finite rank approximation. In particular one crucial thing is that $\tilde{\eta}$ is a log-character on the central group

$$\tilde{\eta}(\tilde{g}\tilde{h}) = \tilde{\eta}(\tilde{g}) + \tilde{\eta}(\tilde{h}). \tag{21.6}$$

Exercise 16. (For the enterprising) Show that the sequence obtained as the kernels of the maps in (21.4):
(21.7)

$$\{\mathrm{Id}\} \longrightarrow G^{-\infty}_{\mathrm{sus,ind}=0}(\mathbb{R}^k) \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus},\tilde{\eta}=0}(\mathbb{R}^k) \xrightarrow{R} G^{-\infty}_{\mathrm{iso,det}=1}(\mathbb{R}^k) \longrightarrow \{\mathrm{Id}\}$$

is a reduced classifying sequence for K-theory – meaning it is exact, that the central group is weakly contractible and that the outer groups each just have the bottom homotopy group removed.

So, we want to do the 'same thing' but one step up in complexity. Be warned, I am planning to do the next step up too! Now we start with the determinant at the beginning of the looping sequence –

$$\begin{tikzcd} \mathbb{C}^* & \mathbb{C} \arrow[d] & \\ G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}) \arrow[u, "\det"] \arrow[r] & \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p) \arrow[r, "\sigma_{0,\mathrm{iso}\,*}"] & G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}(\mathbb{R}^p) \end{tikzcd} \tag{21.8}$$

Here I have added a $\mathbb{C}$ and map to the central group and the final group is supposed to be the image of the 'full' symbol map (which now includes the whole Taylor series at the boundary, hidden in the $\tilde{\sigma}$). The $\mathbb{C}$ is supposed to represent the trivial line bundle, i.e. the top space is really $\mathbb{C} \times \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p)$ but this is a bit messy to write out; $\mathbb{C}$ is of course the fibre so it stands here for the trivial line bundle. The determinant induces a relation on this space

$$\mathbb{C} \times \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p) \ni (z, A) \sim_{\det} (z\det(g), Ag) \in \mathbb{C} \times \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p) \quad \text{if } g \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}). \tag{21.9}$$

This relation is multiplicative, because the determinant is:-
(21.10)

$$(z, A) \sim_{\det} (z\det(g), Ag) \sim_{\det} ((z\det(g))\det(h), (Ag)h) = (z\det(gh), A(gh)).$$

The identifying maps in (21.9) are linear in z so the quotient is a one-dimensional complex vector space for each $\alpha \in G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}(\mathbb{R}^p)$:

$$\mathcal{L}_\alpha = \mathbb{C} \times AG^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p})/\sim_{\det} \ \text{ if } \sigma_{0,\mathrm{iso}\,*}(A) = \alpha \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}) \tag{21.11}$$

which is independent of the choice of A mapping to α. Note that the exactness of the diagram (21.8) means that the inverse image of any point $\alpha \in G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}$ (accepting for the moment that this *is* the image) is of the form $AG^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p})$ for any particular A in the preimage.

This leads to the full diagram

$$\begin{tikzcd} \mathbb{C}^* & \mathbb{C} \arrow[d] \arrow[r, "\sim_{\det}"] & \mathcal{L} \arrow[d] \\ G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}) \arrow[u, "\det"] \arrow[r] & \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R};\mathbb{R}^p) \arrow[r, "\sigma_{0,\mathrm{iso}\,*}"] & G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}(\mathbb{R}^p) \end{tikzcd} \tag{21.12}$$

We need to be a little careful of the sense in which this is a line bundle, because local triviality isn't so clear. However, except for infinite-dimensional effects this is the construction of the vector bundle associated to a representation (the determinant) of the structure bundle of a principal bundle. To discuss local triviality we need to consider the Fréchet topology on the base, etc. However let me state it as a result before we press on to prove this, and more.

Claim 1. *The determinant function on the structure group in* (21.8) *induces the (locally trivial)* determinant line bundle *over the quotient group.*

I hope the relation of this line bundle to Quillen's original definition from [10] will become abundantly clear as we proceed; for now it may seem rather distant. In fact what we are constructing here is a *universal determinant bundle*. One thing I want to come back to and refine is the following

Claim 2. *The determinant bundle, $\mathcal{L}$, over $G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso},\mathrm{ind}=0}(\mathbb{R}^p)$ is* universal *for smooth (complex line) bundles over smooth compact manifolds – i.e. any such smooth line bundle is isomorphic to the pull-back of $\mathcal{L}$ under a smooth map into $G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso},\mathrm{ind}=0}(\mathbb{R}^p)$ (hence defining an even K-class). This much is fairly easy. In fact the same is true for a bundle with connection, it is isomorhic with its connection, to the pull-back of $\mathcal{L}$ with the connection constructed below.*

Note the notion of a *Claim* here is that I believe it to be true but probably do not have a complete proof at hand – so there is always a danger it is not quite right!

Exercise 17. There is a similar construction to this one over the classifying space $\mathcal{H}^{-\infty}(\mathbb{R}^k)$ due I believe to Graeme Segal. This can be found in [9], in the context of trace class operators and groups. In the smooth case we can proceed as follows, and you might like to fill in the details. We have shown that in terms of the action by conjugation on the zero index involutions

$$\mathcal{H}^{-\infty}_{\mathrm{ind}=0}(\mathbb{R}^k) = G^{-\infty}(\mathbb{R}^k;\mathbb{C}^2)/\big(G^{-\infty}(\mathbb{R}^k)\oplus G^{-\infty}(\mathbb{R}^k)\big) \tag{21.13}$$

where the two smaller groups are acting on the first and second components diagonally. So, take the fibre at an involution to be

$$\begin{gathered}\mathcal{L}_I = \big(\mathbb{C}\times\{G\in G^{-\infty}(\mathbb{R}^k;\mathbb{C}^2); I = G^{-1}\gamma_1 G\}\big)\,/\sim_{\det},\\ (z,G)\sim_{\det}\big(\frac{\det(g_1)}{\det(g_2)}z,(g_1\oplus g_2)G\big)\ \forall\ g_i\in G^{-\infty}(\mathbb{R}^k),\ i=1,2.\end{gathered} \tag{21.14}$$

Check that this is a line bundle – linearity and local triviality. Note that if we took the product of the determinants instead of the quotient it we would produce a trivial line bundle (since the determinants are consistent with that on the big group). You could even try to see that the pull-back of the determinant line bundle over the index zero component of $G^{-\infty}_{\mathrm{sus},\mathrm{iso}}(\mathbb{R}^k;\mathbb{C}^2)$ under $\mathrm{cl}_{\mathrm{eo}}$ in (18.16) is isomorphic to the bundle from (21.14) – it is. Report back any success here, since I haven't done this explicitly myself myself (yet). Note that there is a subtlety, since the line bundle above is defined on the $*$ extension of the image group, so you need to do something about that first for this last part. You can see what to do about this, at least in part, from the discussion below.

So, apart from proving the claim above, I want to do a little more – and this is where the $*$ part of the quotient group starts to come into its own. The analogy between the treatment of the determinant via the delooping sequence and the determinant bundle might seem somewhat forced. To make it more apparent that they really are closely related, consider the problem of construction a connection on $\mathcal{L}$ in (21.12). That is, we want to know how to differentiate sections. One can construct a connection using local trivializations, but here we can do it directly because of the quotient construction of the bundle itself. Namely all we need is a connection on the trivial bundle over the 'big group' which descends to a connection on the quotient. Since the trivial bundle is, ahem, trivial, any connection on it is the sum of the trivial connection, d, and a 1-form:

$$\nabla = d - \gamma,\ \gamma \in \mathcal{C}^\infty(\dot{G}^{0,-\infty}_{\text{qiso}}(\mathbb{R};\mathbb{R}^p);\Lambda^1). \tag{21.15}$$

So what do we need for this connection to 'descend' to a connection on $\mathcal{L}$ over $G^{-\infty}_{\text{sus}\,*,\text{iso},\text{ind}=0}(\mathbb{R}^p)$? There are outstanding issues of local triviality etc., but basically we need to know that when applied to the lift of a local section of $\mathcal{L}$ to the trivial bundle over $\dot{G}^{0,-\infty}_{\text{qiso}}(\mathbb{R};\mathbb{R}^p)$ we get the lift of a section. A lifted section, which is just a complex-valued function, must have the transformation law along a fibre of $\dot{G}^{0,-\infty}_{\text{qiso}}(\mathbb{R};\mathbb{R}^p)$ given by

$$e(Ag) = \det(g)e(A)\ \forall\ g \in G^{-\infty}_{\text{iso}}(\mathbb{R}^{1+p}) \tag{21.16}$$

so what we need is that

$$\det(g)^{-1}d\det(g) - \gamma(Ag) = 0 \text{ on } AG^{-\infty}_{\text{iso}}(\mathbb{R}^{1+p}) \tag{21.17}$$

for all A. So, how do we construct such a γ? Basically, (21.17) just says that restricted to a fibre,

$$\gamma = d\log\det = \operatorname{tr}(g^{-1}dg). \tag{21.18}$$

So the 'obvious' thing to do is proceed as we did for $\tilde{\eta}$, somehow regularized the formula on the right to get a 1-form on the central group. This we will do as follows:-

Proposition 19. *The trace functional on* $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^p\times\mathbb{R}^k)$ *has an extension to a continuous linear functional*

$$\overline{\operatorname{Tr}} : \Psi^{0,-\infty}_{\text{iso}}(\mathbb{R}^p : \mathbb{R}^k) \longrightarrow \mathbb{C} \tag{21.19}$$

by Hadamard regularization of the integral and

$$\gamma = -\overline{\operatorname{Tr}}(\tilde{g}^{-1}d\tilde{g}) \in \mathcal{C}^\infty(G^{0,-\infty}_{\text{qiso}}(\mathbb{R};\mathbb{R}^p);\Lambda^1)) \tag{21.20}$$

gives a connection on the trivial bundle through (21.15) *which descends to* $\mathcal{L}$; *the curvature of this line bundle is the 2-form part of the Chern character.*

Question 3. What does the line bundle $\mathcal{L}$ represent – why did Quillen call it the determinant line bundle?

Answer 3. The determinant line at any point consists of all the *possible*, or perhaps one should say *reasonable*, values of the determinant for the operator (or object) in question. If the determinant bundle were trivial then it would be possible to give a global definition of the determinant; if not (which is the case on the whole space) then not. One thing I hope to do in the sequel is to find a big subgroup on which the determinant bundle *is* trivial – although at this stage I am still not quite convinced that it exists in a useful form.

22. Lecture 19: Riesz regularization Friday, 17 October

Today, let me go back and fill in some of the gaps, or perhaps just paper over some of the cracks.

First let me say a little more about symbols. I will probably not go through all of this in the lectures but it may help clarify things a bit to separate the symbol spaces from $\mathbb{R}^n$. (Or it may not, depending on your tendencies!) From our point of view symbols are the same things as 'conormal functions at a boundary'. Suppose we have a compact manifold with boundary M; in the case at hand this is a ball – say $\overline{\mathbb{R}^p}$ or ${}^q\overline{\mathbb{R}^p}$. Such a manifold comes equipped with a space of smooth functions $\mathcal{C}^\infty(M)$ with its Fréchet topology of the uniform norm on derivatives over compact subsets of coordinate neighbourhoods. Since it is a manifold with boundary there is a filtration by ideals which vanish to higher and higher order at the boundary. In particular there is always a boundary defining function $0 \le \rho \in \mathcal{C}^\infty(M)$, $\partial M = \{\rho = 0\}$, $d\rho \neq 0$ on ∂M. Then the boundary ideal is

$$\mathcal{J}(\partial M) = \rho\mathcal{C}^\infty(M) = \{u \in \mathcal{C}^\infty(M); u\big|_{\partial M} = 0\}. \tag{22.1}$$

The successive ideals are the powers, in the sense of finite spans of products

$$\begin{gathered}\mathcal{C}^\infty(M) \supset \mathcal{J}(\partial M) \supset \mathcal{J}^2(\partial M) \supset \cdots \supset \mathcal{J}^k(M),\\ \dot{\mathcal{C}}^\infty(M) = \cap_k \mathcal{J}^k(M).\end{gathered} \tag{22.2}$$

We can also add the negative powers, or 'Laurent functions' at the boundary and then think of

$$\rho^{-k}\mathcal{C}^\infty(M) = \{u \in \mathcal{C}^\infty(M \setminus \partial M); \rho^k u \in \mathcal{C}^\infty(M)\} \tag{22.3}$$

as the classical (perhaps more correctly '1-step-classical') symbols of order k, so elements of $\mathcal{J}^k(\partial M)$ are classical symbols of order $-k$, rather perversely.

Now, we can interpolate with the classical symbols of any complex order by taking complex powers of ρ, $\rho^z = \exp(z\log\rho) \in \mathcal{C}^\infty(M \setminus \partial M)$ and then define

$$\rho^z\mathcal{C}^\infty(M) = \{u \in \mathcal{C}^\infty(M \setminus \partial M); \rho^{-z}u \in \mathcal{C}^\infty(M)\}. \tag{22.4}$$

These are the classical symbols of complex order z. Notice that the only inclusions we have are

$$\rho^{z+k}\mathcal{C}^\infty(M) \subset \rho^z\mathcal{C}^\infty(M)\ \forall\ k \in \mathbb{N}_0. \tag{22.5}$$

The topology on $\rho^z\mathcal{C}^\infty(M)$ is the topology of $\mathcal{C}^\infty(M)$ after division by ρ^z. The common subspace of all these spaces is the 'Schwartz space' of smooth functions vanishing to infinte order at the boundary:-

$$\dot{\mathcal{C}}^\infty(M) \subset \rho^z\mathcal{C}^\infty(M) \text{ is a closed subspace.} \tag{22.6}$$

So there is no hope of it being dense!

To arrange density we introduce the 'symbol (or conormal) spaces with bounds'. Let $\mathcal{V}_{\mathrm{b}}(M)$ be the space of smooth vector fields on M which are tangent to the boundary – so in local coordinates $x = \rho$, $y_1, \dots, y_{n-1}$, $\dim M = n+1$, near the boundary, $\mathcal{V}_{\mathrm{b}}(M)$ is spanned over $\mathcal{C}^\infty$ coefficients by $x\partial_x$, ∂_{y_j}. Then set

$$\mathcal{A}^s(M) = \{u \in \mathcal{C}^\infty(M); V_1 \dots V_k\rho^{-s}u \in L^\infty(M)\ \forall\ V_i \in \mathcal{V}_{\mathrm{b}}(M),\ \forall\ k\}. \tag{22.7}$$

This again is a Fréchet space with supremum norms – $\mathcal{V}_b(M)$ is finitely generated as a module over $\mathcal{C}^\infty(M)$ so there really are only countably many conditions here. In fact

$$\times\rho^t : \mathcal{A}^s(M) \longrightarrow \mathcal{A}^{s+t} \text{ is an isomorphism } \forall\ t, s \in \mathbb{R}. \tag{22.8}$$

Note that $\rho^z \in \mathcal{A}^{\operatorname{Re} z}(M)$ so there is only a real, not a complex, order here and

$$\rho^z\mathcal{C}^\infty(M) \subset \mathcal{A}^s(M),\ \ s \le \operatorname{Re}(z). \tag{22.9}$$

So, now the density result is easy enough. Take a smooth function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ with $\psi = 1$ in $x < \frac{1}{2}$, $\psi = 0$ in $x > 1$ and consider $\psi_\epsilon = (1-\chi)(\rho/\epsilon) \in \mathcal{C}^\infty(M)$ for $\epsilon > 0$. This vanishes for $\rho < \frac{1}{2}\epsilon$ and is eventually equal to 1 on any compact subset of the interior of M as $\epsilon \downarrow 0$. Then

$$u \in \mathcal{A}^s(M) \Longrightarrow \psi_\epsilon u \to u \text{ in } \mathcal{A}^{s'}(M),\ \ s' < s. \tag{22.10}$$

In particular $\dot{\mathcal{C}}^\infty(M)$ is dense in $\rho^z\mathcal{C}^\infty(M)$ in the topology of $\mathcal{A}^s(M)$ for any $s < \operatorname{Re}(z)$.

The algebra of isotropic pseudodifferential operators discussed above is a non-commutative product on the filtration of Fréchet spaces $\mathcal{A}^s({}^q\overline{\mathbb{R}^{2n}})$ for any n that is, it is a consistent associative product

$$\mathcal{A}^s({}^q\overline{\mathbb{R}^{2n}}) \times \mathcal{A}^t({}^q\overline{\mathbb{R}^{2n}}) \longrightarrow \mathcal{A}^{s+t}({}^q\overline{\mathbb{R}^{2n}})\ \forall\ t, s \in \mathbb{R} \tag{22.11}$$

which restricts to define products

$$\rho^z\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2n}}) \times \rho^{z'}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2n}}) \longrightarrow \rho^{z+z'}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2n}})\ \forall\ z, z' \in \mathbb{C}. \tag{22.12}$$

You might ask: How can one characterize $\rho^z\mathcal{C}^\infty(M)$ inside $\mathcal{A}^s(M)$, which contains it for $s < \operatorname{Re}(z)$?

Proposition 20. *For a compact manifold with boundary there exists a vector field $R \in \mathcal{V}_b(M)$ such that $Rf \equiv f$ modulo $\mathcal{J}^2(M)$ for all $f \in \mathcal{J}(M)$ and then for any $s < \operatorname{Re}(z)$,*

$$\begin{gathered}\rho^z\mathcal{C}^\infty(M) = \\ \{u \in \mathcal{A}^s(M); (R-z)(R-z-1)(R-z-2)\dots(R-z-k)u \in \mathcal{A}^{s+k}(M)\ \forall\ k \in \mathbb{N}_0\}\end{gathered} \tag{22.13}$$

Proof. Is not very hard! □

Following the line of thought related to ρ^z for a boundary defining function ρ, I will next consider Riesz-regularized integrals over M – which is a compact manifold with boundary. Suppose $0 < \nu \in \mathcal{C}^\infty(M;\Omega)$ is a smooth density. In case you don't know about densities I will add some exercises so that you can familiarize yourself with them. For the moment just agree that they are objects which in any local coordinates give a smooth (positive) multiple of the Lebesgue measure in coordinates and that under change of coordinates the factor changes by the absolute value of the Jacobian determinant. Alternatively you can assume that M is oriented (which in our case of the balls it is) and that ν is a smooth volume form which is positive, in the sense that it defines the orientation. Either way, this means that

$$\mathcal{C}^\infty(M) \ni u \longmapsto \int_M u\nu \in \mathbb{C} \tag{22.14}$$

is a continuous linear map on the Fréchet space $\mathcal{C}^\infty(M)$. In fact it extends by continuity, and hence unambiguously, from the subspace $\dot{\mathcal{C}}^\infty(M)$ to

$$\int_M \bullet\nu : \mathcal{A}^s(M) \longrightarrow \mathbb{C},\ \forall\ s > -1. \tag{22.15}$$

The limit at $s = -1$ is just the non-integrability of x^{-1} with respect to dx near 0 on the line.

So, how can we extend this functional? Well, the answer really is that one cannot do it on the spaces $\mathcal{A}^s(M)$ for $s \le -1$. However, one can *extend* the integral to

$$\overline{\int_M} \bullet \nu : \rho^{-k}(M) \longrightarrow \mathbb{C},\ k \in \mathbb{N}. \tag{22.16}$$

Lemma 24. *If $u \in \rho^{-k}(M)$ and $\nu \in \mathcal{C}^\infty(M;\Omega)$ then*

$$F(z, u\nu, \rho) = \int_M \rho^z u\nu \text{ is holomorphic in } \operatorname{Re} z > k-1 \tag{22.17}$$

and has a meromorphic extension to $\mathbb{C}\setminus\{k-\mathbb{N}\}$ with only simple poles at the points $k-\mathbb{N}$. The residue at $z=0$ (if any) is independent of the choice of ρ.

The residue at zero is the 'boundary integral' or 'residue integral' and will be denoted

$${}^{\mathrm{R}}\!\!\int u\nu = \lim_{z\to 0} zF(z, u\nu, \rho). \tag{22.18}$$

The regularized value at $z=0$ is the regularized integral

$$\overline{\int_M} u\nu = \lim_{z\to 0}\big(F(z,u\nu,\rho) - \frac{1}{z}\,{}^{\mathrm{R}}\!\!\int u\nu\big). \tag{22.19}$$

In contrast to the residue integral, this functional does depend on the choice of ρ if $k \ge 1$. Note that these are both functionals on $u\nu \in \rho^{-k}\mathcal{C}^\infty(M;\Omega)$ which are consistent on restriction from $\rho^{-k}\mathcal{C}^\infty(M;\Omega)$ to $\rho^{-k+1}\mathcal{C}^\infty(M;\Omega)$.

Proof. For any k holomorphy of $F(z,u\nu,\rho)$ in $\operatorname{Re} z > k-1$ follows from the absolute convergence of the integral defining it in (22.17), the fact that any one deriviative with respect to $\operatorname{Re} z$ or $\operatorname{Im} z$ is also absolutely convergent, since it only introduces another $|\log\rho|$ of growth, and the holomorphy of the integrand. Now, we can split the integral into a part near the boundary and a part away from the boundary using ρ:-

$$\begin{gathered} F(z,u\nu,\rho) = F'(z,u\nu,\rho) + F''(z,u\nu,\rho), \\ F'(z,u\nu,\rho) = \int_{\rho<\delta}\rho^z u\nu,\ F''(z,u\nu,\rho) = \int_{\rho>\delta}\rho^z u\nu \end{gathered} \tag{22.20}$$

where $\delta > 0$ is fixed and is chosen so small that there is a smooth product decomposition, of M, in $\rho \le \delta : \{\rho\le\delta\} \simeq [0,\delta]_\rho \times \partial M$. The term F'' is entire in z, by the same reasoning as above. For the second term we can write the product

$$u\nu = (\sum_{j=0}^{L}\rho^{j-k}u'_j + \rho^{-k+L+1}v_L)d\rho d\nu_{\partial M},\ v_k \in \mathcal{C}^\infty([0,\delta]\times\partial M),\ u_j \in \mathcal{C}^\infty(\partial M) \tag{22.21}$$

for any $L \ge k$. This comes from the product decomposition of the measure $u\nu$, into the product of $d\rho$, a smooth measure, $\nu_{\partial M}$, on ∂M and function $\rho^{-k}u'$ with

$u' \in \mathcal{C}^\infty([0,\delta]\times\partial M)$. Then (22.21) is just the Taylor series expansion up to order L.

The remainder term here makes a contribution to F' of the form

$$F'_L(z) = \int_0^\delta \rho^{z+1+L-k} v'_k d\rho d\nu_{\partial M} \text{ which is holomorphic in } \operatorname{Re} z > k-L-2 \tag{22.22}$$

by the same reasoning. Thus in this half-space, which of course increases with L,

$$F'(z) = F'_L(z) + \sum_{j=0}^{L} (z+j-k+1)^{-1}\delta^{z+j-k+1}\int_{\partial M} u'_j d\nu_{\partial M}. \tag{22.23}$$

Since δ^z is entire, this shows that F' only has simple poles at the points $z \in k-\mathbb{N}$ in the half-plane $\operatorname{Re} z > k-L$. Thus indeed $F(z,u\nu,\rho)$ is meromorphic as claimed.

So, it remains to show that the residue at $z=0$ – which of course can only be non-zero if $k\in\mathbb{N}$ – is independent of the choice of ρ. The other residue-functionals are not independent in this way. Any two boundary defining fucntions are positive smooth multiples of each other so a second can be connected to the first by a smooth curve

$$\rho_s = ((1-s)1+sa)\rho = A(s)\rho,\ 0<a\in\mathcal{C}^\infty(M),\ 0<A\in\mathcal{C}^\infty([0,1]_s\times M). \tag{22.24}$$

Inserting this into the definition of F we get

$$F(z,u\nu,\rho_s) = \int_M \rho^z A(s)^z u\nu \tag{22.25}$$

in the half-plane of holomorphy.

The arguments above now apply uniformly in $s\in[0,1]$, with the extra, entire, factor of $A(s)^z$. It follows by differentiation that all residues and the analytic continuation are smooth in s. Now

$$\frac{d}{ds}F(z,uv,\rho_s) = z\int_M \rho^z \frac{dA(s)}{ds}A(s)^{z-1}uv. \tag{22.26}$$

The same argument regarding meromorphy can now be applied to the integral on the right, so it can only have a simple pole at $z=0$. The extra factor of z ensures that there is no such pole, so the residue of $F(z,uv,\rho_s)$ at $z=0$ is constant in s and hence indeed independent of the choice of ρ. $\square$

23. Lecture 20: Trace defect formula Monday, 20 October, 2008

Reminder. *I am currently examining the algebra* $\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ *to establish, and check various things about, the looping sequence* (20.27) *and hence the construction of the determinant bundle in* (21.12).

Last time I discussed the Riesz regularized integral of classical symbols on any compact manifold with boundary and the residue integral at the boundary. Let us apply this discussion to define a regularized trace functional and a residue trace functional on isotropic pseudodifferential operators

$$\begin{gathered}\overline{\mathrm{Tr}}:\Psi^{m,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)\longrightarrow\mathbb{C},\\ \mathrm{Tr}_{\mathrm{R}}:\Psi^{m,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)\longrightarrow\mathbb{C},\ m\in\mathbb{Z}.\end{gathered}\tag{23.1}$$

The first is supposed to be an extension of the trace functional which is given on smoothing operators by

$$\mathrm{Tr}:\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})\ni a\longmapsto(2\pi)^{-k}\int_{\mathbb{R}^{2k}}\mathrm{tr}_{\mathbb{R}^p}(F(t,\tau))dtd\tau.\tag{23.2}$$

Here the two parts of the space are treated differently as far as the kernel is concerned, with Weyl coordinates and Fourier transform used in the first part

$$F(t,\tau,z,z')=\int_{\mathbb{R}^k}e^{-is\cdot\tau}f(t,s,z,z'),\ a(Z,Z',z,z')=f(\frac{Z+Z'}{2},Z-Z',z,z').\tag{23.3}$$

Passing to $\Psi^{k,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ amounts to replacing the Schwartz condition by the partially-Schwartz space $\rho_q^{-m/2}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2k}};\mathcal{S}(\mathbb{R}^{2p})$. Now, in terms of this quadratic compactification to a ball we know that $\rho_q=(1+|(t,\tau)|^2)^{-1}$ is a boundary defining function – which is to say that $x=R^{-2}$ is also a boundary defining function near the boundary. The symplectic volume form is therefore

$$\begin{gathered}|dtd\tau|=R^{2k-1}|dRd\theta|=\frac{1}{2}x^{-k+1}|dxd\theta|\Longrightarrow\\ |dtd\tau|=\rho_q^{-k+1}\nu,\ 0<\nu\in\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2k}};\Omega).\end{gathered}\tag{23.4}$$

Thus the symplectic volume form is the product of a smooth non-vanishing volume form and an element of $\rho_q^{-k+1}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2k}})$. Only in the case $k=1$ is this a smooth volume form. Thus we can use the Riesz regularized index, choosing[4] $\rho_q^{\frac{1}{2}}$ as the defining function, to define

So we could just take the regularized integral of the 'symbol' (which is the whole operator) and this would give a regularized trace. However, it is better to follow an idea which comes originally from Seeley [11] but was effectively improved by Guillemin [3]. Namely we observe that $\rho_q^{z/2}\in\Psi^{-z}_{\mathrm{qiso}}(\mathbb{R}^k)$ – with no stabilization and remembering the annoying $\frac{1}{2}$'s. So we can consider the operator product, with A :

$$\rho_q^{z/2}\circ A\in\Psi^{s+m,-\infty}_{\infty\text{-iso}}(\mathbb{R}^k;\mathbb{R}^p),\ s>-\operatorname{Re}z,\tag{23.5}$$

[4]This business about the square-roots and quadratic defining functions is quite irritating; I will have to think of a clearer course of action

where I am using the fact, which I forgot to include earlier, that the stabilized operators are a module over the ustabilized ones (since the unstabilized ones just act 'as a multiple of the identity' in the second variables).

I have just claimed that the composite is an operator with symbol-with-bounds in (23.5). Of course a lot more is true, since we know that the product is given by a bidifferential operator up to any preassigned order[5]

$$\begin{gathered}A \circ B = Q_N(A,B) + Q_{(N)}(A,B),\ Q_N(A,B) = \sum_{|\alpha|+|\beta|\le N} c_{\alpha,\beta} D^\alpha A \cdot D^\beta B,\\ Q_{(N)} : \mathcal{A}^s(\overline{\mathbb{R}^{2k}}) \times \mathcal{A}^t(\overline{\mathbb{R}^{2k}}) \longrightarrow \mathcal{A}^{s+t+2N}(\overline{\mathbb{R}^{2k}})\ \forall\ s,t \in \mathbb{R},\end{gathered} \tag{23.6}$$

where we actually know the coefficients. The 'remainder term' is continuous – as is the explict expansion. The same formula applies to the suspended algebra provided we interpret the product as the composition of smoothing operators.

Applying this to the product in (23.5) we conclude that as a function

$$\rho_q^{z/2} \circ A = \rho_q^{z/2} P_N(z,\rho_q,A) + Q_{(N)}(z) \tag{23.7}$$

where the leading term is a differential operator applied to A and a polynomial in z while the remainder term is holomorphic as a map

$$\{\operatorname{Re} z > L\} \longrightarrow \Psi^{m-2N-L,-\infty}_{\infty\text{-iso}}(\mathbb{R}^k;\mathbb{R}^p). \tag{23.8}$$

Lemma 25. *For any* $A \in \Psi^{m,-\infty}_{\text{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$

$$\operatorname{Tr}(\rho_q^{z/2} \circ A) \tag{23.9}$$

is meromorphic in the complex plane with at most simple poles at $z \in m + n - \mathbb{N}$.

Proof. The result follows in $\operatorname{Re} z > L$ for any L by using the splitting (23.7) for large enough N, applying the discussion of Riesz regularization of the integral to the first part and the holomorphy in (23.8) to the second part. □

So, now we can define

$$\begin{gathered}\operatorname{Tr}_{\mathrm{R}}(A) = \lim_{z\to 0} z \operatorname{Tr}(\rho_q^{z/2} \circ A),\\ \overline{\operatorname{Tr}}(A) = \lim_{z\to 0}\left(\operatorname{Tr}(\rho_q^{z/2} \circ A) - \frac{1}{z}\operatorname{Tr}_{\mathrm{R}}(A)\right)\end{gathered} \tag{23.10}$$

as respectively the residue and the regularized value of the analytic continuation of the trace to $z = 0$. The residue trace was defined in the case of the usual pseudodifferential algebra on a compact manifold by Wodzicki [12].

Proposition 21. *If the order of A is less than $-2k$ then* $\overline{\operatorname{Tr}}(A) = \operatorname{Tr}(A)$. *The residue trace is a trace functional,* $\operatorname{Tr}_{\mathrm{R}}([A,B]) = 0$, *it vanishes on operators of order less than $-2k$, is given explicitly by the residue integral*

$$\operatorname{Tr}_{\mathrm{R}}(A) = (2\pi)^{-k} \,{}^{R}\!\!\int A\omega, \tag{23.11}$$

and the regularized trace satisfies the trace defect formula

$$\overline{\operatorname{Tr}}([A,B]) = \frac{1}{2}\operatorname{Tr}_{\mathrm{R}}\left([B,\log\rho_q]A\right),\ \forall\ A,B \in \Psi^{\mathbb{N},-\infty}_{\text{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p). \tag{23.12}$$

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[5]Again I should have included this earlier, I will!

Proof. When the order of A is less than $-2k$, the trace of $\rho_q^{z/2} \circ A$ is holomorphic in a neighbourhood of $z = 0$. Evaluating there, $\rho_q z/2 = 1$ is the identity in the (unstabilized) algebra so indeed $\overline{\mathrm{Tr}}(A) = \mathrm{Tr}(A)$. Thus $\overline{\mathrm{Tr}}$ is an extension of the trace functional.

To compute $\mathrm{Tr}_{\mathrm{R}}([A,B])$ observe that this is, by definition, the residue at $z = 0$ of the analytic continuation of

$$\mathrm{Tr}(\rho_q^{z/2} \circ ([A,B]) = \mathrm{Tr}([B,\rho_q^{z/2}]A) - = \mathrm{Tr}([A,\rho_q^{z/2}]B) \tag{23.13}$$

where we have used the trace identity for $\mathrm{Re}\, z >> 0$ and the uniqueness of analytic continuation. Using the decomposition of the product in (23.6) the commutator here can be written as a sum

$$[B,\rho_q^{z/2}] = Q_N(B,\rho_q^{z/2}) - Q_N(\rho_q^{z/2},B) + Q_{(N)}(z) \tag{23.14}$$

where if N is large enough the second term is holomorphic and uniformly of order less than $-2k$ up to $z = 0$ after composition with A. Thus, the trace of this term is regular at 0 so does not contribute to the residue; only the first two terms in (23.14) contribute for N large enough. The leading, commutative product, term cancels in the commutator so in every remaining term, $\rho_q^{z/2}$ is differentiated at least once. This produces a factor of z with the trace of the coefficient having at most a simple pole at $z = 0$ by the discussion above. Thus there is no pole at $z = 0$ and $\mathrm{Tr}_{\mathrm{R}}([A,B]) = 0$ always.

Again if A is of order less than $-2k$ then so is $\rho_q^{z/2} \circ A$ near $z = 0$ where it is holomorphic. The reside, and hence the residue trace of A, therefore vanishes. Similarly for general A the difference between $\mathrm{Tr}(\rho \circ A)$ and the integral of the commutative product $\rho_q^{z/2}A$ involves all the terms in $Q_N(\rho_q^{z/2}, A)$ after the constant term and the remainder. For N large enough, the latter can not contribute to the residue at $z = 0$. All the other terms involve at least one derivative falling on $\rho_q^{z/2}$ so again cannot contribute to the residue trace of A. Thus (23.11) follows by the definition of the Riesz regularization of the integral.

It remains to prove the trace defect formula (23.12). Following the discussion above, especially (23.13), $\overline{\mathrm{Tr}}([A,B])$ is the regularized value of the analytic continuation of the trace of the product of (23.14) and A. For large N the second term is holomorphic near $z = 0$ and of low order so

$$\mathrm{Tr}(Q_{(N)}(z)A) = \mathrm{Tr}(Q_{(N)}(0)A) \text{ is regular near } z = 0. \tag{23.15}$$

However, $Q_{(N)}(z)$ is the 'low order part of the comutator $[B,\rho_q^{z/2}]$. At $z = 0$ $\rho_q^{z/2} = 1$ is the identity operator so all the leading terms vanish (since they involve differentiation of 1 so this low order part also vanishes, since the whole commutator vanishes. It follows that the right side of (23.15) vanishes at $z = 0$. Thus $\overline{\mathrm{Tr}}([A,B])$ is the regularized value at $z = 0$ of the analytic continuation of

$$\mathrm{Tr}\left(\left(Q_N(B,\rho_q^{z/2}) - Q_N(\rho_q^{z/2},B)\right) \circ A\right). \tag{23.16}$$

Again, $\rho_b^{z/2}$ is differentiated at least once, producing a factor of z. Thus the analytic continuation is regular at $z = 0$. Writing

$$d\rho_q^{z/2} = (\frac{z}{2}\frac{d\rho_q}{\rho_q})\rho_q^{z/2} \tag{23.17}$$

it follows that if any subsequent derivative falls on the last factor then this produces an overall factor of z^2 and hence does not contribute to the regularized trace. Thus the effect is the same as if all derivatives acting on the appropriate factor in Q_N in (23.16) fall on $\log \rho_q$. That is, the regularized trace is the same as the regularized value of

$$\frac{z}{2}\operatorname{Tr}\left(([B,\log\rho_q]\cdot\rho_q^{z/2})\circ A\right). \tag{23.18}$$

Again expanding out the product with A, the low order term is holomorphic – so does not contribute – and any differentiation of $\rho_q^{z/2}$ produces another factor of z so also does not contribute. Thus the value of the integral at $z=0$ reduces to the residue trace and (23.12) follows. □

The case $k=1$ is particularly simple, since then we can easily compute the right side of (23.12). The difficulty of this computation is greater when $k>1$ since the residue trace occurs higher and higher in the Taylor series expansion of the symbol of an element of $\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p)$ as k increases.

Proposition 22. *If* $k=1$ *the trace defect formula* (23.12) *involves only the* principal symbols *of* A *and* B *:*

$$\overline{\operatorname{Tr}}([A,B]) = c\int_{\mathbb{S}}\operatorname{tr}(\frac{\partial b(\theta)}{\partial\theta}a(\theta))d\theta = -c\int_{\mathbb{S}}\operatorname{tr}(\frac{\partial a(\theta)}{\partial\theta}b(\theta))d\theta, \tag{23.19}$$
$$a=\sigma_0(A),\ b=\sigma_0(b),\ A,\ B\in\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R};\mathbb{R}^p).$$

24. Lecture 21: Curvature and Chern class Wednesday, 22 October, 2008

Reminder. *Last time I computed the trace defect for the extension of the trace functional to* $\Psi^{\mathbb{N},-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ *given by Riesz regularization. Today I want to use this to compute the curvature of the determinant line bundle. First I have to make sure of a few technical points, one of these is the proof that the range of the symbol map in* (20.27) *is indeed the index zero component – I will use the trace defect formula to compute the index.*

Recall that $\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p)$ is the space of functions with $\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2k}};\mathcal{S}(\mathbb{R}^{2p})$ with a non-commutative product. Choosing a point on the boundary of the quadratic compactification here – the North Pole – we can consider the subalgebra, $\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p)$, of functions that vanish in Taylor series there, so identified with

$$\left\{a\in\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2k}};\mathcal{S}(\mathbb{R}^{2p});a\equiv 0\text{ at }\{N\}\times\mathbb{R}^{2p},\ N\in\partial{}^q\overline{\mathbb{R}^{2k}}\right\}. \tag{24.1}$$

I will probably skip the proof of the following result in the lecture. Not that it is unimportant. However, its proof is generally similar to ones we have seen before, although not reducible to them. In particular it makes use of the L^2 boundedness of these operators, which is significant in itself (and one reason the proof is so long).

I should have used this notation before (but only on the insistence of Frédéric Rochon am I introducting it now).

Definition 6. A Fréchet algebra, $\mathcal{A}$, without identity, is said to be a Neumann-Fréchet algebra if the group of invertible elements in $\mathrm{Id}+\mathcal{A}$ is open – it behaves as if the Neumann series converges near the identity. These are often called 'good' Fréchet algebras. Good grief! If it contains the identity then it is Neumann-Fréchet if the elements in a neighbourhood of the identity are invertible.

Proposition 23. *The group* $\dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ *of operators* $A\in\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ *such that* $\mathrm{Id}+A$ *has an inverse of the same form, is open in* $\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$, *i.e. this is a Neumann-Fréchet algebra, and* $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ *is a (relatively) closed normal subgroup.*

Proof. The Fréchet topology is on $\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ comes from the $\mathcal{C}^\infty$ standard toplogy of the supremum norms of derivatives, in this case on $\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2k}})\times\overline{\mathbb{R}^{2p}})$. The rapid vanishing at the boundary in the second factor (of course uniformly in the first factor) to give the Schwartz subspace and at one point in the first factor, give a closed subspace. So we need to show that for a in a small neighbourhood of 0 in this topology, $\mathrm{Id}+b$ has an inverse with $b\in\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$.

To show this we first need to prove the L^2 boundedness of elements this algebra, which acts on $\mathcal{S}(\mathbb{R}^{k+p})$. This can be done using Hörmander's approach, which I will outline at some point. In particular this shows that the L^2 operator norm is continuous on $\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p)$, from which it follows that $\mathrm{Id}+a$ is invertible on L^2 for a in a neighbourhood of 0. So, it still needs to be seen that this inverse is of the form $\mathrm{Id}+b$, $b\in\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p)$.

To see this we first show that if $\mathrm{Id}+a$ is L^2 invertible then it must be elliptic, in the sense that $\mathrm{Id}+\sigma_{0,\mathrm{iso}}(a)\in G^{-\infty}_{\mathrm{sus}(2k-1),\mathrm{iso}}(\mathbb{R}^p)$. This can be done by constructive contradiction. That is, non-ellipticity means that $\mathrm{Id}+a(p)$ must be non-invertible as a smoothing perturbation of the identity on $\mathcal{S}(\mathbb{R}^p$ for some $p\in\mathbb{S}^{2k-1}\setminus\{N\}\simeq\mathbb{R}^{2k-1}$

(since at $p = N$ it is the identity). This in turn means there must be an element of the null space. The constructive part is to use this to generate a sequence $u_j \in \mathcal{S}(\mathbb{R}^{k+p})$ which has norm one in L^2 but is such that $(\mathrm{Id} + a)u_j \to 0$ in L^2. The idea here is that the solution should 'concentrated near p' in a sense that can be understood in terms of the symbol of the operator. This violates invertibility. Essentially what is being shown here is that the symbol map extends by continuity to the closure of these operators in the bounded operators on L^2, and so it must be invertible. Again I plan to add a bit about this at some point.

Once we know that if $\mathrm{Id} + a$ is an invertible operator on L^2 then it is necessarily elliptic, we can apply usual methods. Namely we can construct a parameterix for $\mathrm{Id} + a$, $\mathrm{Id} + b'$, such that

$$(\mathrm{Id} + b')(\mathrm{Id} + a) - \mathrm{Id} = R_R,\ (\mathrm{Id} + a)(\mathrm{Id} + b') - \mathrm{Id} = R_L \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}). \tag{24.2}$$

Then it follows that the inverse is of the expected form since applying (24.2) on the left and right

$$(\mathrm{Id} + a)^{-1} = \mathrm{Id} + b' - (\mathrm{Id} + a)^{-1}R_L = \mathrm{Id} + b' - R_L - b'R_L + R_R(\mathrm{Id} + a)^{-1}R_L. \tag{24.3}$$

The last term is in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ because of the corner property of these operators.

This shows that $\dot\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ (and indeed $\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$) is a Neumann-Fréchet algebra.

The last closure property follows directly from the characterization of the kernels. □

Proposition 24. *The quotient*

$$\dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)/G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}) = G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso}}(\mathbb{R}^p) \tag{24.4}$$

is the group of formal power series in ρ_q, with leading terms forming the component of the identity (i.e. the part on which the index functional vanishes) in $G^{-\infty}_{\mathrm{sus}(2k-1),\mathrm{iso}}(\mathbb{R}^p)$, with arbitrary lower order terms and a $*$-product; for the moment we only prove this for $k = 1$ although it is true in general.

Proof. From Proposition 23 above, first group in (24.4) is an open subset of the smooth functions in (24.1). The subgroup is just the subspace which vanishes in Taylor series on the whole of the product of the boundary of the first factor with the second factor. This is simply because a smoothing perturbation of the identity is invertible if and only if it has an inverse in the larger space, i.e. it is an element of the larger group. Thus the quotient is certainly a subset, and necessarily open, of the space of formaly power series at the boundary of the first factor. The leading term, which is the identity plus a function on the $2k-1$ sphere, valued in smoothing operators, must be invertible, and the perturbation vanishes to infinite order at the point N so it can be identified with an element of $G^{-\infty}_{\mathrm{sus}(2k-1),\mathrm{iso}}(\mathbb{R}^p)$. The induced product on the formal power series is given by the loop product of the leading terms, since this is just the principal symbol, and a $\star$ product in the lower order terms, in the sense that the k the term in the product is give by bidifferential operators on the first k terms of the factors. More about star products below!

If $g \in \dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ then any lower order, formal power series, perturbation of the image in the quotient can be realized as an actual function, $a \in \Psi^{-1,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ with the correct Taylor series. Cutting off near the boundary, this can be seen to have arbitrariy small L^2 norm, and, as discussed in the proof of (23) this implies that $G + a \in \dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$.

Thus it remains to show that the leading part of the quotient in $G^{-\infty}_{\mathrm{sus}(2k-1),\mathrm{iso}}(\mathbb{R}^p)$ is the component of the identity, i.e. on the elements on which the index functional vanishes. It is at this point that we reduce to the case $k=1$. If $g\in \dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p)$ and h is its inverse then certainly

$$\overline{\mathrm{Tr}}([g,h])=0=c\int_{\mathbb{S}}\mathrm{tr}(\frac{\partial h_0}{\partial\theta}g_0(\theta))d\theta=-c\int_{\mathbb{S}}\mathrm{tr}(g_0^{-1}(\theta)\frac{\partial g_0}{\partial\theta}(\theta))d\theta,\ c\neq 0. \tag{24.5}$$

Here the explicit trace defect formula in Proposition 22 has been used. The last integral is the winding number of the determinant of g_0, the leading term of g, i.e. the index functional. Thus the range is contained in the component on which the index vanishes.

To see the converse, we need to do a little analysis to reverse the argument above. Namely given the symbol $\mathrm{Id}+a_0$ we can quantize it to a not-necessarily-invertible operator $\mathrm{Id}+A$. However, we know that this has a parametrix $\mathrm{Id}+B$ up to smoothing errors,

$$(\mathrm{Id}+B)(\mathrm{Id}+A),\ (\mathrm{Id}+A)(\mathrm{Id}+B)\in\mathrm{Id}+\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}). \tag{24.6}$$

Indeed, with an error term of order -1 this follows by choosing B with symbol $b=(\mathrm{Id}+a)^{-1}-\mathrm{Id}$. Taking the formal Neumann series for the error term and summing it allows the parametrix to be improved to (24.6). Now, if Π_N is the projector onto the first N eigenfunctions of the harmonic oscillator on $\mathbb{R}^{k+p}$ it follows that

$$(\mathrm{Id}+B)(\mathrm{Id}+A)(\mathrm{Id}-\Pi_N)=(\mathrm{Id}+E(\mathrm{Id}-\Pi_N))(\mathrm{Id}-\Pi_N). \tag{24.7}$$

Since E is a smoothing opertor, $\mathrm{Id}+E(\mathrm{Id}-\Pi_N)\in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ for N large enough. Composing on the left with the inverse gives, with a different operator B',

$$(\mathrm{Id}+B')(\mathrm{Id}+A')=(\mathrm{Id}-\Pi_N),\ A'=A-A\Pi_N-\Pi_N. \tag{24.8}$$

Here A' has the same symbol as A but now must have null space precisely the range of Π_N. Proceeding in the same manner with the adjoint, it follows that $\mathrm{Id}+A'$ must have range of finite codimension. Composing with an element of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+n})$ it can be arranged seen that, a possibly different, $\mathrm{Id}+A'$, but with the same symbol, has null space the range of Π_N and range that of $\mathrm{Id}-\Pi_{N'}$. Then B' can be shifted by a smoothing operator so that (24.8) holds and also

$$(\mathrm{Id}+A')(\mathrm{Id}+B')=\mathrm{Id}-\Pi_N. \tag{24.9}$$

Then we see that the index, in the conventional sense,

$$\begin{aligned}\mathrm{ind}(\mathrm{Id}+A')&=N-N'=\mathrm{Tr}(\Pi_N-\Pi_{N'})\\&=\overline{\mathrm{Tr}}((\mathrm{Id}-\Pi_{N'})-(\mathrm{Id}-\Pi_N))=\overline{\mathrm{Tr}}([\mathrm{Id}+B',\mathrm{Id}+A']).\end{aligned} \tag{24.10}$$

Here we have used the fact that $\overline{\mathrm{Tr}}$ is an extension of the trace functional. Now we can apply the same argument as in (24.5) to see that this is a non-vanishing multiple (-1 I think) of the 'index' in the sense of the winding number of the determinant of g, the symobl of $\mathrm{Id}+A'$. Thus if this vanishes then $N=N'$ and adding Π_N to A' gives an invertible lift of the symbol. □

The use of the trace defect formula in the proof above is very close to the discussion of the curvature below and presages the treatment of 'η forms' later (I hope).

Proposition 25. *The subgroups*

$$\dot{G}^{m,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^{k+p}) \subset \dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p), \tag{24.11}$$

of elements where the perturbation of the identity is of order $m \in -\mathbb{N}$, are normal and the quotient map gives a fibration

$$\begin{tikzcd} \dot{G}^{m,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^{k+p}) \arrow[r] & \dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p) \arrow[d] \\ & G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso}}(\mathbb{R}^p)/\big(\rho_q^{m+1}\equiv 0\big), \end{tikzcd} \tag{24.12}$$

i.e. there is a smooth section in a neighbourhood of each point of the quotient.

Proof. The normality of these subgroups follows directly from the order properties of the product. The handy thing is that the quotients all give fibrations, whereas in the case $m=-\infty$ it is a Serre fibration only – not locally trivial. It suffice to give a section of the projection over a neighbourhood of the identity, since this can be translated to any other point. Since there are only finitely many terms in the power series in the quotient, they can be summed and cut off near the boundary on ${}^q\overline{\mathbb{R}^{2k}}$. Provided the terms are small enough, this gives an invertible perturbation of the identity, following the arguments of Proposition 23, and hence a section – meaning the sequence is a fibration. □

Lemma 26. *The construction in* (21.11), (21.12) *carries over to the sequence* (24.12) *for any $m<-2k$ and constructs a locally trivial line bundle $\mathcal{L}_m$ over the quotient in* (24.12)*. Under any of the projection maps for $m'<m$, $m'\in -\mathbb{N}\cup\{-\infty\}$*

$$G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso}}(\mathbb{R}^p)/\big(\rho_q^{-m'+1}\equiv 0\big) \longrightarrow G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso}}(\mathbb{R}^p)/\big(\rho_q^{-m+1}\equiv 0\big), \tag{24.13}$$

$\mathcal{L}_m$ pulls back to be canonically isomorphic to $\mathcal{L}_{m'}$, where $\mathcal{L}=\mathcal{L}_{-\infty}$.

In particular $\mathcal{L}$ as constructed originally is indeed locally trivial, so Claim 1 is vindicated.

Moreover the same is true of the connections. Namely, for $m<-k$ each of the determinant line bundles $\mathcal{L}_m$ has a connection given as the quotient of the connection

$$d-\gamma,\ \gamma=\frac{1}{2}\overline{\mathrm{Tr}}\big(g^{-1}dg+(dg)g^{-1}\big) \tag{24.14}$$

where the regularized trace is defined above.

Proposition 26. *The 1-form γ on $\dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p)$ defined in* (24.14) *induces a connection on each $\mathcal{L}_m$, $m<-2k$. In case $k=1$, the curvature is*

$$c\int_{\mathbb{R}}(\mathrm{tr}\,\big(g^{-1}\frac{dg}{dt}(g^{-1}dg)^2\big)dt \tag{24.15}$$

Proof. Let me concentrate on the case $m=-\infty$. The case for finte m corresponds to the extension of the determinant to the groups $G^{-m}_{\mathrm{iso}}(\mathbb{R}^{k+p}$ by continuity – where it continues to be multiplicative.

So, even though the projection in the delooping sequence may not be a fibration, the determinant bundle itself is locally trivial. Namely over any small open set in one of the finite order quotients there is a section and this lifts to the preimage of the open set in $G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso}}(\mathbb{R}^p)$ and hence to a section of the trivial bundle over the preimage in $\dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k:\mathbb{R}^p)$ with the transformation law

$$f(Ag)=f(A)\det(g),\ \forall\ g\in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}). \tag{24.16}$$

Conversely, a local section of the trivial bundle on the preimage of a set descends to be a section of $\mathcal{L}$ over that set if and only if (24.16) holds. Now, the connection on the trivial bundle is

$$\nabla(f)(A)=df-\gamma(A)f \tag{24.17}$$

so

$$\begin{aligned}\nabla(f)(Gg)&=d(f\det(g))-\gamma(Ag)f(Gg)=\det(g)(df-\gamma(A)f)\\&+\operatorname{tr}(g^{-1}dg)f-\frac{1}{2}\overline{\operatorname{Tr}}(g^{-1}A^{-1}d(Ag)+d(Ag)g^{-1}A^{-1}-A^{-1}dA-(dA)A^{-1}).\end{aligned} \tag{24.18}$$

The combination of the last terms vanishes, since commutation by g preserves $\overline{\operatorname{Tr}}$ (since the smoothing term permits approximation by smoothing terms) so

$$\begin{aligned}\overline{\operatorname{Tr}}(g^{-1}A^{-1}d(Ag))&=\overline{\operatorname{Tr}}(A^{-1}dA)+\operatorname{Tr}(g^{-1}dg),\\ \overline{\operatorname{Tr}}(d(Ag)g^{-1}A^{-1})&=\overline{\operatorname{Tr}}((dA)A^{-1})+\operatorname{Tr}(g^{-1}dg).\end{aligned} \tag{24.19}$$

Thus the connection descends to $\mathcal{L}$. The curvature can be compute on the central group – of course it must also descend to the quotient. It will be exact on the large group (which is actually weakly contractible as we shall see) but not on the quotient – reconcile yourself with this as necessary!

The curvature is $-d\gamma$, or $-d\gamma/2\pi i$ according to taste. Anyway, it is enough to compute:-

$$\begin{aligned}d\gamma&=-\frac{1}{2}\overline{\operatorname{Tr}}(A^{-1}(dA)A^{-1}dA-(dA)A^{-1}(dA)A^{-1})\\&=\frac{1}{2}\overline{\operatorname{Tr}}\big([(dA)A^{-1}dA,A^{-1}]\\&=-\operatorname{Tr}_{\mathrm{R}}\big(A^{-1}[A,\log\rho](A^{-1}(dA))^2\big)\\&=-c\int_{\mathbb{R}}\operatorname{tr}(a^{-1}\frac{da}{dt}(a^{-1}da)^2)dt,\ a=\sigma_{0,\mathrm{iso}}(A),\end{aligned} \tag{24.20}$$

using the trace defect formula and in the last line the explicit formula in case $k=1$. Here, you may recognize the 2-form part of the Chern character on the suspended group (of course I have lost track of the constants at the moment, but it is actually equal to it – at least up to sign). □

It is very unlikely I will get to this until some time later!

There is another property of this construction of the determinant line bundle which is a consequence of a wider conjugation-invariance property of the determinant.

Proposition 27. *The determinant line bundle is* primitive *in the sense that for any two elements* $\alpha,\ \beta\in G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso,ind}=0}(\mathbb{R}^p)$ *there is a natural isomorphism*

$$\mathcal{L}_\alpha\otimes\mathcal{L}_\beta\simeq\mathcal{L}_{\alpha\beta} \tag{24.21}$$

which makes the complement of the zero section $\mathcal{L}^* \subset \mathcal{L}$ into a group which is a $\mathbb{C}^*$ extension of the base, so

$$\mathbb{C}^* \longrightarrow \mathcal{L}^* \longrightarrow \dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p) \tag{24.22}$$

is a short exact sequence of groups.

Proof. The additional conjugation invariance referred to above, is that

$$\det(A^{-1}gA) = \det(g)\ \forall\ A \in G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p),\ g \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}). \tag{24.23}$$

Given this the statement of the Proposition follows readily. Namely if A, $B \in \dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ project to α, $\beta \in G^{-\infty}_{\mathrm{sus}(2k-1)*,\mathrm{iso}}(\mathbb{R}^p)$ then for $g \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ the map

$$(z\det(g), Ag)(z'\det(g'), Bg') \longmapsto (zz'\det(gg'), AB(B^{-1}gB)g') \tag{24.24}$$

descends to an identification of $\mathcal{L}_\alpha \otimes \mathcal{L}_\beta$ with $\mathcal{L}_{\alpha\beta}$. This can clearly be interpreted as a group product

$$(l,\alpha)\cdot(l',\beta) \longmapsto (l\otimes l', \alpha\beta) \tag{24.25}$$

on $\mathcal{L}^*$ which reduces to $\mathbb{C}^*$ on the trivial fibre above Id. This gives the short exact sequence (24.22).

So, it remains to prove (24.23). There is a smooth curve, $g(t)$, connecting g to the identity in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$. Certainly

$$A^{-1}gA = A^{-1}(\mathrm{Id}+a)A = \mathrm{Id}+A^{-1}aA \in G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}) \tag{24.26}$$

so consider

$$\begin{aligned} &\frac{d}{dt}\log\big(\det(g(t))^{-1}\det(A^{-1}g(t)A)\big) \\ &= \frac{d}{dt}\log\big(\det(g^{-1}(t)A^{-1}g(t)A)\big) = \mathrm{Tr}\,\big(A^{-1}g^{-1}(t)Ag(t)\frac{d}{dt}(g^{-1}(t)A^{-1}g(t)A)\big) \\ &= \mathrm{Tr}\,\big(A^{-1}g^{-1}(t)g'(t)A - A^{-1}g^{-1}(t)Ag'(t)g^{-1}A^{-1}g(t)A\big) = 0. \end{aligned} \tag{24.27}$$

In the last step the commutation of factors of A is justifed by the fact that there is one factor, g', which is smoothing so A can be approximated by smoothing operators, in the topology of symbols-with-bounds, of marginally positive order, in such a way that the product converges in smoothing operators. Since the determinants are equal when $g = \mathrm{Id}$ they are equal everywhere. □

Exercise 18. Did I explain (24.27) clearly enough? If not, go through the following. If $A \in \Psi^{s,-\infty}_{\infty\text{-iso,iso}}(\mathbb{R}^k : \mathbb{R}^p)$ and $a \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ then $Aa \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ depends continuously on A, as does aA, in the topology of $\Psi^{s,-\infty}_{\infty\text{-iso}}(\mathbb{R}^k : \mathbb{R}^p)$ for any $s' \ge s$. Since $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})$ is dense in $\Psi^{s,-\infty}_{\infty\text{-iso}}(\mathbb{R}^k : \mathbb{R}^p)$ in this weaker topology, for any $s' > s$, and on this dense subspace $\mathrm{Tr}(Aa) = \mathrm{Tr}(aA)$ it follows in general.

Exercise 19. Write out in some reasonably explicit form the product on the 'star-extended' loop group $G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}(\mathbb{R}^p)$ and show that it can be extended to the whole of the group $G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso}}(\mathbb{R}^p)$ – which is defined to have the 'same' product but arbitrary invertible leading term (rather than index zero) and arbitrary lower order terms as before. Show that the determinant line bundle can be transferred from the component of index zero to the other components by choosing a base point in each. Is it possible to extend the group property to the whole thing? Assuming that you agree with me that this does not seem possible, can you explain why?

25. Lecture 22: Isotropic families index ($k = 1$) Friday, 24 October, 2008

Today I am supposed to be proving the weak contractibility of the central group in the looping sequence. With any luck I will get to that as a by-product of the families isotropic index, in K-theory for $k = 1$ (and untwisted). Namely, what I want to do is to define the istropic index map

$$\operatorname{ind}_{\mathrm{iso}} : [X; G^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^p)]_{\mathrm{c}} \longrightarrow [X; \mathcal{H}^{-\infty}_{\mathrm{iso}}]_{\mathrm{c}} \tag{25.1}$$

for any manifold X. Here I have written out the homotopy groups explicitly, since both represent even K-theory, as we already know. The restriction to $k = 1$ shows up in the single suspension on the left (but these arguments do carry over with only relatively minor changes to $2k-1$ suspension, meaning isotropic operators on $\mathbb{R}^k$ as we will see next week).

For the definition I will use same sort of set up as for Bott periodicity and define some larger spaces. Thus, let

$$\begin{aligned} &\dot{\mathrm{Ell}}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p) \\ &\quad = \{\mathrm{Id} + A \in \dot{\Psi}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p); \mathrm{Id} + \sigma_{\mathrm{iso}}(A) \in G^{-\infty}_{\mathrm{sus}(2k-1);\mathrm{iso}}(\mathbb{R}^p)\}. \end{aligned} \tag{25.2}$$

So, this is just the set of elliptic elements perturbations of the identity, the operators with invertible symbols. Then consider a similar space of pairs which are parameterices of each other

$$\begin{aligned} &\dot{\mathcal{P}}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p) \\ &\quad = \{(\mathrm{Id} + A, \mathrm{Id} + B) \in \dot{\Psi}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p); (\mathrm{Id} + A)(\mathrm{Id} + B) - \mathrm{Id}, \\ &\qquad\qquad (\mathrm{Id} + B)(\mathrm{Id} + A) - \mathrm{Id} \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}). \end{aligned} \tag{25.3}$$

Proposition 28. *In the diagram*

$$\begin{tikzcd} & \dot{\mathcal{P}}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p) \arrow[dl, "p_1"'] \arrow[dr, "\operatorname{ind}_{\mathrm{iso}}"] & \\ \dot{\mathrm{Ell}}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p) & & \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}) \end{tikzcd} \tag{25.4}$$

where

$$\begin{aligned} \operatorname{ind}_{\mathrm{iso}}(A,B) &= \begin{pmatrix} 1 - 2R_L^2 & 2R_L(\mathrm{Id} + R_L)(\mathrm{Id} + B) \\ 2R_R(\mathrm{Id} + A) & -\mathrm{Id} + 2R_R^2 \end{pmatrix}, \\ &R_L = \mathrm{Id} - (\mathrm{Id} + A)(\mathrm{Id} + B),\ R_R = \mathrm{Id} - (\mathrm{Id} + B)(\mathrm{Id} + A), \end{aligned} \tag{25.5}$$

the left map is surjective with the lifting property for compactly supported maps and the right map (is well-defined and) has the lifting property up to homotopy, so every compactly supported smooth map into $\mathcal{H}^{-\infty}_{\mathrm{iso}}$ is homotopic to the image of a map into $\dot{\mathcal{P}}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$.

Proof. Of course the first assertion is that the two maps are well-defined. Certainly the left map is, since it is just projection onto the first factor and this must be elliptic, since the existence of a parameterix implies that the symbol is invertible. I will not dwell on the lifting property for this map, since I discussed the construction earlier. (Where exactly?) A smooth map with compact support into the elliptics can be quantized smoothly, to the identity outside a compact set. Then a smooth family of parametrices can be constructed, also reducing to the identity outside a compact set.

So, to the index map. This is in the rather obscure form (25.4) because I have been remiss about filling in the details about the relationship between involutions and vector bundles. It would be more usual (perhaps, it depends a bit on the circles you move in) to express the index map in terms of null bundle and null bundle of parametrix, after stabilization. I did do this quickly earlier too. The explicit map (25.4) has the advantage that it is explicit and defined for any parametrix, without stabilization. Of course, I invite you to do the algebra to show that $\operatorname{ind}_{\mathrm{iso}}(A,B)$ so defined *is an involution*:

$$\operatorname{ind}_{\mathrm{iso}}(A,B)^2 = \operatorname{Id}. \tag{25.6}$$

(Which is a strange looking identity.) In doing so it is helpful to note that

$$\begin{gathered} R_L Q = Q R_R,\ R_R P = P R_L,\ PQ = \operatorname{Id} - R_R, \\ QP = \operatorname{Id} - R_L \text{ if } Q = (\operatorname{Id} + B),\ P = (\operatorname{Id} + A). \end{gathered} \tag{25.7}$$

At this point it only remains to show the lifting property up to homotopy. Given the normal form for involutions which can be achieved for families, as discussed in Proposition 16 it is really enough to show that the involution corresponding to any pair of families of finite rank, commuting projections can be recovered under the index map. So, as in the periodicity construction we need a 'Bott element'. In this case it is easier – and it is certainly possible I should have done this much earlier. Namely, we know that the annihilation operator, $A = \partial_z + z$, has one dimensional null space but is surjective on Schwartz functions. Of course the order is wrong, but we can just divide by a square root of the harmonic oscillator to make it of order zero with the same property. Then its symbol is not flat at N, but it is equal to 1 at one point. So, deforming it a little more we can find $\operatorname{Id} + A$, $A \in \dot{\Psi}^0_{\mathrm{qiso}}(\mathbb{R})$ which is surjective, with one dimensional null space. Now to recover a given pair of projections $P_+(x)$, $P_-(x)$ which are smooth families of $N \times N$ matrices, consider

$$\begin{pmatrix} AP_+(x) + \operatorname{Id}(\operatorname{Id} - P_+(x)) & 0 \\ 0 & CP_-(x) + \operatorname{Id}(\operatorname{Id} - P_-(x)) \end{pmatrix} \in \dot{\Psi}^0_{\mathrm{qiso}}(\mathbb{R};\mathbb{C}^{2N}). \tag{25.8}$$

Here C is the adjoint of A. It is easy to check (but I have not actually done it ..) that $\operatorname{ind}_{\mathrm{iso}}$ is a family of involutions which 'recovers' these two projections. Of course one should stabilize $\mathbb{C}^{2N}$ into smoothing operators first. □

So, the existence of index map in K-theory, (25.1), follows from the uniqueness up to homotopy of the lifting on the left extended to:

$$\begin{array}{ccc} \mathcal{C}^\infty_c(X;\dot{\mathcal{P}}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)) & \xrightarrow{\operatorname{ind}_{\mathrm{iso}}} & \mathcal{C}^\infty_c(X;\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p})) \\ \Big\downarrow{\scriptstyle p_1} & & \\ \mathcal{C}^\infty_c(X;\dot{\operatorname{Ell}}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)) & & \\ \Big\downarrow{\scriptstyle \sigma_{\mathrm{iso}}} & & \\ \mathcal{C}^\infty_c(X;G^{-\infty}_{\mathrm{sus}(2k-1),\mathrm{iso}}(\mathbb{R}^p)). & & \end{array} \tag{25.9}$$

Namely the linear homotopies between quantizations and parameterices shows the uniqueness up to homotopy.

Finally then this isotropic index is rather precise:-

Theorem 7. *This isotropic index map* (25.1) *is an isomorphism with right inverse the clutching map* $\mathrm{cl}_{\mathrm{eo}}$.

Proof. This requires one to check that the symbols constructed in (25.8) are homotopic to that given by the map $\mathrm{cl}_{\mathrm{eo}}$. This implies that $\mathrm{ind}_{\mathrm{iso}}\,\mathrm{cl}_{\mathrm{eo}} = \mathrm{Id}$. Since we know that $\mathrm{cl}_{\mathrm{eo}}$ is an isomorphism at the level of homotopy, i.e. as an inverse to (25.1) it follows that the index is also an isomorphism. □

So, this is the $k = 1$ case of the isotropic families index. We want to generalize it in two ways. First to $k > 1$. In fact the restriction to $k = 1$ only occurs in the construction of the annihilation/creation operators and $\mathrm{cl}_{\mathrm{eo}}$. The more significant extension (which is slightly different in form) is to twist the spaces on which the isotropic algebra acts to be a vector bundle over X, instead of just considering straight families. This leads to the Thom isomorphism. We can think of the group on the left in several ways because of Bott periodicity. The most obvious is to identify it as $\mathrm{K}^1_{\mathrm{c}}(X \times \mathbb{R})$ and hence as $\mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}^2)$. This is how the map is usualy described, as

$$\mathrm{ind}_{\mathrm{iso}} : \mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}^2) \longrightarrow \mathrm{K}^0(X) \tag{25.10}$$

implementing Bott periodicity. In fact it is precisely this relationship I want to exploit in order to define the Thom isomorphism

$$\mathrm{K}^0_{\mathrm{c}}(W) \longrightarrow \mathrm{K}^0(X) \tag{25.11}$$

for any complex, or symplectic, vector bundle over X (so it has even rank as a real bundle).

Now, what does this tell us about $\dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$? Basically half of what we want, I would say the hard half given where we are. Namely

Corollary 4. *If* $f : X \longrightarrow G^{-\infty}_{\mathrm{sus;iso}}(\mathbb{R}^k)$ *(is compactly supported) then* $f = \sigma_{\mathrm{iso}}(F)$ *for a compactly supported family* $F : X \longrightarrow \dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ *implies that* f *is homotopic to the identity.*

Proof. The index vanishes, so $[f] \in [X; G^{-\infty}_{\mathrm{sus;iso}}(\mathbb{R}^k)]_{\mathrm{c}}$ must vanish. □

In fact the converse is also true but this involves another argument which we can subsume into what is the last part of the looping sequence.

Proposition 29. *The group* $\dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R};\mathbb{R}^p)$ *is weakly contractible; any compactly supported map into it is homotopic to the constant map to the identity.*

Proof. Given such a map, F, it follows from the corollary that $\sigma_{\mathrm{iso}}(f)$ is homotopic to the identity. Let f_t be such an homotopy, with $f_0 = \mathrm{Id}$, $f_1 = f$. Since it is an elliptic family, and has an invetible quantization at $t = 0$, this can be lifted to an homotopy $F_t : X \times [0,1] \longrightarrow \dot G^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R};\mathbb{R}^p)$ with symbol family f_t and $F_0 = \mathrm{Id}$. This might seem to solve the problem, but not so fast! It follows that F_1 is *a* lift of f to be a family of invertibles, but it is not clear that it is the one we started with. Since we can deform lower order terms in the symbol away, we can arrange that

$$F_1^{-1}F \in \mathcal{C}^\infty_c(X; G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{p+k})) \tag{25.12}$$

since the two quantizations then differ by smoothing terms. So the remaining problem is to show that the image of a family of smoothing perturbations can be deformed away in the bigger group $\dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R};\mathbb{R}^p)$. In fact with the use of a bit more homotopy theory we already have enough to show this. However, I think it is worth doing directly. So the proof of this Proposition is completed by the next. □

Proposition 30. *If $g : X \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{p+k})$ is a compactly supported smooth map then there is an homotopy $G_t \in \mathcal{C}^\infty_c(\dot{G}^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R};\mathbb{R}^p))$ with $G_1 = g$ and $G_0 = \mathrm{Id}$.*

Proof. This can be done quite explicitly using the creation and annihilation operators. Here is the idea but I have not checked the details at all. The crucial point is that we have found an operator of index 1.

First retract g to be of finite rank, on some $\mathbb{C}^N$. Then take two copies and consider the creation and annihilation operators acting on $\mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^2$ as a 2×2 matrix with matrix-operator values

$$\begin{pmatrix} \pi_1 & C \\ A & 0 \end{pmatrix} \tag{25.13}$$

This is invertible, since the null space of A and the lack of range of C are compensated for by the off-diagonal π_1, projecting onto the null space. Now, tensor with matrices on $\mathbb{C}^N$ and consider

(25.14)

$$\begin{pmatrix} \cos(\theta) & g(x)\sin(\theta) \\ -g^{-1}(x)\sin(\theta) & \cos(\theta) \end{pmatrix}^{-1} \begin{pmatrix} \pi_1 \otimes g(x) + \cos(\theta)(\mathrm{Id} - \pi_1) & g(x)\sin(\theta)C \\ -g^{-1}(x)\sin(\theta)A & \cos(\theta). \end{pmatrix}$$

These are invertible matrices, the first normalizes the symbol to be the identity at N, the point where A and C are both flat to 1.

Then, consider the reverse homotopy through invertibles

$$\begin{pmatrix} g(x)\cos(\theta) & g(x)\sin(\theta) \\ -g^{-1}(x)\sin(\theta) & g^{-1}(x)\cos(\theta) \end{pmatrix}^{-1} \times \begin{pmatrix} g(x)\pi_1 + g(x)\cos(\theta)(\mathrm{Id} - \pi_1) & g(x)\sin(\theta)C \\ -g^{-1}(x)\sin(\theta)A & g^{-1}(x)\cos(\theta). \end{pmatrix} \tag{25.15}$$

This starts at the same matrix, at $\theta = \pi/2$ and deforms back to the identity. As I say, I haven't checked this. □

There is another approach to proving Proposition 30 which is a bit more machine-heavy but has other advantages, as I hope we will see. To do this we need the adiabatic limit for operators of order 0.

When talking about the isotropic product, leading to the algebra $\Psi^{0,-\infty}_{\mathrm{qiso,iso}}(\mathbb{R}^k;\mathbb{R}^p)$ that is underlying the looping sequence, I carried along, at least for a while, the adiabatic version of it. Namely the product in (20.3) is actually smooth down to $\epsilon = 0$, just as in the case of smoothing operators discussed earlier. This means we can set up an algebra of adiabatic operators of order 0 which is just the space of functions

$$\mathcal{C}^\infty([0,1]_\epsilon \times {}^q\overline{\mathbb{R}^{2k}} \times ; \mathcal{S}(\mathbb{R}^{2p})) \tag{25.16}$$

with the product given by (20.3). Without going into details this will have both an adiabatic symbol, at $\epsilon = 0$, extending the one in the smoothing case, and now also a 'regular' symbol at the boundary in the second variable (or a 'full symbol if we take Taylor series at this boundary). This second symbol depends on $\epsilon \in [0,1]$, but just as a parameter since the leading term is always just the product (including the operator product in the last variables of course). The lower order terms, the star product, does depend on ϵ. We also have the restriction to $\epsilon = 1$ and if you recall this is how I finally talked about Bott periodicity. I will denote this adiabatic algebra

$$\Psi^{0,-\infty}_{\mathrm{ad\,qiso,iso}}(\mathbb{R}^k; \mathbb{R}^p) \tag{25.17}$$

Now to this we can add the 'dot' conditon of having the functions vanish to infiniite order at the fixed point N on the boundary of the quadratic compactification of $\mathbb{R}^{2k}$. This leads to the subalgebra

$$\dot{\Psi}^{0,-\infty}_{\mathrm{ad\,qiso,iso}}(\mathbb{R}^k; \mathbb{R}^p). \tag{25.18}$$

Proposition 31. *The adiabatic algebras of order* 0 *in* (25.17) *and* (25.18) *(which are non-unital) are Neumann-Fréchet algebras.*

Proof. This requires a combination of the proofs of the earlier cases. □

So now I have at least described the corresponding group of invertible perturbations of the identity by elements of (25.18)

$$\dot{G}^{0,-\infty}_{\mathrm{ad\,qiso,iso}}(\mathbb{R}^k; \mathbb{R}^p). \tag{25.19}$$

What good is it? Well, it has the same symbol maps as the algebra but now valued in the groups. This gives us a diagram where the bottom line is the looping sequence, the middle line is the corresponding sequence with this new group in the middle and the top line is in principle the adiabatic part – what we get by taking the adiabatic symbol (i.e. restricting to $\epsilon = 0$ to the extent that it makes sense).

$$\begin{tikzcd}
G^{-\infty}_{\mathrm{sus}(2),\mathrm{iso}}(\mathbb{R}^p) \arrow[r] & \tilde{G}^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}; \mathbb{R}^p) \arrow[r] & G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}(\mathbb{R}^p) \\
G^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R} : \mathbb{R}^p) \arrow[r] \arrow[u, "\sigma_{\mathrm{ad}}"] \arrow[d, "R"] & \dot{G}^{0,-\infty}_{\mathrm{ad,iso}}(\mathbb{R}; \mathbb{R}^p) \arrow[r, "\sigma_{0,\mathrm{iso}}"] \arrow[u, "\sigma_{\mathrm{ad}}"] \arrow[d, "R"] & \mathcal{C}^\infty([0,1]; G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}(\mathbb{R}^p)) \arrow[u, "\epsilon=0"] \arrow[d, "R"] \\
G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}) \arrow[r] & \dot{G}^{0,-\infty}_{\mathrm{qiso}}(\mathbb{R}; \mathbb{R}^p) \arrow[r, "\sigma_{0,\mathrm{iso}}"] & G^{-\infty}_{\mathrm{sus}\,*,\mathrm{iso,ind}=0}(\mathbb{R}^p)
\end{tikzcd} \tag{25.20}$$

Now, I need to discuss what is happening here carefully, but the – maybe somewhat surprising – point to grasp is that the top row is in fact the delooping sequence for the (terminal) group which is the component of the identity in the suspended group we are by now getting familiar with. So what this diagram is supposed to show conceptually is that the delooping sequence is 'just' the partly quantized delooping sequence for the loop group. Now, before I go into a term by term discussion of (25.20) let me show why it might help us.

Claim 3. *All the vertical arrows, up and down, in* (25.20) *are surjective weak homotopy equivalences with the lifting property for compact families, all three rows are exact and the central column consists of weakly contractible groups.*

So this is just making the same claim but more so! In fact the very central group here is easily seen to be contractible by hand. The surjectivity of the middle R

would imply directly the weak contractibility of the middle group for the looping sequence, but this is where I have failed to find a direct proof.

However, just look at the bottom left rectangle and see that we can use it, if we know a bit – particularly the commutativity – to give a proof of Proposition 30. What we are given is a compactly supported map into the bottom left group. By the lifting property property this comes from a map into the 'adiabatic' group on the left of the middle row. This can then be sent into the really central group. As I said above, it is easy to see geometrically that this group is weakly contractible. Thus it can be deformed away here. Mapping the homotopy forward to the central group on the bottom row gives, by commutativity, an homotopy trivializing the image of the original map.

26. Lecture 23: Iterated periodicity maps Monday, 27 October, 2008

We have earlier shown that there is a semi-classical relation between involutions on $X \times \mathbb{R}^2$ and X in the form of surjective maps with the homotopically-unique lifting property for compactly-supported families

$$\begin{tikzcd} & \mathcal{H}^{-\infty}_{\mathrm{ad,iso}}(\mathbb{R}^k : \mathbb{R}^p) \arrow[dl, "R"'] \arrow[dr, "\sigma_{\mathrm{ad}}"] & \\ \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}) & & \mathcal{H}^{-\infty}_{\mathrm{sus}(2k),\mathrm{iso}}(\mathbb{R}^p) \end{tikzcd}, \ k = 1. \tag{26.1}$$

This generates the periodicity isomorphism (going from right to left)

$$[X; \mathcal{H}^{-\infty}_{\mathrm{sus}(2);\mathrm{iso}}(\mathbb{R}^p)] \longrightarrow [X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{1+p}]_{\mathrm{c}}. \tag{26.2}$$

These constructions can be iterated in slightly different sense. We can increase k in (26.1) and iteratively replace X by $X \times \mathbb{R}^2$ in (26.2). However we need to be a little careful to check that these give the same result. In particular we have not shown the surjectivity or lifting property for the map to the left in (26.1) when $k > 1$. Now we will. We have already shown that the adiabatic map, the one to the right, is surjective and has the homotopically-unique lifting property for every k. Thus the diagram (26.1) does generate a map

$$[X; \mathcal{H}^{-\infty}_{\mathrm{sus}(2k);\mathrm{iso}}(\mathbb{R}^p)] \longrightarrow [X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^{k+p}]_{\mathrm{c}} \text{ for each } k \geq 1. \tag{26.3}$$

Proposition 32. *For any $k > 1$ the map* (26.3) *is the isomorphism given by iteration of* (26.2).

Proof. We need to show is that the quantization of involutions (with relatively compact support) on $\mathbb{R}^{2k} \times X$ to involutions on X can be carried out in k steps, quantizing an $\mathbb{R}^2$ each time.

The clearest way to really check that this is possible is to think about the doubly-adiabatic calculus. So, we are interested in operators on $\mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \times \mathbb{R}^p$ which are separately adiabatic in the two first sets of variables. So the kernels in question are determined by elements

$$F \in \mathcal{C}^\infty([0,1]_{\epsilon_1} \times [0,1]_{\epsilon_2}; \mathcal{S}(\mathbb{R}^{2k_1+2k_2+2p}) \tag{26.4}$$

through the adiabatic-Weyl quantization where the actually kernels for $\epsilon < 1 > 0$ and $\epsilon_2 > 0$ are the

$$\begin{aligned} f(\epsilon_1, \epsilon_2, z_1, z_1', z_2, z_2', z, z') \\ = \epsilon_1^{-k_1}\epsilon_2^{-k_2} F(\epsilon_1, \epsilon_2, \frac{\epsilon_1(z_1+z_1')}{2}, \frac{z_1 - z_1'}{\epsilon_1}, \frac{\epsilon_1(z_2+z_2')}{2}, \frac{z_2-z_2'}{\epsilon_2}, z, z'). \end{aligned} \tag{26.5}$$

Since the two adiabatic parameters are in different variables the same argument as before shows that these doubly-adiabatic famililes form an algebra under composition and that function, as in (26.4), determining the composite can be written down quite directly – namely taking the Fourier transform in the second of the adiabatic variables leads to

$$\hat{H}(t_1, \tau_1, t_2, \tau_2, Z, Z') = \int \hat{F}(t_1, \tau_1, t_2, \tau_2, Z, Z'')\hat{G}(t_1, \tau_1, t_2, \tau_2, Z'', Z'). \tag{26.6}$$

So, this is just two adiabatic limits going on independently. There are really three different symbol maps. One where $\epsilon_1 = 0$ but $\epsilon_2 > 0$ – but this needs to be understood as an adiabatic family. Another one the other way round. And then the doubly-adiabatic symbol at $\epsilon_1 = \epsilon_2 = 0$. Clearly this latter one is the adiabatic symbol of *each* of the other ones. Said more formally, there are three homomorphisms of algebras:

$$
\begin{gathered}
\Psi^{-\infty}_{\mathrm{ad,ad,iso}}(\mathbb{R}^{k_1} : \mathbb{R}^{k_2} : \mathbb{R}^p) \longrightarrow \Psi^{-\infty}_{\mathrm{sus}(2k_1),\mathrm{ad,iso}}(\mathbb{R}^{k_2} : \mathbb{R}^p),\\
\Psi^{-\infty}_{\mathrm{ad,ad,iso}}(\mathbb{R}^{k_1} : \mathbb{R}^{k_2} : \mathbb{R}^p) \longrightarrow \Psi^{-\infty}_{\mathrm{ad,sus}(2k_2),\mathrm{iso}}(\mathbb{R}^{k_1} : \mathbb{R}^p),\\
\Psi^{-\infty}_{\mathrm{ad,ad,iso}}(\mathbb{R}^{k_1} : \mathbb{R}^{k_2} : \mathbb{R}^p) \longrightarrow \Psi^{-\infty}_{\mathrm{sus}(2(k_1+k_2)),\mathrm{iso}}(\mathbb{R}^p).
\end{gathered}
\tag{26.7}
$$

Here the suspended algebras are just the old algebras depending in a Schwartz manner on the additional parameters.

All three maps are surjective, but they are not jointly surjective. Rather they satisfy precisely the relationships given by the commutative diagram

$$
\begin{tikzcd}
& \Psi^{-\infty}_{\mathrm{sus}(2k_1),\mathrm{ad,iso}}(\mathbb{R}^{k_2} : \mathbb{R}^p) \arrow[dr] & \\
\Psi^{-\infty}_{\mathrm{ad,ad,iso}}(\mathbb{R}^{k_1} : \mathbb{R}^{k_2} : \mathbb{R}^p) \arrow[ur] \arrow[rr] \arrow[dr] & & \Psi^{-\infty}_{\mathrm{sus}(2(k_1+k_2)),\mathrm{iso}}(\mathbb{R}^p) \\
& \Psi^{-\infty}_{\mathrm{ad,ad,iso}}(\mathbb{R}^{k_1} : \mathbb{R}^{k_2} : \mathbb{R}^p) \arrow[ur] &
\end{tikzcd}
\tag{26.8}
$$

where all the maps are 'adiabatic symbols'.

So, here is what we need and a good deal more:

Lemma 27. *For (families of) involutions of the form $\gamma_1 + A$, $A \in \Psi^{-\infty}_{\mathrm{ad,ad,iso}}(\mathbb{R}^{k_1} : \mathbb{R}^{k_2} : \mathbb{R}^p; \mathbb{C}^2)$, the four maps corresponding to 'restriction' to $\epsilon_1 = 0 = \epsilon_2 = 0$, $\epsilon_1 = 0, \epsilon_2 = 1$, $\epsilon_1 = 1, \epsilon_2 = 0$ and $\epsilon_1 = \epsilon_2 = 1$ are all surjective with the homotopically-unique lifting property for compactly supported families, and hence induce weak homotopy equivalences between all five spaces of involutions.*

Proof. There is actually nothing really new here, we just have to apply the old procedures thoughtfully. □

In particular of course this completes the proof of Proposition 32. □

Assuming that I have not run out of time, let me now start to discuss the setting of the Thom isomorphism, and closely related isotropic index theorem. This concerns the case of a complex or symplectic vector bundle over a manifold

$$
\begin{tikzcd}
W \arrow[d] \\
X.
\end{tikzcd}
\tag{26.9}
$$

In the symplectic case this is a real vector bundle with a symplectic structure on each fibre, varying smoothly with the base point. Since a symplectic structure on a vector space is just a non-degenerate antisymmetric real bilinear form, the fibres must certainly be even-dimensional.

The relationship between the symplectic and complex structures can be seen geometrically by defining a metric which is compatible with the symplectic structure. Take any real fibre metric h and look at the duality map it defines relative to ω, the symplectic structure:

$$\omega_x(v,w) = g_x(v, J'_x w),\ J'_x : W_x \longrightarrow W_x. \tag{26.10}$$

This uses the non-degeneracy of each of the forms, the one symmetric the other antisymmetric, from which it follows that J'_x is a smooth isomorphism. In fact it is necessary skew-adjoint (it is real) with respect to g since

$$g_x(v, J'_x w) = \omega_x(v,w) = -\omega_x(w,v) = -g_x(w, J'_x v) = -g(J'_x v, w) \tag{26.11}$$

for all v, w. Thus it follows that the eigenvalues of J'_x are pure imaginary and non-zero, since J'_x is invertible from the non-degeneracy of ω and g. Now, applying the same procedure as we did earlier in turning near projections to projections, we can 'compress' J'_x to J_x, also a smooth family of isomorphism which have eigenvalues $\pm i$. Now, J_x is a complex structure on W_x and the metric

$$h(v,w) = -\omega(v, J_x w) \tag{26.12}$$

is the real part of an hermitian parametrix with imaginary part ω.

Exercise 20. Show conversely that on a complex vector space, a choice of positive-definite hermitian inner product generates a symplecti structure, as the imaginary part of this inner product, on the underlying real vector space such that the given complex structure on that even-dimensioanl vector space over the reals is the one constructed above.

Now, we have constructed a complex structure on the vector space which is consistent with the symplectic structure we can introduce the higher dimensional analogue of the annhilation and creation operators and in particular the harmonic oscillator. We already know these in local coordinates, i.e. on $\mathbb{R}^{2n}$. The annihilation operators, $A_j = \partial_{x_j} + x_j = x_j + iD_{x_j}$, where $D_x = \frac{1}{i}\partial/\partial x$, can be assembled into the creation complex:

$$\begin{gathered} A : \mathcal{S}(\mathbb{R}^{2n}; \Lambda^{k,0}) \longrightarrow \mathcal{S}(\mathbb{R}^{2n}; \Lambda^{k+1,0}), \\ u = \sum_\alpha u_\alpha dz^\alpha \longmapsto \sum_\alpha \sum_j A_j u_\zeta dz_j \wedge dz^\alpha. \end{gathered} \tag{26.13}$$

Lemma 28. *With $\Lambda^k = \Lambda^k V$ for an hermitian vector space V, the annihilation complex* (26.13) *is well-defined, i.e. is independent of the choice of complex orthonormal basis used to define it.*

The adjoint complex is the 'creation complex'. Check that $AC + CA = \Box$ acts on complex forms in each degree and reduces to the harmonic oscillator on zero forms.

Corollary 5. *The choice of an hermitian structure on a complex vector space, or of a compatible metric on a symplectic vector space, fixes the associated harmonic oscillator which reduces to the standard harmonic oscillator in local coordinates in which the structures are reduced to the standard ones on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$.*

What we most want from this is uniform finite-rank approximability of smoothing operators. Let $\Psi^{-\infty}_{\text{iso}}(W/X) = \mathcal{S}({}^q\overline{W})$ denote the smooth functions on the quadratic compactification of (the fibres of) a symplectic vector space, made into the space of smooth sections of the bundle of Schwartz-smoothing algebras on the fibres.

Corollary 6. *On the fibres of a complex, or a real symplectic, vector bundle* W *over a manifold* X *there is a sequence of Schwartz-smoothing projections,* $\Pi_N \in \Psi^{-\infty}_{\mathrm{iso}}(W/X)$, *on the fibres, smooth over the base, such that for any* $A \in \Psi^{-\infty}_{\mathrm{iso}}(W/X)$, $\chi\Pi_N A$, $\chi A\Pi_N \to A$ *in* $\Psi^{-\infty}_{\mathrm{iso}}(W/X)$ *for any* $\chi \in \mathcal{C}^\infty_c(X)$.

27. Lecture 24: Thom isomorphism Wednesday, 29 October, 2008

Reminder. *Last time I talked a little about symplectic and complex vector bundles and recalled that on a complex vector bundle with hermitian structure one has a well-defined smooth family of harmonic oscillators on the fibres.*

For a symplectic vector bundle over a manifold X we have shown that there is a well-defined non-commuatative product given on $\mathcal{C}^\infty([0,1];\mathcal{S}(W/X))$, the space of smooth functions on $[0,1]\times {}^q\overline{W}$ which are Schwartz on the fibres, see (20.4) for the explicit formula. Thus we have a bundle of algebras where the fibre above $x\in X$ is $\mathcal{C}^\infty([0,1];\mathcal{S}(W))$ and we will denote the space of global sections of this algebra by $\Psi^{-\infty}_{\mathrm{sl,iso}}(W)$. In fact if we can consider two, or even three, symplectic vector bundles, W_1, W_2 and W_3 over X and observe that the algebra defined last time – separately adiabatic in each of the first two variables and just the product at $\epsilon=1$ in the last, is well-defined. We can denote the algebra of global sections of this bundle of algebras as

$$\Psi^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X\ W_2/X\ W_3/X;\mathbb{C}^N) \tag{27.1}$$

where I have thrown in matrix values for good measure. The fibre at some point $x\in W$ is just $\Psi^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}((W_1)_x\ (W_2)_x\ (W_3)_x;\mathbb{C}^N)$ which is essentially the algebra we were looking at last time.

Now there is one important thing to note. Even though this bundle of algebras is 'twisted', because the bundles W_i need not be trivial, its homotopy properties haven't changed much. To see let us note that, given two symplectic vector bundles over X there is a reasonably natural map of algebras:-

$$B:\Psi^{-\infty}_{\mathrm{iso}}(W_1/X)\ni A\longmapsto A\otimes\pi_B(x)\in\Psi^{-\infty}_{\mathrm{iso}}((W_1\times W_2)/X). \tag{27.2}$$

Here π_B is the self-adjoint projection onto the ground state of some chosen smooth family of harmonic oscillators corresponding to a compatible complex structure on W_2. Different choices of compatible complex structure are homotopic and this homotopy lifts to π_B.

We can modify (27.2) to give us a similar map on involutions and invertibles, where I will add a c subscript to indicate that things are compactly supported – appropriately trivial outside a compact set:-

$$\begin{aligned}B_H:\mathcal{H}^{-\infty}_{\mathrm{iso,c}}(W_1/X)\ni I(x)=\gamma_1+a(x)\longmapsto&\\ \gamma_1+a\otimes\pi_B(x)\in\mathcal{H}^{-\infty}_{\mathrm{iso,c}}((W_1\times W_2)/X)&\\ B_H:G^{-\infty}_{\mathrm{iso,c}}(W_1/X)\ni I(x)=\mathrm{Id}+a(x)\longmapsto&\\ \mathrm{Id}+a\otimes\pi_B(x)\in G^{-\infty}_{\mathrm{iso,c}}((W_1\times W_2)/X).&\end{aligned} \tag{27.3}$$

Here the 'stabilization' if different in each case – remember for $\mathcal{H}$ all the objects are in 2×2 matrices over the obvious one. Of course the homotopies implicit in the definition of the components have to have uniformly compact support.

Proposition 33. *The stabilization maps in* (27.3) *induce homotopy equivalences*

$$\begin{aligned}\Pi_0\big(\mathcal{H}^{-\infty}_{\mathrm{iso},c}(W_1/X)\big)&\simeq\Pi_0\big(\mathcal{H}^{-\infty}_{\mathrm{iso},c}((W_1\times W_2)/X)\big)\\ \Pi_0\big(G^{-\infty}_{\mathrm{iso},c}(W_1/X)\big)&\simeq\Pi_0\big(G^{-\infty}_{\mathrm{iso},c}((W_1\times W_2)/X)\big).\end{aligned} \tag{27.4}$$

Note that these objects are the spaces of sections, and we can only map from the base.

Proof. For a change look at the odd case. The main point is that we certainly know how to do this in case the bundles are trivial, using finite rank approximation, and we have such finite rank approximation available in the general case. Thus the projections, π_N to the span of the eigenfunctions for eigenvalues less than $\operatorname{rank}(W)+2N+1$ are all well-defined and smooth. Thus if $a\in\Psi^{-\infty}_{\text{iso}}(W/X)$ then

$$a\pi_N,\ \pi_N a\to a\in\Psi^{-\infty}_{\text{iso}}(W/X). \tag{27.5}$$

The range of π_N is a vector bundle over X which is readily seen to be isomorphic to the Nth symmetric power of the W as a complex bundle. We don't really need to know this, just that it is a vector bundle. It is a standard result that a (compactly supported) vector bundle can be embedded as a subbundle of any vector bundle over the same set with sufficiently large rank – this can be proved by the same sort of crude method as used to embed into a trivial bundle if the rank required is not needed, as it isn't here. Anyway, this means that we can think of the range of $\pi_N(W_1)$, as a vector bundle over $\pi_M(W_2)$ for some M. This allows us to 'identify' the cut-off operator on W_1, $g_N=\pi_N+\pi_N a\pi_N$, which is invertible if N is large enough, as an of invertible family of homorphisms g'_M of π_M on W_2, extending as the identity off the subbundle. This means we can rotate in the usual way between

$$g_N(x)\otimes\pi_B(W_2)\text{ and }\pi_B(W_1)\otimes g'_M(x)\text{ extended as the identity.} \tag{27.6}$$

This is not quite what we want, since we really want to go the other way, but it is easy to extend it a bit so that it is. Namely any element $\operatorname{Id}+a(x)\in G^{-\infty}_{\text{iso,c}}((W_1\times W_2)/X))$ can be deformed by homotopy to a finite rank perturbation of the identity which acts on the range of $\pi_N(W_1)\otimes\pi_N(W_2)$ for some N. Then $\pi_N(W_2)$ can be complemented to be trivial, and the complement can be embedded in $\pi_M(W_1)-\pi_N(W_1)$ and $\pi_M(W_2)-\pi_N(W_2)$ respectively, for M large enough. Thus we can think of the perturbation as acting on the tensor product of the ranges of two globally trivial families of projections. This actually reduces us to the case right at the beginning (although I did not write down a very elegant proof then). In particular it follows by embedding $\dim\pi_N(W_2)$ copies of the same trivial bundle in W_1 repeatedly in higher 'tranches' $\pi_{M_i}(W_1)-\pi_{M_{i-1}}(W_1)$. Then the argument in the flat case works just as well here to show that such an element can be rotated to act on the range of $\pi_M(W_1)\otimes\pi_1(W_2)$ and hence is in the range of the second map in (27.4).

The part for idempotents can be proved similary. $\square$

Corollary 7. *For any symplectic bundle W (of positive fibre rank of course) over a manifold there are natural idenifications*

$$\begin{aligned}\Pi_0\big(\mathcal{H}^{-\infty}_{\text{iso},c}(W/X)\big)&\simeq \mathrm{K}^0_c(X)\\ \Pi_0\big(G^{-\infty}_{\text{iso},c}(W/X)\big)&\simeq \mathrm{K}^1_c(X).\end{aligned} \tag{27.7}$$

Proof. Apply the preceeding proposition twice in each case to the product of W with a trivial bundle, say with symplectic fibre $\mathbb{R}^2$. $\square$

Now, let us proceed to the Thom isomorphism. This concerns a vector bundle W over a manifold X. As note above, the Thom isomorphism

$$\text{Thom}:\mathrm{K}^0_{\mathrm{c}}(X)\longrightarrow\mathrm{K}^0_{\mathrm{c}}(X) \tag{27.8}$$

is fixed by the homotopy class of the smoothly varying symplectic form on the fibres of W, so it depends on some orientation information, but otherwise it is well-defined. We will get it, by semiclassical quantization.

I have alread briefly discussed the space

$$\Psi^{-\infty}_{\mathrm{ad,iso}}(W\times\mathbb{R}^{2p}/X;M(N,\mathbb{C}))\subset\mathcal{C}^\infty([0,1]\times\mathcal{S}(W\times\mathbb{R}^{2p}/X;M(N,\mathbb{C})) \tag{27.9}$$

which consists of all the smooth functions on $[0,1]\times W\times\mathbb{R}^{2p}$ with values in $M(N,\mathbb{C})$ and which have entries which are uniformly Schwartz functions on the fibres of $[0,1]\times W\times\mathbb{R}^{2p}$ as a bundle over $X\times[0,1]$. The product is the adiabatic product with respect to the symplectic structure on each fibre W_x. For $N=2$ we consider the space of adiabatic perturbations of γ_1 and which are involutions

(27.10)

$$\mathcal{H}^{-\infty}_{\mathrm{ad,iso}}(W\times\mathbb{R}^{2p}/X)=\left\{I=\gamma_1+a,\ a\in\Psi^{-\infty}_{\mathrm{ad,iso}}(W\times\mathbb{R}^{2p}/X;M(N,\mathbb{C})),\ I^2=\mathrm{Id}\right\}.$$

The symbolic properties of this space should by now be fairly clear. Namely the adiabatic sybol map restricts to

(27.11)

$$\sigma_{\mathrm{ad}}:\mathcal{H}^{-\infty}_{\mathrm{ad,iso,c}}(W\times\mathbb{R}^{2p}/X)\longrightarrow\mathcal{C}^\infty_c(X;\mathcal{H}^{-\infty}_{\mathrm{sus}(W),\mathrm{iso}}(\mathbb{R}^p))=\mathcal{S}_{\mathrm{c}}(W;\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p))$$

which is the involutions of the form γ_1+b, where $b:W\longrightarrow\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p)$ is uniformly Schwartz on the fibres of W and has support in the preimage of a compact set in the base.

Lemma 29. *The symbol map* (27.11) *is surjective and the preimage of each element is connected.*

Since we are now talking about families, necessarily defined on X rather than an arbitrary manifold, this is the analogue of the homotopically-unique lifting property.

Proof. At some point I will write out a general result for these symbolic lifting constructions. This is no different to most others. Namely we use the adiabatic symbol map, the properties if which follow from the case we have been discussing where W is the trivial product $X\times\mathbb{R}^{2k}$ since locally over X there is a trivialization in which the symplectic form reduces to the Darboux form on $\mathbb{R}^{2k}$. Thus we have a short exact and multiplicative symbol sequence

$$\begin{aligned}\epsilon\Psi^{-\infty}_{\mathrm{ad,iso,c}}(W\times\mathbb{R}^{2p}/X;M(N,\mathbb{C}))\longrightarrow\Psi^{-\infty}_{\mathrm{ad,iso,c}}(W\times\mathbb{R}^{2p}/X;M(N,\mathbb{C}))&\\ \xrightarrow{\sigma_{\mathrm{ad}}}\mathcal{S}_{\mathrm{c}}(W;\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p)).&\end{aligned} \tag{27.12}$$

Thus, we can lift the an element in the targen space in (27.11) to $I_0'\gamma_1+b_0$ where $(I_0')^2=\mathrm{Id}+\epsilon c$. The same iteration argument used earlier for involutions and Borel's Lemma allows us to improve this to $(I')^2=\mathrm{Id}+c'$ where c' vanishes to infinite order with ϵ, so is just smooth (and rapidly vanishing) down to $\epsilon=0$ in the ordinary sense. Then the same integral formula as before allows this to be corrected to an involution in $[0,\epsilon_0]$ for some $\epsilon_0>0$ and finally stretching the parameter space we find a lift of the symbol as desired and (27.11) is therefore surjective.

A modification of this argument shows that any two lifts are homotopic as families, i.e. the set of lifts is connected. □

Now, the Thom isomorphism comes from looking at the restriction operator R to $\epsilon=1$ as before. This gives a diagram (written the other way compared to the perioidicity case) where my notation is a bit out of hand

$$
\begin{tikzcd}
& \mathcal{H}^{-\infty}_{\mathrm{ad,iso,c}}(W/X : \mathbb{R}^{2p}) \arrow[dl, "\sigma_{\mathrm{ad}}"'] \arrow[dr, "R"] & \\
\mathcal{S}_{\mathrm{c}}(W; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p)) \arrow[d, "\Pi_o"] & & \mathcal{H}^{-\infty}_{\mathrm{iso,c}}(W \times \mathbb{R}^{2p}/X) \arrow[d, "\Pi_o"] \\
\mathrm{K}^0_{\mathrm{c}}(W) \arrow[rr, "\mathrm{Thom}"] & & \mathrm{K}^0_{\mathrm{c}}(X).
\end{tikzcd}
\tag{27.13}
$$

Here the vertical 'Π_0' maps are just the passage to components. Thus we know that both maps on the left side are surjective and that the lift is unique up to homotopy. So the map along the bottom is well-defined

Theorem 8. *[Thom isomorphism] For any symplectic vector budle the map on the bottom in* (27.13) *is an isomorphism and R on the right is surjective.*

So, this is just the same as Bott periodicity in case $W = \mathbb{R}^{2k} \times X$ which we finally discussed properly last time. We already know how to change the dimension 'p' of the isotropic image space, so the Thom map really is well-defined. To prove that it is an isomorphism we will bring out the main tool we have used so far, which is the ability to do two things at once.

As mentioned above, we can consider adiabatic families, as we did last time, with respect to two symplectic bundles W_1 and W_2 over X. The hardes thing here is really the notation! I will let $\mathrm{sus}(W)$, as a subscript, replace $\mathrm{sus}(2p)$ and here mean that we are considering sections which are Schwartz, in an appropriate sense, on the fibres of W. I have already used this without discussing it in (27.11). The doubly adiabatic algebra, which has two parameters, one in the W_1 slots and the other in the W_2 slots, is a non-commutative product on the spaces of sections which have compact support in the base (or arbitrary support, both work but we mostly want the compact case)

$$\{a \in \mathcal{C}^\infty_c([0,1] \times [0,1] \times {}^q\overline{W_1} \times_X {}^q\overline{W_2} \times {}^q\overline{\mathbb{R}^{2p}}); a \equiv 0 \text{ at all boundaries}\}. \tag{27.14}$$

Here the quadratic compactifications can be replaced by the radial ones, since we are considering functions flat at the boundaries anyway. The $\times_X$ means the fibre product, so really we are taking the products of the compactifications of $(W_1)_x$ and $(W_2)_x$ and making them into a bundle over X.

I leave it to you to carefully define the corresponding space of involutions

$$\mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X : W_2/X : \mathbb{R}^p) \tag{27.15}$$

which are doubly-adiabatic-smoothing perturbations of γ_1, so as usual are 2×2 matrices. Now there are a total of seven 'restriction maps' we wish to consider. Six of them correspond to restricting to one of $\epsilon_1 = \epsilon_2 = 0$ (the doubly-adiabatic symbol), to $\epsilon_i = 0$, $i = 1, 2$, the two single adiabatic symbols, $\epsilon_i = 1$, $i = 1, 2$, the two restriction maps and $\epsilon_1 = \epsilon_2 = 1$. The seventh map is the restriction to $\epsilon_1 = \epsilon_2$

($=\epsilon$ if you like). These are of the form

(27.16)
$$
\begin{aligned}
\sigma_{\mathrm{ad}(W_1\times W_2)} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X : W_2/X : \mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{sus}(W_1\times W_2),\mathrm{iso}}(\mathbb{R}^p)\\
\sigma_{\mathrm{ad}(W_1)} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X : W_2/X : \mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{sus}(W_1),\mathrm{ad,iso}}(W_2/X : \mathbb{R}^{2p})\\
\sigma_{\mathrm{ad}(W_2)} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X : W_2/X : \mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{ad,sus}(W_2),\mathrm{iso}}(W_1/X : \mathbb{R}^{2p})\\
R_{\epsilon_2=1} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X : W_2/X : \mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{ad,iso}(\omega)}(W_1/X : (W_2\times\mathbb{R}^{2p})/X)\\
R_{\epsilon_1=1} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X : W_2/X : \mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{ad,iso}(\omega)}(W_2/X : (W_1\times\mathbb{R}^{2p})/X)\\
R_{\epsilon_1=\epsilon_2=1} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X : W_2/X : \mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{iso}(\omega)}((W_1\times_X W_2\times\mathbb{R}^{2p})/X)\\
R_{\epsilon_1=\epsilon_2} &: \mathcal{H}^{-\infty,-\infty}_{\mathrm{ad,ad,iso}}(W_1/X : W_2/X : \mathbb{R}^p) \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{ad,iso}}((W_1\times_X W_2)/X : \mathbb{R}^{2p}).
\end{aligned}
$$

Here I just thought of the idea of using $\mathrm{iso}(\omega)$ instead of iso to mean that the space is a symplectic vector space instead of operators on a vector space. The main thing to swallow is that all these maps exist. Here is a diagram in ϵ_1, ϵ_2 space:-

So, there are lots of things we could easily prove about this picture. However, recall that at the level of functions these maps really are restrictions to the sets in question. So they have the obvious consistency properties that I will not write out but will use below.

Now recall what we want to use this set-up for. We want to consider three bundles, namely

$$
\begin{gathered}
W_1\times_X W_2 \longrightarrow X \text{ bundle over X}\\
W_1\times_x W_2 \longrightarrow W_1 \text{ bundle over } W_1\\
W_1 \longrightarrow X \text{ bundle over } X.
\end{gathered}
\tag{27.17}
$$

although for vector bundles the notation $W_1\times W_2$ for the fibre product $W_1\times_X W_2$ is conventional, here I am just trying to emphasize what things really are. In all three case we have the Thom isomorphism and what we will use the doubly-adiabatic set up to show:-

Proposition 34. *For any pair of symplectic vector bundles over a manifold the three Thom maps give a commutative diagram*

(27.18)
$$
\begin{tikzcd}
\mathrm{K}^0_c(W_1\times_X W_2) \arrow[d, "\mathrm{Thom}"'] \arrow[dr, "\mathrm{Thom}"] & \\
\mathrm{K}^0(X). & \mathrm{K}^0_c(W_1) \arrow[l, "\mathrm{Thom}"]
\end{tikzcd}
$$

Proof. The main claim, that I am not for the moment going to write down, is that the *third* map above is surjective. This is the same sort of argument as in Lemma 29 above, with a few extra twists because of the two parameters – but not essentially harder. It follows from this that the first map is also surjective, because this, by consistency of the symbols, is the adiabatic symbol map applied to the range of the third map.

Thus we can start of with an element of $\kappa \in \mathcal{H}^{-\infty}_{\mathrm{sus}(W_2),\mathrm{iso}}(W_1 \times \mathbb{R}^{2p}/X)$ which is in the image, under restriction to $\epsilon_1 = 1$ of an element $K \in \mathcal{H}^{-\infty}_{\mathrm{ad},\mathrm{sus}(W_2),\mathrm{iso}}(W_1/X : \mathbb{R}^{2p})$, meaning its class is in the image of the top sloping Thom map. Using the surjectivity discussed above, this is the image under the third map in (27.17) of an element $\tilde{K}$ in the doubly-adiabatic space. The image of this under the fifth map in (27.17) is therefore a lift of κ which defines the lower Thom map on the right, i.e. restricting this element to $\epsilon_2 = 1$ gives the image of κ in $\mathrm{K}^0(X)$. However the restriction of $\tilde{K}$ under the last map gives the image of K in this space, and by the consistency of these restricition maps these are the same. Thus the Proposition is proved. $\square$

Proof of Theorem 8 – Thom isomorphism. Consider any symplectic vector bundle W. We know that this can be given a complex structure compatible with its symplectic structure (and determined up to homotopy). As a complex vector bundle, W can be embedded as a subbundle of $\mathbb{C}^N$ for some N and hence complemented to this trivial bundle with another complex bundle $\tilde{W}$. Conversely $\tilde{W}$ has a symplectic structure so we can arrange, after a homotopy of the symplectic structure which does not affect the Thom map, that

$$W \times_X \tilde{W} = W \oplus \tilde{W} = \mathbb{R}^{2N} \times X \tag{27.19}$$

with consistent symplectic structures. Now Proposition 34 gives the commutative diagram

$$\begin{tikzcd} \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2N} \times X) \arrow[dd, "\mathrm{Thom}=p_{\mathrm{sl}}"'] \arrow[dr, "\mathrm{Thom}"] & \\ & \mathrm{K}^0_{\mathrm{c}}(W) \arrow[dl, "\mathrm{Thom}"] \\ \mathrm{K}^0(X). & \end{tikzcd} \tag{27.20}$$

Thus the vertical map is Bott periodicity, so is an isomorphism. It follows that the lower map on the right, the Thom map for W as a bundle over X is surjective. It also follows that the upper map on the right, which is the Thom map for $W \times_X \tilde{W} = \mathbb{R}^{2N} \times X$ as a symplectic bundle over W is injective.

We have shown that the Thom map is *universally* surjective. However the upper map on the right, which we know to be injective, is an example of such a map. So it must also be surjective, hence an isomorphism. Hence the general Thom map on the lower right is also always an isomorphism. $\square$

Here is some material that seems to have been orphaned; I will work out where to put it some time!

Let me recall, and extend, some of the basic results about the space $\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p)$; especially since the treatment I gave was rather brief, to say the least.

Proposition 35. *Two compactly supported smooth maps $I_i : X \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p)$, $i = 0, 1$, (through such maps) if and only if they are conjugate under a smooth compactly supported map $g : X \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p; \mathbb{C}^2)$ which is homotopic through such* *maps to the identity:*

$$I_1(x) = g^{-1}(x)I_0(x)g(x) \ \forall \ x \in X. \tag{27.21}$$

Proof. Certainly if $g = g_1$ for a compactly supported homotopy $g_\cdot : [0,1] \times X \longrightarrow G^{-\infty}_{\text{iso}}(\mathbb{R}^p; \mathbb{C}^2)$ with $g_0 = \text{Id}$ then $I_t = g_t^{-1} I_0 g_t(x)$ is an homotopy from I_0 to I_1.

To see the converse, we will solve a differential equation. Recall that for an homotopy of involutions

$$\dot{I}_t I_t + I_t \dot{I}_t = 0 \Longrightarrow \dot{I}_t = I_t^+ \dot{I}_t I_t^- + I_t^- \dot{I}_t I_t^+, \tag{27.22}$$

meaning that the derivative must be off-diagonal with respect to the involution. Now, if we want to solve $I_t = g_t^{-1} I_0 g_t$, for the moment for fixed x, we can differentiate and try to solve

$$\dot{I}_t = \gamma(t) = -(g_t^{-1}\dot{g}_t)I_t + I_t(g_t^{-1}\dot{g}_t). \tag{27.23}$$

Now, it follows from (27.22) that this identity is satisfied if we take arrange that

$$g_t^{-1}\dot{g}_t = \frac{1}{2}\big(- I_t^+ \dot{I}_t I_t^- + I_t^- \dot{I}_t I_t^+ \big). \tag{27.24}$$

Thus, we can simply (try to) solve

$$\begin{gathered} \dot{g}_t = \frac{1}{2} g_t \gamma(t), \ g_0 = \text{Id} \\ \Longleftrightarrow g_t = \text{Id} + a(t), \ a(t) = \int_0^t \big(\gamma(s) + a(s)\gamma(s)\big) ds. \end{gathered} \tag{27.25}$$

Now, the integral equation has a unique solution by standard contraction arguments, and it follows from this uniqueness that the solution is smooth in the parameters. Moreover it follows that $g_t(x) = \text{Id} + a(t,x)$ is always invertible, and is equal to the identity outside a compact set in X. For instance the invertibility follows by following the determinant since

$$\frac{d}{dt} \log \det(g_t(x)) = \operatorname{tr}(\gamma(t,x)). \tag{27.26}$$

Exercise 21. Do it – check that it works in each seminorm and from uniqueness the solution to (27.25) is Schwartz.

Now, going backwards it follows that g_t implements the conjugation we want. □

Lemma 30. *For a symplectic vector bundle W over X, two elements*

$$I_i \in \mathcal{H}^{-\infty}_{\text{iso},c}(W/X; \mathbb{C}^N) \ i = 0, 1,$$

are in the same component if and only if there exists $g \in G^{-\infty}_{\text{iso}}(W/X; \mathbb{C}^N \otimes \mathbb{C}^2)$ in the component of the identity such that $I_1 = g^{-1} I_0 g$.

Proof. The uniqueness of the method used in the previous proof means that it works in the same way for sections of these bundles over X and then this is simply a restatement of the conclusion. □

I did show earlier that any smooth map of compact support $I : X \longrightarrow \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^p)$ are homotopic to simple sections of the form

$$
\begin{gathered}
\tilde I = \gamma_1 - 2E_+ \otimes P_-(x) + 2E_+ \otimes P_+(x),\ P_\pm(x)^2 = P_\pm(x), \\
\Pi_N P_+(x) = P_+(x)\Pi_N = P_+(x), \\
(\Pi_M - \Pi_N)P_-(x) = P_+(x)(\Pi_M - \Pi_N) = P_+(x)\ \forall\ x \in X, \\
P_\pm(x) = A_\pm P_\mp(x) B_\pm \text{ are constant for } x \in X \setminus K,\ K \Subset X.
\end{gathered}
\tag{27.27}
$$

Here M and N are integers and $A_\pm$ and $B_\pm$ are matrices acting on the range of π_M.

To extend this to the case of sections of a symplectic bundle W is straightforward except that we cannot demand the constancy outside a compact set unless we demand that the bundle W itself is trivial, and has constant symplectic structure outside such a set – I have been a bit cavalier about this. Fortunatly it is not really a problem. Instead we just demand the conjugation equivalence in the complement of a compact set (and I will change things retrospectively at some point).

Proposition 36. *Any family of involutions, $I \in \mathcal{H}^{-\infty}_{\mathrm{iso}}(W/X)$, is homotopic over any open subset $\Omega \subset X$ with compact closure to one of the form*

$$
\begin{gathered}
\tilde I = \gamma_1 - 2E_+ \otimes P_-(x) + 2E_+ \otimes P_+(x),\ P_\pm(x)^2 = P_\pm(x), \\
\Pi_N(x) P_+(x) = P_+(x)\Pi_N(x) = P_+(x), \\
(\Pi_M(x) - \Pi_N(x))P_-(x) = P_+(x)(\Pi_M(x) - \Pi_N(x)) = P_+(x)\ \forall\ x \in X, \\
P_\pm(x) = A_\pm(x) P_\mp(x) B_\pm(x) \text{ in } \Omega' \setminus \Omega,\ \Omega' \text{ open},\ \overline{\Omega'} \Subset X,\ \overline{\Omega} \subset \Omega'.
\end{gathered}
\tag{27.28}
$$

28. Lecture 25: Isotropic families index theorem Friday, 31 October, 2008

This lecture did not go over so well, there were definitely blank stares at what I thought was the punchline! I think one problem was my insistence on working in the generality of a symplectic bundle W over X instead of working on operators on (Schwartz functions on the fibres of) a real vector bundle U over X – which is the special case where $W = U \oplus U'$ with the symplectic form coming from the pairing. It is a bit late now to undo this. In fact I clearly tried to include too much in this lecture, as I discovered when I tried to write it up! Sorry about that, but I will press on to the Atiyah-Singer Theorem. The argument I will use there is essentially the same as this one, so maybe it will become clearer as we go on. This isotropic index theorem is not actually needed in the proof.

Here is the outline I had originally for the lecture, which is pretty much what I did. I will try to write out a different version below in the hopes that it will be more helpful. Of course, part of the problem was that I did not take the time to do things in detail.

Outline:-

(1) Index for $P \in \Psi^k_{\mathrm{iso}}(\mathbb{R}^k)$, elliptic, is an integer.
(2) Index for an elliptic family $P \in \mathcal{C}^\infty(X; \Psi^k_{\mathrm{iso}}(\mathbb{R}^k))$, X compact, is an element of $\mathrm{K}^0(X)$ determined by choosing a parametrix and defining

$$\begin{gathered}\operatorname{ind}_{\mathrm{iso}}(P) = [I(P,Q)] \\ I(P,Q)(x) = \begin{pmatrix} 1-2R_L^2 & 2R_L(\mathrm{Id}+R_L)Q \\ 2R_RP & -\mathrm{Id}+2R_R^2 \end{pmatrix} \in \mathcal{C}^\infty(X; \mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^k), \\ R_L = \mathrm{Id}-PQ,\ R_R = \mathrm{Id}-QP,\end{gathered} \tag{28.1}$$

from (25.5).
(3) More generally we want to consider a symplectic bundle $W \longrightarrow X$ (I will take the base here to be compact to avoid having to qualify various statements and come back to the non-compact case if necessary – it isn't seriously harder). Then we could take an elliptic family

$$P \in \Psi^k_{\mathrm{iso}}(W/X; \mathbb{C}^N) \tag{28.2}$$

where this stands for the space of sections – so for each $x \in X$ we have an element $P(x) \in \Psi^k_{\mathrm{iso}}(W_x; \mathbb{C}^N)$ which varies smoothly with $x \in X$. In fact, for reasons of generality but also it turns out for topolical reasons that I will mention somewhere, we will consider a pair of complex (smooth of course) bundles over X, $\mathbb{E} = (E_+, E_-)$ which I write as a superbundle for fun and brevity. Then we want an elliptic family

$$P \in \Psi^k_{\mathrm{iso}}(W/X; \hom(\mathbb{E})) \text{ elliptic,} \tag{28.3}$$

which means 'formally mapping sections of E_+ to sections of E_-.' Of course the things we have are not operators so what this means is

$$\begin{gathered}P(x) \in \Psi^k_{\mathrm{iso}}(W_x; \hom(E_+(x), E_-(x)) = \mathcal{C}^\infty({}^q\overline{W_x}; \hom(E_+(x), E_-(x)), \\ P \in \Psi^k_{\mathrm{iso}}(W/X; \hom(E_+, E_-) = \mathcal{C}^\infty({}^q\overline{W}; \pi^* \hom(E_+, E_-)\end{gathered} \tag{28.4}$$

where I have written out what these are as spaces of functions (well, sections of bundles). Thus $\hom(\mathbb{E}) = \hom(E_+, E_-)$ is the bundle of homomorphisms on the fibres. The form algebras when $\mathbb{E} = (E, E)$ is a fixed bundle and more generally form modules and can be composed when the bundles 'in the middle' are the same.

(4) So, what is the index of an elliptic operator (28.3)? It is supposed to be given by the same formula (28.1). Of course we have to remember the bundles. Still, the construction of a smooth parametrix goes through unchanged to give

$$\begin{gathered} Q \in \Psi^{-k}_{\mathrm{iso}}(W/X; \hom(\mathbb{E}^-)),\ \mathbb{E}^- = (E_-, E_+), \\ R_L = \operatorname{Id} - PQ \in \Psi^{-\infty}_{\mathrm{iso}}(W/X; \hom(E_-)),\ R_R = \operatorname{Id} - QP \in \Psi^{-\infty}_{\mathrm{iso}}(W/X; \hom(E_+)). \end{gathered} \tag{28.5}$$

So, let us embed $\mathbb{E} \longrightarrow (\mathbb{C}^N, \mathbb{C}^N)$ which is to say, embed both bundles in trivial bundles which can be taken to have the same rank. Let $\pi_\pm(x)$ be the projections onto the ranges of the embeddings of $E_\pm$ in $\mathbb{C}^N$. Now if we look at (28.1) we get the central block in

$$\tilde{I}(P, Q) = \begin{pmatrix} \operatorname{Id} - \pi_+(x) & 0 & 0 & 0 \\ 0 & 1 - 2R_L^2 & 2R_L(\operatorname{Id} + R_L)Q & 0 \\ 0 & 2R_R P & -\operatorname{Id} + 2R_R^2 & 0 \\ 0 & 0 & 0 & -(\operatorname{Id} - \pi_-(x)) \end{pmatrix}. \tag{28.6}$$

So, this has been stabilized into a $\mathbb{C}^2 \otimes \mathbb{C}^N$ and hence can be mapped into $\mathcal{H}^{-\infty}_{\mathrm{iso}}(W \times \mathbb{R}^2/X)$. This then is our index from the point of view of involutions. Later I will do the more conventional (but of course very closely related) stabilization to projections.

(5)

Theorem 9. *For an elliptic family* (28.4) *the class of the involution* (28.6) *does not depend on the choices involved and so defines an index class which only depends on the symbol (and of course the bundles):*

$$\operatorname{ind}_{\mathrm{iso}}(P) = \operatorname{ind}_{\mathrm{iso}}(\sigma(P); \mathbb{E}) \in \mathrm{K}^0(X). \tag{28.7}$$

The index map (say from elliptic-parametrix pairs) factors through the semiclassicl index map giving the Thom isomorphism in terms of a map $[\sigma]$ *to be described giving a commutative diagram*

$$\begin{tikzcd} \{(P, Q)\} \arrow[rr, "{[\sigma]}"] \arrow[dr, "\operatorname{ind}_{\mathrm{iso}}"'] & & \mathrm{K}^0_c(W) \arrow[dl, "\operatorname{ind}_{\mathrm{sl}} = \mathrm{Thom}"] \\ & \mathrm{K}^0(X). & \end{tikzcd} \tag{28.8}$$

Furthermore the vanishing of $\operatorname{ind}_{\mathrm{iso}}(P)$ *is a necessary and sufficient condition for the existence of a perturbation* $T \in \Psi^{-\infty}_{\mathrm{iso}}(W/X; \mathbb{E})$ *such that* $P + T$ *is invertible with inverse* $(P + T)^{-1} \in \Psi^{-k}_{\mathrm{iso}}(W/X; \mathbb{E}^-)$.

(6) Let me proceed to the idea of the proof without first defining the K-class of the symbol, but that is really what I am working towards. For the moment I will work with projections, but it might be better to do it with involutions. So, what we have done above is embedd the two bundles as

projection-valued sections of $\mathbb{C}^N$ over X. So that is our data:

$$\begin{gathered}\pi_\pm : X \longrightarrow M(N;\mathbb{C}),\ \pi_\pm^2 = \pi_\pm \text{ and } p,\ q \in \mathcal{C}^\infty(\mathbb{S}W; M(N;\mathbb{C})),\\ \pi_-(x)p(w_x) = p(w_x) = p(w_x)\pi_+(x),\ q(w_x)\pi_-(x) = p(w_x) = \pi_+(x)q(w_x),\\ p(w_x)q(w_x) = \pi_-(x),\ q(w_x)p(w_x) = \pi_+(x),\ w_x \in \partial^q\overline{W_x}.\end{gathered} \tag{28.9}$$

Here q is just the inverse of $p = \sigma(P)$ extended as zero outside the bundles. So we need to understand how this gives an element of $\mathrm{K}^0_{\mathrm{c}}(W)$.

(7) To do so, let me generalize the symbolic data in (28.9). Namely we can take exactly the same thing except that we allow the projections to depend on the variables on W but require them to be smooth up to the boudary of the quadratic compactification:-

$$\begin{gathered}\pi_\pm : {}^q\overline{W} \longrightarrow M(N;\mathbb{C}),\ \pi_\pm^2 = \pi_\pm \text{ and } p,\ q \in \mathcal{C}^\infty(\mathbb{S}W; M(N;\mathbb{C})),\\ \pi_-p = p = p\pi_+,\ q\pi_- = p = \pi_+q,\ pq = \pi_-,\ qp = \pi_+ \text{ on } \partial^q\overline{W}.\end{gathered} \tag{28.10}$$

(8) Now what we want to do is to *quantize* this more general data. We can do this in the following way:-

Proposition 37. *For general data as in* (28.10) *there exist semiclassical families of projections*

$$\begin{gathered}\Pi_\pm \in \Psi^0_{\mathrm{ad}}(W/X;\mathbb{C}^N) \textit{ with } \sigma_{\mathrm{ad}}(\Pi_\pm) = \pi_\pm,\ \sigma_{\mathrm{iso}}(\Pi_\pm) = \pi_\pm \textit{ on } \partial^q\overline{W} \textit{ and}\\ P,\ Q \in \Psi^0(W/X;\mathbb{C}^N) \textit{ s.t. } \sigma_0(P) = p,\ \sigma_0(Q) = q,\\ R(\Pi_-)P = PR(\Pi_+),\ R(\Pi_+)Q = QR(\Pi_-),\\ R_R = R(\Pi_-) - PQ,\ R_L = R(\Pi_+) - QP \in \Psi^{-\infty}_{\mathrm{iso}}(W/X;\mathbb{C}^N).\end{gathered} \tag{28.11}$$

Here R is our usual restriction to $\epsilon = 1$ of the adiabatic family.

Proof. We can quantize the $\pi_\pm$ to semiclassical families, with constant standard symbol (as a function of ϵ.) Such a family is unique up to homotopy through such families. Then we can quantize p and q to operators P' and Q', replace these by $R(\Pi_-)P'R(\Pi_+)$ and $R(\Pi_+)Q'R(\Pi_-)$. Then we need to correct a little to get the remainder terms to be smoothing. $\square$

Really this is just a variant of the elliptic construction.

(9) So, we can write down the 'same' involution still defining a K-class:-

$$\tilde{I}(P,Q,\Pi_\pm) = \begin{pmatrix} \mathrm{Id}-\Pi_+(x) & 0 & 0 & 0\\ 0 & 1-2R_L^2 & 2R_L(\mathrm{Id}+R_L)Q & 0\\ 0 & 2R_RP & -\mathrm{Id}+2R_R^2 & 0\\ 0 & 0 & 0 & -(\mathrm{Id}-\Pi_-(x))\end{pmatrix}. \tag{28.12}$$

The uniqueness of the construction up to homotopy shows that this defines an index from the 'general data' in (28.10).

(10) I now need to define the map for the 'generalized data' into $\mathrm{K}^0_{\mathrm{c}}(W)$.

(11) So, how does it improve things to make the problem harder? Well, there is one thing to notice here.

Proposition 38. *The data where $\pi_\pm$ are equal to the same constant projection outside a compact subset of W and $p = q = \mathrm{Id}$ on the range of this projection* *'generates' all generalized data up to operations, stability and homotopy, under which the index is constant.*

So, the proof is really this observation plus the fact that the index in this case is given by semiclassical quantization.

So that was my original outline. Let me approach things from the other end writing out the maps that do exist more carefully. Thank you Paul for pointing out that I had lost a lot qiso's – I was using iso instead. This really does not make any *significant* difference; it just makes a difference. Note that X is taken compact below.

First of all, what exactly is the families isotropic index? Consider the subspace of elliptic operators

$$\mathrm{Ell}^0_{\mathrm{qiso}}(W/X;\mathbb{E}) \subset \Psi^0_{\mathrm{qiso}}(W/X;\mathbb{E}) = \mathcal{C}^\infty({}^q\overline{W};\pi^*\hom(\mathbb{E})). \tag{28.13}$$

As a space of 'functions' the space on the right consists of the smooth sections over ${}^q\overline{W}$, the quadratic compactification of W, of the pull-back of the homomorphism bundle from E_- to E_+ over X. Thus at each point of $w \in {}^q\overline{W}$ one has a linear map from $E_+(x)$ to $E_-(x)$ where $\pi(w) = x$ and this depends smoothly on w. The elliptic elements are those for which the symbol, p, just the restriction to the sphere at infinity, is invertible. In particular the ranks of $E_\pm$ must be equal for there to be any elliptic elements.

Lemma 31. *Any elliptic family $P \in \mathrm{Ell}^0_{\mathrm{qiso}}(W/X;\mathbb{E})$ has a parametrix*

$$Q \in \mathrm{Ell}^0_{\mathrm{qiso}}(W/X;\mathbb{E}^-,)\ \mathbb{E}^- = (E_-,E_+),$$

meaning that

$$R_L = \mathrm{Id} - QP \in \Psi^{-\infty}_{\mathrm{iso}}(W/X;E_+),\ R_R = \mathrm{Id} - PQ \in \Psi^{-\infty}_{\mathrm{iso}}(W/X;E_-) \tag{28.14}$$

and any two parametrices are smoothly homotopic within parametrices.

Proof. Pretty much the same old constructions using the symbol map, iteration and asymptotic summation. □

So we will consider the big set of pairs, of elliptic elements and parametrices as in (28.14), together with smooth embeddings $\pi_\pm$ of the bundles $E_\pm$ as subbundles of $\mathbb{C}^N$ over X and denote this $\mathcal{P}^0_{\mathrm{qiso}}(W/X;\mathbb{E}) = \{P,Q,\pi_\pm\}$.

Definition 7. Let $\mathcal{D}(W)$ be the collection of *elliptic data* for W with elements $(p',\pi_\pm)$ where $\pi_\pm \in \mathcal{C}^\infty(X;M(N,\mathbb{C})$ are projection-valued and $p \in \mathrm{Iso}(\mathbb{S}W;\pi)$ is a smooth isomorphism between the pull-back of the range of π_+ and the pull-back of the range of π_- over $\mathbb{S}W$.

Each element of $\mathcal{P}^0_{\mathrm{qiso}}(W/X;\mathbb{E})$ defines an involution through

$$I(P,Q,\pi_\pm) = \\ \begin{pmatrix} \mathrm{Id} - \pi_+(x) & 0 & 0 & 0 \\ 0 & 1-2R_L^2 & 2R_L(\mathrm{Id}+R_L)Q & 0 \\ 0 & 2R_RP & -\mathrm{Id}+2R_R^2 & 0 \\ 0 & 0 & 0 & -(\mathrm{Id}-\pi_-(x)) \end{pmatrix} \\ \in \mathcal{H}^{-\infty}(W/X;\mathbb{C}^N) \tag{28.15}$$

where, since we have chosen an embedding of E_+ and E_- into the trivial bundle $\mathbb{C}^N$, so all terms can be regared as 'operators on' $\mathbb{C}^N$. Thus, the 4-fold decomposition of $\mathbb{C}^2 \otimes \mathbb{C}^N = \mathbb{C}^N \oplus \mathbb{C}^N$ in (28.15) is in terms of the ranges of $(\mathrm{Id} - \pi_-(x))$, $\pi_-(x)$, $\pi_+(x)$ and $\mathrm{Id} - \pi_+(x))$. In particular the operators in the central 2×2 block are of the form

$$\begin{pmatrix} \Psi^{-\infty}(W/X; E_-) & \Psi^{-\infty}(W/X; \mathbb{E}) \\ \Psi^{-\infty}(W/X; \mathbb{E}^-) & \Psi^{-\infty}(W/X; E_+) \end{pmatrix}. \tag{28.16}$$

Exercise 22. Make sure that (28.15) is an involution.

Proposition 39. *Corollary 7 shows that the involution* (28.15) *defines an element of* $\mathrm{K}^0(X)$ *and this induces a commutative diagram*

$$\begin{tikzcd} & \mathcal{P}^0_{\mathrm{qiso}}(W/X; \mathbb{E}) \arrow[dl, "\sigma"'] \arrow[dr, "{[I(P,Q,\pi_\pm)]}"] & \\ \mathcal{D}(W) \arrow[rr, "\mathrm{ind}_{\mathrm{iso}}"'] & & \mathrm{K}^0(X). \end{tikzcd} \tag{28.17}$$

Proof. We have to check that the class $[I(P, Q, \pi_\pm)]$ defined by applying Corollary 7 to (28.15) is independent of the choices of parametrix, Q, of quantization P and of embedding of $E_\pm$ in $\mathbb{C}^N$, including the indepence of N. In fact the last of these is the simplest since increasing N just corresponds to stabilization which is already part of the definition of the map in $\mathrm{K}^0(X)$. By definition the class $[I(P, Q, \pi_\pm)]$ is homotopy invariant – notice that the notation really is inadequate since it depends on the *identification* of $E_\pm$ with the ranges of $\pi_\pm$. Fixing everything else, independence of the choice of P and Q follows from the fact that the linear family $(1 - t)P_0 + tP_1$ between any two quantization (operators with symbol p) consists of quantizations and the construction of parametrices can be carried out uniformly in an additional parameter (which can be hidden in X). Thus, it suffices to suppose that P is fixed. Then the linear homotopy $(1-t)Q_0 + tQ_1$ consists of parametrices. Thus it suffices to consider P and Q fixed and change the embeddings. Changing the embedding of $E_\pm$ with $\pi_\pm$ fixed means conjugating by isomorphisms on the ranges of π_+ and π_- in the middle block (but not the outer block) in (28.5). These can be rotated away after stabilising a bit. On the other hand, if $\pi'_\pm$ is another family of projections with range bundle isomorphic to $E_\pm$, for the same N then there are necessarily elements $F_\pm \in \mathcal{C}^\infty(X; \mathrm{GL}(N, \mathbb{C}))$ conjugating $\pi_\pm(x)$ to $\pi'_\pm(x)$ for each x and homotopic to the identity, see Proposition 40 – these can be deformed away. □

Proposition 40. *If* $F_i : E \longrightarrow \mathbb{C}^N$ *are two embeddings of a complex vector bundle into a trivial bundle then, after stabilizing by further embedding as* $\tilde{F}_i = F_i \oplus 0 : E \longrightarrow \mathbb{C}^N \oplus \mathbb{C}^N$ *there is an element* $A \in \mathcal{C}^\infty(X; \mathrm{GL}(N+M, \mathbb{C}))$ *which is homotopic to the identity and conjugates the range of* F_1 *to the range of* F_2.

Proof. Let $\pi_i \in \mathcal{C}^\infty(X; M(N, \mathbb{C})$ be the orthogonal projections onto the ranges of the F_i. We can consider the joint embedding $F_1 \oplus F_2 : E \oplus E \longrightarrow \mathbb{C}^{2N}$ which has range $\pi_1(x) \oplus \pi_2(x)$ at each point. Consider the 'rotation' on $\mathbb{C}^{2N}$ obtained by decompositing into the four pieces

$$
\begin{aligned}
(28.18)\quad G_t : (v,w) &\longmapsto \pi(x)v + (\mathrm{Id} - \pi_1(x))w + \pi_2(x)w + (\mathrm{Id} - \pi_2(x))w \\
&\longmapsto (\cos t)\pi(x)v + (\sin t)F_1(F_2)^{-1}(\pi_2(x)v) + (\mathrm{Id} - \pi_1(x))w \\
&\qquad - (\sin t)F_2(x)F_1^{-1}\pi(x)v + (\cos t)\pi_2(x)w + (\mathrm{Id} - \pi_2(x))w.
\end{aligned}
$$

Clearly, $G_0 = \mathrm{Id}$ and $G_{\pi/2}$ conjugates the range of $\pi_1 \oplus 0$ to the rang of $0 \oplus \pi_2$. Following this by a 1-parameter family of rotations between the two factors, starting at the identity and finishing at a map which exchanges the factors (and reverses one sign) finally gives a bundle isomorphism of $\mathbb{C}^{2N}$ which intertwines the two projections and is connected to the identity – where it is easy to make the family smooth:

$$
(28.19)\qquad g_0 = \mathrm{Id},\ \ g_1^{-1}(\pi_1 \oplus 0)g_1 = \pi_2 \oplus 0.
$$

□

Theorem 10. *(Essentially Theorem 9 above). The relation $\sim$ on $\mathcal{D}(W)$ generated by stability, bundle isomorphisms and homotopy on p, gives a natural isomorphism $\mathcal{D}(W)/\sim = \mathrm{K}^0_c(W)$ which leads to a commutative diagram*

$$
(28.20)\qquad
\begin{tikzcd}
 & \mathcal{P}^0_{\mathrm{qiso}}(W/X;\bullet) \arrow[dl, "\sigma"'] \arrow[dr, "{[I(P,Q,\pi_\pm)]}"] & \\
\mathcal{D}(W) \arrow[rr, "\mathrm{ind}_{\mathrm{iso}}"'] \arrow[dr, "\sim"'] & & \mathrm{K}^0(X) \\
 & \mathrm{K}^0_c(W) \arrow[ur, "\mathrm{Thom}=p_{\mathrm{sl}}"'] &
\end{tikzcd}
$$

under which the isotropic index map factors through the semiclassical realization of the Thom isomorphism. The vanishing of $\mathrm{ind}(\sigma(P)$ *in* $\mathrm{K}^0(X)$ *is a necessary and sufficient condition for the existence of a perturbation $T \in \Psi^{-\infty}_{\mathrm{iso}}(W/X;\mathbb{E})$ such that $P+T$ is invertible with inverse in $\Psi^0_{\mathrm{iso}}(W/X;\mathbb{E}^-)$ and is also equivalent to the exists of an homotopy, through elliptic elements starting from a stabilization of P, to the identity.*

So this is the families isotropic index theorem.

Now the main aim is to prove (28.20) identifying the isotropic index map with the Thom isomorphism. It would be logical to discuss the relation $\sim$ on the symbol data, however as in the outline above, I prefer to launch into a discussion of 'generalized symbol data'. The key ingredient in the proof of (28.20) is then that the index map can be extended to this more general data.

Definition 8. The space $\widetilde{\mathcal{D}}^0(W)$ of *generalized elliptic data* for W consists of the elements $(p,\pi_\pm)$ where $\pi_\pm \in \mathcal{C}^\infty({}^q\overline{W}; M(N,\mathbb{C})$ are projection-valued and $p \in \mathrm{Iso}(\mathbb{S}W;\pi)$ is a smooth isomorphism between the range of π_+ and the range of π_- over $\mathbb{S}W$.

So the only sense in which this is generalized compared to Definition 7 is that the projections are smooth on the whole of the quadratic compactification of W – rather than being pulled-back from X and so constant on the fibres. Of course

$$
(28.21)\qquad \mathcal{D}(W) \subset \widetilde{\mathcal{D}}^0(W).
$$

The main thing we need, and at this stage it may seem just like a strange generalization, is to define the index map on the whole of this generalized data. So let me consider a big version of $\mathcal{P}^0_{\mathrm{qiso}}(W/X;\bullet)$ discussed above.

Definition 9. Let $\widetilde{\mathcal{P}}^0(W/X)$ consist of all elements $(P,Q,\Pi_\pm)$ where for some N,

$$\begin{gathered}\Pi_\pm\in\Psi^0_{\mathrm{ad,qiso}}(W/X;\mathbb{C}^N),\ \Pi_\pm^2=\Pi_\pm,\\ \sigma_{\mathrm{iso}}(\Pi_\pm)\text{ is independent of }\epsilon,\\ P,\ Q\in\Psi^0_{\mathrm{qiso}}(W/X;\mathbb{C}^N)\text{ satisfy}\\ P=R(\Pi_-)P=PR(\Pi_+),\ Q=R(\Pi_+)Q=QR(\Pi_-),\\ R_L=R(\Pi_+)-QP,\ R_R=R(\Pi_-)-PQ\in\Psi^{-\infty}_{\mathrm{iso}}(M/X;\mathbb{C}^N)\end{gathered}\tag{28.22}$$

where R is the restriction of the adiabatic family to $\epsilon=1$.

Again we certainly have

$$\mathcal{P}^0(W)\subset\widetilde{\mathcal{P}}^0(W/X)\tag{28.23}$$

where an element $(P,Q,\pi_\pm)$ can be regarded as an element of $\widetilde{\mathcal{P}}^0(W/X)$ since $\pi_\pm$ are just smooth matrices over X so can be thought of as adiabatic families, just constant matrices on each fibre, which are then completely independent of ϵ. In particular in this case $\Pi_\pm=\pi_\pm$ with the isotropic symbol reducing to $\pi_\pm$ again and this is constant in ϵ.

Lemma 32. *The adiabatic and istropic symbol maps induce a surjective map*

$$\widetilde{\mathcal{P}}^0(W/X)\ni(P,Q,\Pi_\pm)\longmapsto(p,\pi_\pm=\sigma_{\mathrm{sl}}(\Pi_\pm))\in\widetilde{\mathcal{D}}^0(W).\tag{28.24}$$

Proof. The existence of the map (28.24) is just a matter of checking the consistency conditions. Namely, the adiabatic symbols of $\Pi_\pm$ are projections $\pi_\pm\in\mathcal{C}^\infty({}^q\overline{W};M(N,\mathbb{C})$. The compatibility between adiabatic and istropic symbols means that the istropic symbol restricted to $\epsilon=0$ is $\pi_\pm$ restricted to the boundary, $\mathbb{S}W$. Then the insistence in (28.22) that the isotropic symbol be constant in ϵ means this holds everywhere. Then it follows from the other conditions in (28.22) that $p=\sigma_{\mathrm{iso}}(P)$ satisfies

$$\pi_-p=p\pi_+=p\tag{28.25}$$

and the existence and properties of Q then it means it is an isomorphism from the range of π_+ to the range of π_- at each point of $\mathbb{S}W$.

So, the surjectivity is just the converse, that every such triple in $\widetilde{\mathcal{D}}^0(W)$ arises this way. This is our usual constructive task – to find $\Pi_\pm$ and P, Q as in (28.24) given p and $\pi_\pm$. First think about the $\pi_\pm$. The joint surjectivity of semiclassical and isotropic symbols means that we can choose $\Pi'_\pm\in\Psi^0_{\mathrm{ad,iso}}(W/X;\mathbb{C}^N)$ with the symbolic conditions in (28.24) – since the compatibility condition is evidently satisfied. Now, I leave it to you to go back and see that $\Pi'_\pm$, which are necessary projections up to leading order, in both the semiclassical and the isotropic sense, can be deformed to be actual projections with the same symbols, i.e. by adding terms which are lower order in both senses. In brief this comes from the same iterative arguments with the symbols as before. Do it first at the semiclassical face, checking that the resulting correction (after summing the Taylor series) is of order at most -1 in the isotropic sense. Then do the same iterative correction using the isotropic symbol and note that all terms, and hence the asymptotic sum, can be chosen to vanish to infinite order at $\epsilon=0$ (so they aren't really semiclassical, just smooth in ϵ). This corrects $\Pi'_\pm$ to be projections modulo and error $(\Pi'_\pm)^2 - \Pi'_\pm$ which is of order $-\infty$ and vanishes to infinite order at $\epsilon = 0$. Now, the integral argument allows this to be corrected to a family of projections in $\epsilon < \epsilon_0$ for some $\epsilon_0 > 0$. Since the isotropic symbol is constant anyway, reparameterizing allows the family to be 'extended' all the way to $\epsilon = 1$.

Now, having constructed $\Pi_\pm$ we need to construct P and Q to satisfy the remaining conditions. We can certainly choose $P' \in \Psi^0_{\mathrm{qiso}}(W/X;\mathbb{C}^N)$ with $\sigma_{\mathrm{iso}}(P') = p$. Replacing it by $P = R(\Pi_-)P'R(\Pi_+)$ does not change the symbol, given the properties of p and the $\pi_\pm$ and of course implies that $P = R(\Pi_-)P = PR(\Pi_+)$. So, it remains to construct Q satisfying the remaining properties. By assumption p has a generalized inverse $q \in \mathcal{C}^\infty(\mathbb{S}W; M(N,\mathbb{C}))$ such that $pq = \pi_-$, $qp = \pi_+$ on $\mathbb{S}W$. First take $Q' \in \Psi^0_{\mathrm{qiso}}(W/X;\mathbb{C}^N)$ with $\sigma_{\mathrm{iso}}(Q') = q$ and set $Q_0 = R(\Pi_+)Q'\Pi_-$. We have everything but the last line in (28.22) and we have this to first order, because of the properties of p and q – namley

$$
\begin{aligned}
(28.26)\quad R'_L = R(\Pi_+) - Q_0P \in \Psi^0_{\mathrm{qiso}}(W/X;\mathbb{C}^N) \text{ has}&\\
\sigma_{\mathrm{iso}}(R'_L) = \pi_+ - qp = 0 \Longrightarrow R'_L \in \Psi^{-1}_{\mathrm{qiso}}(W/X;\mathbb{C}^N).&
\end{aligned}
$$

Thus we wish to successively add lower order terms to make successive symbols vanish. We can in fact add any term of order -1 or the form $R(\Pi_+)Q_1R(\Pi_-)$ to Q_0 and this has arbitrary symbol of order -1 of the form $\pi_+q_1\pi_-$. Moreover it follows by composing the identity in (28.26) on the right with Π_- and the left with Π_+ that the symbol of R'_L of order -1 is of this form. So, iterating this argument and asymptotically summing we can arrange that Q satisfies the first identity on the last line in (28.22) with everything else still holding. So, it remains to ensure the last condition – which can certainly be done by the obvious variant of the preceeding argument, but we need both to hold at once! So, go back to the previous construction and proceed by induction. The extra step is that at stage p we have arranged that R'_L is of order $-k-1$ and R'_R is of order $-k$ and we want to correct the second without destroying the first. The term we add to Q' is $\Pi_+Q'_k\Pi_-$ where $\sigma_{-k}(Q'_k) = q_k = -\sigma_{-k}R'_R$. However, from the definition of $R'_R = \Pi_+ - PQ'$, $R'_RP = PR'_L$ has vanishing symbol of order $-k$, so $q'_kp = 0$ from which it follows that both conditions then hold at order $-k$ and the induction can continue.

Thus indeed the operators in (28.22) can be constructed and surjectivity follows, proving the Lemma. □

Proposition 41. *The analogue of Proposition 39 holds for the 'generalized' parametrix sets and elliptic data, so inducing a commutative diagram which restricts to* (28.17)*:-*

$$
(28.27)\qquad
\begin{tikzcd}
& \widetilde{\mathcal{P}^0}_{\mathrm{qiso}}(W/X;\mathbb{E}) \arrow[dl, "\sigma"'] \arrow[dr, "{[I(P,Q,\Pi_\pm)]}"] & \\
\widetilde{\mathcal{D}}(W) \arrow[rr, "\mathrm{ind}_{\mathrm{iso}}"'] & & \mathrm{K}^0(X).
\end{tikzcd}
$$

Proof. The main thing to notice is that modifying (28.15) to

$$I(P,Q,\Pi_\pm)= \begin{pmatrix} \operatorname{Id}-\Pi_+(x) & 0 & 0 & 0 \\ 0 & 1-2R_L^2 & 2R_L(\operatorname{Id}+R_L)Q & 0 \\ 0 & 2R_RP & -\operatorname{Id}+2R_R^2 & 0 \\ 0 & 0 & 0 & -(\operatorname{Id}-\Pi_-(x)) \end{pmatrix} \in \mathcal{H}^{-\infty}(W/X;\mathbb{C}^N) \tag{28.28}$$

gives a family of involutions with essentially the same homotopy properties as in the proof of Proposition 39. I will write a little more, especially about the conjugation which shows up when we change things – here it is a bit more general but the same arguments work. □

So, if you believe all that, observe that something rather pleasant happens in that we have another 'extreme' subset of $\widetilde{\mathcal{D}}^0(W)$ (the other one being $\mathcal{D}^0(W)$. Namely consider

$$\begin{aligned} \mathcal{D}^{-\infty}(W)=\{(\pi_\infty,\pi_\pm);\\ \pi_+=\pi_\infty+a_+,\ a_+\in\mathcal{S}(W/X;M(N,\mathbb{C}),\ \pi_-=\pi_\infty\in M(N,\mathbb{C})\}. \end{aligned} \tag{28.29}$$

Thus in this subset, π_- is constant, π_+ is a Schwartz perturbation of $\pi_-=\pi_\infty$ and p is the identity map on the range of π_∞. Thus the $-\infty$ is denotes, that the elements are smoothing perturbations of constant objects (we could allow π_- to have a Schwartz term too).

Lemma 33. *The equivalence relation $\sim_{-\infty}$ on $\mathcal{D}^{-\infty}(W)$ in which elements can be stabilized, by the addition of the identity or of zero on a complementary subspace (so increasing N), or subject to homotopies within $\mathcal{D}^{-\infty}(W)$, so preserving the constancy of π_- on ${}^q\overline{W}$ etc, but allowing π_∞ to vary in $M(N,\mathbb{C})$ with the parameter, gives a natural isomorphism*

$$\mathcal{D}^{-\infty}/\sim_{-\infty}\xrightarrow{=}\mathrm{K}_c^0(W). \tag{28.30}$$

Proof. We define the map (28.30) directly. For an element (π_+,π_∞) in $\mathcal{D}^{-\infty}(W)$ (so $\pi_-=\pi_\infty$ is a projection in $M(N,\mathbb{C})$ and π_+ is a family of projections on W which is a Schwartz perturbation of π_∞) let $M=\operatorname{rank}(\pi_\infty$. The cases $M=N$ and $M=0$ are trivially globally constant. So, we can add either an identity block of size $N-2M$ if this is positive or a zero block of size $2M-N$ in the opposite case, to arrange that $N=2N$ keeping equivalence under $\sim_{-\infty}$. Now, all projections in $M(N,\mathbb{C})$ of given rank are are homotopy where the curve is obtained by conjugation with a curve in $\mathrm{GL}(N,\mathbb{C})$. So after an admissible homotopy under $\sim_{-\infty}$ we can arrange that $N=2p$ is even and under a decomposition $\mathbb{C}^{2p}=\mathbb{C}^2\otimes\mathbb{C}^p$

$$\pi_\infty=E_+\otimes\operatorname{Id}_{p\times p}\Longrightarrow I(w)=\pi_+(w)-(\operatorname{Id}-\pi_+(w))\longrightarrow\mathcal{S}(W;\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R})) \tag{28.31}$$

where as usual we are further stabilizing by using the harmonic oscillator basis of $\mathcal{S}(\mathbb{R})$ to map into $\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R})$.

It remains to check that passing to homotopy classes in (28.31) projects to a map (28.30) into $\mathrm{K}_c^0(W)$ and that this map is an isomorphism. I omit the details but surjectivity is clear enough by finite rank approximation and after stabilization homotopy on the left in (28.31) exhausts the freedom on the right. □

Proof of Theorem 10. The crucial observation is that we know already how to quantize the data in $\mathcal{D}^{-\infty}(W)$ in (28.29). So we prove the main result by looking at the expanded and rearranged version of (28.17):-

(28.32)

$$
\begin{tikzcd}
\mathcal{P}^0_{\mathrm{qiso}}(W/X;\bullet) \arrow[r, hook] \arrow[d, "\sigma"'] & \widetilde{\mathcal{P}}^0_{\mathrm{qiso}}(W/X;\bullet) \arrow[d, "\sigma"'] & \mathcal{P}^{-\infty}_{\mathrm{iso}}(W/X;\bullet) \arrow[l, hook'] \arrow[d, "\sigma"'] & \\
\mathcal{D}(W) \arrow[r, hook] \arrow[d, "{[\sigma]}"'] & \widetilde{\mathcal{D}}(W) \arrow[d, "{[\sigma]}"'] & \mathcal{D}^{-\infty}(W) \arrow[l, hook'] \arrow[d, "{[\sigma]}"'] & \\
\mathcal{D}(W)/\sim \arrow[r, equal] \arrow[dr, "\mathrm{ind}_{\mathrm{iso}}"'] & \widetilde{\mathcal{D}}(W)/\sim \arrow[r, equal] \arrow[d, "\mathrm{ind}_{\mathrm{iso}}"'] & \mathcal{D}^{-\infty}(W)/\sim_{-\infty} \arrow[r, equal] \arrow[dl, "\mathrm{ind}_{\mathrm{iso}}"'] & \mathrm{K}^0_{\mathrm{c}}(W) \arrow[dll, "\mathrm{Thom}"] \\
& \mathrm{K}^0(X). & &
\end{tikzcd}
$$

So, we proceed to check that this diagram commutes.

Going down the left side is repeating the discussion above, that we know how to define the isotropic families index by looking at the family of involutions $I(P,Q,\pi_\pm)$. The image, in $\mathrm{K}^0(X)$ of this isotropic index map factors through the symbol data, into $\mathcal{D}^0(W)$ and further under the equivalence relation $\sim$ to the quotient. So the map from top left down the left side and to $\mathrm{K}^0(X)$ is the isotropic index in the sense of $[I(P,Q,\pi_\pm)]$. The same is true down the middle column, except that the problem has been 'aggrandized' by includion of semiclassical Toeplitz objects – the $\Pi_\pm$. We also know the top two maps from the left column, given by inclusion, give commutative squares. Similarly for the next map down we know there is an inclusion-induced map from left to right where I have equality, and this gives a commutative left side. Everything is the same on the right side, again with a map now from right to left third down; the equality to $\mathrm{K}^0_{\mathrm{c}}(W)$ is Lemma 33.

So the only things left to show are that the maps in the third row are isomorphism, and given their naturality as inclusion maps can then be regarded as equalities plus the proof of the commutativity of the lower right triangle.

Let's do the last part first. Going way back to the discussion of the Bott element. Modulo checking the details it is clear enough. The 'isotropic index' in this case is obtained by semiclassical quantization of the one projection π_+ – the isotropic part of the quantization is trivial since there we just have π_∞ itself. So, this map and the Thom isomorphism are given by semiclassical quantization. Unfortunately on one side it is given by quantization of a projection and the other side by an involution. Of course the projection is supposed to be the positive part of the involution, so the exact correspondence needs to be checked.

By Lemma 33 it is enough to consider the case where $\pi_\infty = E_+$ on $\mathbb{C}^2 \otimes \mathbb{C}^N$ where E_+ is the projection onto the first element of $\mathbb{C}^2$, tensored with the identity of course. The semiclassical quantization of $\pi_+ \in \mathcal{C}^\infty({}^q\overline{W}; M(N,\mathbb{C}))$ to a family of projections Π_+ also gives a semiclassical quantization of the involution $\pi_+ - (\mathrm{Id} - \pi_+) = \gamma_1 + a$ with a Schwartz. So, it is only necessary to check that the isotropic index, which is $[I(P,Q,\Pi_\pm)]$ for this very special data, is the same as the class of semiclassical quantization of the involution at $\epsilon = 1$ which defines the Thom isomorphism.

A bit more detail needed here.

Proving the horizontal maps are equalities is showing that every class in $\widetilde{\mathcal{P}}/\sim$ can be represented uniquely by an element either in $\mathcal{P}/\sim$ or $\mathcal{P}^{-\infty}/\sim_{-\infty}$. It is straightforward.

There is still more to Theorem 10 apart from (28.20) – which certainly follows from (28.32). Namely, what happens if the index vanishes. Going through the proof of the equalities of the quotients in (28.32) and of course using the fact that the Thom map is an isomorphism, one concludes that the vanishing of the isotropic index implies that $(P, Q, \pi_\pm) \sim 0$. Looking at the equivalence relation, the implication is that the symbol p, between the original bundles, is, after stabilization by the identity on some additional bundle, homotopic to a bundle isomorphism – can be deformed to be constant on the fibres of $\mathbb{S}W$ over X. This is one of the two claims.

The other one is more interesting analytically so I will extract it for later reference. □

Proposition 42. *If $P \in \Psi^0_{\mathrm{qiso}}(W/X; \mathbb{E})$ is elliptic then there is a perturbation $T \in \Psi^{-\infty}_{\mathrm{qiso}}(W/X; \mathbb{E})$ such that $P+T$ has a generalized inverse $Q \in \Psi^{-\infty}_{\mathrm{qiso}}(W/X; \mathbb{E}^-)$ meaning*

$$\begin{cases} \varpi_L = \mathrm{Id}_+ - Q(P+T) & \in \Psi^{-\infty}_{\mathrm{iso}}(W/X; E_+) \\ \varpi_R = \mathrm{Id}_- - (P+T)Q & \in \Psi^{-\infty}_{\mathrm{iso}}(W/X; E_-) \end{cases} \text{ are projections} \tag{28.33}$$

and the index is represented by the K-class $\mathrm{ind}_{\mathrm{iso}}(P) = \mathrm{Ran}(\varpi_L) \ominus \mathrm{Ran}(\varpi_R)$. If this K-class vanishes then T can be chosen so that $P + T$ is invertible.

So the isotropic index is the precise obstruction to perturbative invetibility without any need to stabilize. This is basically because there is 'enough room' in the smoothing terms.

Proof. Replace P by $P(\mathrm{Id} - \Pi_N) = P + T$ where Π_N is the sequence of harmonic oscillator projections and check that for large N this has 'null space the range of Π_N' (these aren't actual operators) and (28.33) can be arranged. If the index K-class vanishes this means that after increasing N enough (effectively stabilizing) the bundles defined by $\Pi_N = \Pi_L$ and Π_R are isomorphic. This there is a further perturbation T which maps precisely from the 'null space' of $P(\mathrm{Id} - \Pi_N)$ to the range of Π_R and hence $P(\mathrm{Id} - \Pi_N) + T$ is invertible with the invserse as claimed. □

29. Lecture 26: Semiclassical push-forward for fibrations Monday, 3 November, 2008

At this point I want to start the transition to geometric settings and in particular the Atiyah-Singer index theorem. This can be paraphrased in the form: 'The push-forward in K-theory for fibrations is realized by the index of pseudodifferential operators' – although this is slightly misleading since the push-forward is from the fibre-cotangent bundle of the fibration. That is what I want to examine today.

So, let me start with a single compact manifold Z. In fact I will allow it to be a manifold with corners later, but for the moment let us require that it not have a boundary. The basic commutative object is $\mathcal{C}^\infty(Z)$, the space of smooth functions on Z. I will also assume that you know about $\Lambda^k Z$, the bundle of k-forms on Z.

One thing we need to be able to do is to integrate, invariantly. Given the transformation law for integrals under coordinate changes we can only integrate, at least in the usual sense, objects which transform with a factor of the absolute value of the Jacobian unless we assume that the manifold is oriented. The latter works because we then only make coordinate changes with positive Jacobian matrices anyway and volume forms $v \in \mathcal{C}^\infty(Z; \Lambda^n Z)$, $n = \dim Z$, transform with a factor of the Jacobian:-

$$F^*(dz_1 \wedge \cdots \wedge dz_n) = \det\left(\frac{\partial F_i(z)}{\partial z'_j}\right) dz'_1 \wedge \cdots \wedge dz'_n,\ z_j = F_j(z'). \tag{29.1}$$

Here the fibre at a given point $\Lambda^n_z Z$ can be viewed as the space of totally antisymmetic multilinear forms

$$T_zZ \times T_zZ \cdots \times T_zZ \longrightarrow \mathbb{C} \text{ or } \mathbb{R}. \tag{29.2}$$

In general $\Lambda^k_z Z$ is a contraction for $\Lambda^k(T^*_z Z)$ and alternatively on can identify $\Lambda^n_z Z$ as the space of *linear* functions on, i.e. the dual of, $\Lambda^n(T_z Z)$. The latter is a one-dimensional vector space so we can apply the self-proving

Lemma 34. *For any one dimensional real vector space L the space of absolutely homogoneous functions of degree α*

$$f : L \setminus \{0\} \longrightarrow \mathbb{R},\ f(tv) = |t| f(v)\ \forall\ t, \in \mathbb{R} \setminus \{0\},\ v \in V \setminus \{0\} \tag{29.3}$$

is a well-defined one-dimensional vectors space, denoted $\Omega^\alpha V^*$, or ΩV^* when $\alpha = 1$.

It follows easily enough that the fibres $\Omega(\Lambda^n_z Z)$ form a smooth one-dimensional vector bundle ΩZ over Z. This is the space of densities.

Exercise 23. If you have not done this before, check that the integral is well-defined by reference to local coordinates and a partition of unity:

$$\int_Z : \mathcal{C}^\infty(Z; \Omega Z) \longrightarrow \mathbb{R} \text{ or } \mathbb{C}. \tag{29.4}$$

Note that if $v \in \mathcal{C}^\infty(Z; \Lambda^n Z)$ then $|v| \in \mathcal{C}^0(Z; \Omega Z)$ can be integrated and if Z is oriented and $v > 0$ this gives the integral back again (and in that case $|v| \in \mathcal{C}^\infty(Z; \Omega Z)$).

Now consider the product, $Z_1 \times Z_2$, of two compact manifolds. The density bundle on Z_2 can be pulled back to the product, where we can again denote it ΩZ_2 or $\pi^*_R \Omega$ where $\pi_R : Z_1 \times Z_2 \longrightarrow Z_2$ is the projection and we drop, as obvious, the reminder that the bundle comes from the second factor. Fubini's theorem then shows that

$$\int_{Z_2} : \mathcal{C}^\infty(Z_1 \times Z_2; \pi_R \Omega) \longrightarrow \mathcal{C}^\infty(Z_1). \tag{29.5}$$

Exercise 24. Try to write down a clean proof of the existence, and natural properties, of the integration map

$$\int_{Z_2} : \mathcal{C}^\infty(Z_1 \times Z_2; \pi_L^* E \otimes \pi_R \Omega) \longrightarrow \mathcal{C}^\infty(Z_1; E),\ \pi_L : Z_1 \times Z_2 \longrightarrow Z_1 \tag{29.6}$$

being the projection onto the first factor and E being any vector bundle over Z_1.

We will make extensive use of smoothing operators. Let me set these up first for any pair of compact manifolds Z_i, $i = 1, 2$ with complex vector bundles E_i over them. Namely a smoothing operator is a continuous linear map

$$A : \mathcal{C}^\infty(Z_2; E_2) \longrightarrow \mathcal{C}^\infty(Z_1; E_1) \tag{29.7}$$

which is given by the generalization of (29.6). Namely there must exist a Schwartz kernel $A \in \mathcal{C}^\infty(Z_1 \times Z_2; \mathrm{Hom}(E_2, E_1) \otimes \pi_R^* \Omega)$ such that

$$Au(z_1) = \int_{Z_2} A(z_1, z_2) u(z_1) \tag{29.8}$$

Here $\mathrm{Hom}(E_2, E_1)$ is a bundle over $Z_1 \times Z_2$ which has fibre at a point (z_1, z_2) the linear space of linear maps $T : (E_2)_{z_2} \longrightarrow (E_1)_{z_1}$ – it is unfortunate about the reversals here. Standard linear algebra gives a natural isomorphism

$$\mathrm{Hom}(E_2, E_1) = E_1 \boxtimes (E_2)^*. \tag{29.9}$$

Then (29.8) reduces to (29.6) since it means we 'contract' in E_2 – or apply the homomorphism – to get

$$A(z_1, z_2) u(z_2) \in (E_1)_{z_1} \otimes \Omega(Z_2)_{z_2} \tag{29.10}$$

and then we can integrate.

Exercise 25. I have not discussed the Fréchet topology on $\mathcal{C}^\infty(Z; E)$ for a vector bundle over Z – it is basically the same as the earlier spaces such as $\mathcal{S}(\mathbb{R}^n)$. In fact it is isomorphic to this space! You might wish to go through the topology carefully (and think about the isomorphism which will appear a little later).

For the moment we are most interested in the case $Z_1 = Z_2 = Z$ and $E_1 = E_2 = E$, then $\mathrm{Hom}(E) = \mathrm{Hom}(E, E)$ is a bundle over Z^2. The 'usual' homomorphism bundle, $\mathrm{hom}(E) = \mathrm{hom}(E, E)$ is a bundle over Z which is the restriction of $\mathrm{Hom}(E)$ to the diagonal.

Lemma 35. *The space $\mathcal{C}^\infty(Z^2; \mathrm{Hom}(E) \otimes \pi_R^* \Omega)$ is an associative, non-commutative, Neumann-Fréchet algebra, denoted $\Psi^{-\infty}(Z; E)$ under the operator product*

$$AB(z, z') = \int_Z A(z, z'') B(z'', z'). \tag{29.11}$$

Proof. I leave it to you to check that similar arguments as in the isotropic case show that the product is continuous and that it has the corner property. For the moment this means that with seminorms (based on continuous derivatives in local coordinates and trivializations)

$$\|A_1 A_2 A_3\|_k \le C \|A_1\|_k \|A_2\|_0 \|A_3\|_k. \tag{29.12}$$

This again is Fubini's theorem. It follows that for A in a neighbourhood of the zero in $\Psi^{-\infty}(Z;E)$ the operator $\operatorname{Id}+A$ is invertible with inverse $\operatorname{Id}+B$, $B\in\Psi^{-\infty}(Z;E)$. □

Exercise 26. Let $G^{-\infty}(Z;E)$ be the corresponding Fréchet group – it is in fact an open dense subset of $\Psi^{-\infty}(Z;E)$ with the 'drop the Id' identification. I invite you to check that many things we have done previously hold for this group. It has a determinant function, admits finite rank approximation and it is a classifying space for K^1. I will write down the Chern forms and so on later.

This is our replacement in the geometric setting for $\Psi^{-\infty}(\mathbb{R}^k;\mathbb{C}^N)$. Moreover we can generalize at least some of the things we have done before. First we can introduce the corresponding semiclassical algebra. The scaling here is slightly different to the isotropic case, but this does not in the end make very much difference. Of course we immediately know what smooth dependence on a parameter, even one in a manifold, means.

For the semiclassical calculus we want to consider the appropriate subspace of kernels

$$A_\bullet\in\mathcal{C}^\infty((0,1];\Psi^{-\infty}(Z;E)=\mathcal{C}^\infty((0,1]\times Z^2;\operatorname{Hom}(E)\otimes\pi_R^*\Omega)$$

where I am even too lazy to write the extra pull-backs from Z^2 to $(0,1]\times Z^2$. So, of course the crucial thing is to specify exactly what happens as $\epsilon\downarrow 0$. We demand two things of the kernels A_ϵ. First assume $E=\mathbb{C}$:

$$\begin{gathered}\text{If } K\subset Z^2 \text{ is closed and } K\cap\operatorname{Diag}=\emptyset \text{ then}\\ A_\epsilon\to 0 \text{ rapidly with all derivatives as } \epsilon\downarrow 0 \text{ on } K.\end{gathered}\tag{29.13}$$

$$\begin{gathered}\text{If } U\subset Z \text{ is a coordinate chart then } \exists\ F_U\in\mathcal{C}^\infty([0,1]\times U\times\mathbb{R}^n) \text{ s.t.}\\ A_\epsilon(z,z')=\epsilon^{-n}F_U(\epsilon,z,\frac{z-z'}{\epsilon})|dz'| \text{ on } (0,\epsilon_0(K))\times K\times K,\ \epsilon_0(K)>0\ \forall\ K\Subset U.\end{gathered}\tag{29.14}$$

So, there are two main changes compared to the Euclidean case. First we need to specify the rapid vanishing away from the diagonal – this is true in the Euclidean case anyway – since we do not have global coordinates. Secondly, the scaling is different in (29.14) – it simply does not make sense to scale the base variable since they lie in U. I have also not made the 'Weyl' change from z to $(z+z')/2$ but this is only because I would have to put a double covering – since $(z+z')/2$ is not in U in general.

Exercise 27. Work out the wording for (29.14) in terms of Weyl coordinates.

This is a seriously overspecified definition. Even so, this would not make much sense if it wasn't really local:-

Exercise 28. Check that (29.14) for *all* coordinate charts is equivalent to the same definition for a covering by coordinate charts, given (29.13). Check at the same time that the bundle E can be put back in where in (29.14) U should be such that E is trivial over it and then f should take values in $M(N,\mathbb{C})$ where N is the rank of E.

Proposition 43. *The semiclassical families form an algebra under operator composition, denoted $\Psi^{-\infty}_{\mathrm{sl}}(Z;E)$ with a well-defined symbol map giving a multiplicative short* *exact sequence*

$$\epsilon\Psi_{\text{sl}}^{-\infty}(Z;E) \longrightarrow \Psi_{\text{sl}}^{-\infty}(Z;E) \xrightarrow{\sigma_{\text{sl}}} \mathcal{S}(T^*Z;\hom(E)), \tag{29.15}$$

where in any local coordinate system

$$\sigma_{\text{sl}}(z,\zeta) = \int_{\mathbb{R}^n} e^{-iZ\cdot\zeta} F_U(0,z,Z). \tag{29.16}$$

Proof. Use the preceeding exercise to reduce the problem to local coordinates and then check directly by changing variable as we did before. Most importantly, check that the leading part of F_U, $F_U(0,z,Z)|dZ|$ is actually a well-defined density on T_zZ for each $z \in Z$, so (29.16) makes sense and gives a well-defined function on the cotangent bundle – the density is absorbed by the Fourier transform. □

Digression 1. I am pretty unhappy having to do local coordinate proofs like the one above – that I have not done. So, I now resort to global definititons of things like the semiclassical calculus described above, where the composition and symbolic properties become geoemtrically compleling. In this case it is convenient to use the notion of real blow up. Since I do not have the time to discuss this in the course I have not used it, although I have come close. You could look at the notes from my introductory lectures at MSRI this year but it can also be found in lots of other places. So, let me just assume you know what blow up is. The manifold we want to consider, a manifold with corners, is by definition

$$Z^2_{\text{sl}} = [[0,1]\times Z^2; \{0\}\times \text{Diag}],\ \beta_{\text{sl}} : Z^2_{\text{sl}} \longrightarrow [0,1]\times Z^2. \tag{29.17}$$

That is, it is the kernel and parameter space, blown up at the diagonal at $\epsilon = 0$. This is a manifold with corners with a 'front face' corresponding to the blow up – it is a bundle over $\{0\}\times\text{Diag}$ which is naturally isomorphic to $\overline{TZ}$, the radial compactification of the tangent bundle of Z. The other, or 'old', boundary hypersurface is the closure of the preimage of $\{0\}\times(Z^2\setminus\text{Diag})$. It is naturally diffeomorphic to $[Z^2;\text{Diag}]$, the product with the diagonal blown up. The intersection of these two faces, the corner, is naturally the sphere bundle, the boundary of the radial compactification of the tangent bundle of Z.

The blow-down map can be composed with the projections to get, for instance

$$\tilde{\pi}_R = \pi_R\circ\beta : Z^2_{\text{sl}} \longrightarrow Z \text{ and } \tilde{\beta} = \pi_{Z^2}\circ\beta : Z^2_{\text{sl}} \longrightarrow Z^2 \tag{29.18}$$

which are also smooth.

Proposition 44. *The kernels of semiclassical operators, forming $\Psi_{\text{sl}}^{-\infty}(Z;E)$ can be identified naturally (by continuity from $\epsilon > 0$) with*

$$\big\{A \in (\beta^*(\epsilon))^{-n}\mathcal{C}^\infty(Z^2_{\text{sl}}; \tilde{\beta}^*\operatorname{Hom}(E)\otimes\tilde{\pi}_R^*\Omega);\\ (\tilde{\beta}^*\epsilon)^n A \equiv 0 \text{ at } \beta^{-1}(\{0\}\times(Z^2\setminus\text{Diag})\big\}. \tag{29.19}$$

Thus, except for the power of ϵ (which can be hidden in the density if one prefers) the kernels are smooth on Z^2_{sl}. The semiclassical symbol then comers from the restriction of the kernel to the front face. If the ϵ factor is absorbed into the density, this is naturally a Schwartz function on TZ with values in the fibre density. The Fourier transform along the fibres then gives the function $\sigma_{\text{sl}}(A)$. The exactness of (29.15) is then just the fact that vanishing at the front face produces a similar kernel with an extra factor of ϵ – since the kernel vanishes rapidly at the 'old' face by assumption.

The product itself can be usefully viewed in this picture too. I may put in a description here if I have an idle moment!

So, all this is setting up the semiclassical calculus of smoothing operators on a compact manifold. Naturally we want to go further and the main immediate extension is to operators on the fibres of a fibration. This is the setting of the Atiyah-Singer theorem.

Recall that a smooth map $\phi : M \longrightarrow Y$ between manifolds (compact or not) is a fibration if there exists another manifold Z such that each point $\bar{y} \in Y$ has an open neighbourhood $U \subset Y$ corresponding to which there is a diffeomorphism F_U giving a commutative diagram

$$\begin{tikzcd} \phi^{-1}(U) \arrow[rr, "F_U"] \arrow[dr, "\phi"'] & & U \times Z \arrow[dl, "\pi_L"] \\ & U. & \end{tikzcd} \tag{29.20}$$

Exercise 29. Recall that the implicit function theorem shows that if M and Y are compact and connected then the condition that ϕ be a submersion, that its differential be surjective at each point of M, implies that it is a fibration. The connectedness conditon can be dropped with minor consequences. Dropping compactness is more serious.

If Y is connected, or by *fiat* in the definition above, the manifold Z is fixed, up to diffeomorphism. I will use the notation

$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & Y \end{tikzcd} \tag{29.21}$$

for a fibration and denote the fibre above $y \in Z$ by $Z_y = \phi^{-1}(y)$. There is no specific map from Z to a given fibre, but such a diffeomorphism does exist by hypothesis from (29.20).

Now, if I am to get this far I will have to be quick. Let me just say, the coordinate invariance of the smoothing algebra and the semiclassical algebra on a fixed manifold, Z, means that it can be transferred to the fibres of ϕ in such a way that we know what

$$\Psi^{-\infty}(M/Y;E) \text{ and } \Psi^{-\infty}_{\mathrm{sl}}(M/Y;E) \tag{29.22}$$

are, where E is a bundle over M (not necessarily coming from a bundle over Y). They are the spaces of smooth sections of bundles of (families of) operators. In the first case we get for each $y \in Y$ an element of $\Psi^{-\infty}(Z_y;E_y)$ where $Z_y = \phi^{-1}(y)$ and E_y is the restriction of E to this submanifold (which of course is diffeomoprhic to Z). In the second case we get a semiclassical family, an element of $\Psi^{-\infty}_{\mathrm{sl}}(Z_y;E_y)$. Enough said, well not quite. We need a little of the geometry of fibrations.

The pull-back of the cotangent bundle of the base $\phi^*T^*Y \longrightarrow T^*M$ is a subbundle and the quotient is denoted

$$T^*(M/Y) = T^*M/\phi^*T^*Y,\ \pi : T^*(M/Y) \longrightarrow M. \tag{29.23}$$

Its fibre can be thought of as the space of fibre-differentials at that point of M.

We will let $G^{-\infty}(M/Y;E)$ be the group of invertibles in $\operatorname{Id}+\Psi^{-\infty}(M/Y;E)$ and $\mathcal{H}^{-\infty}(M/Y;E)$ the space of involutions of the form γ_1+a, $a\in\Psi^{-\infty}(M/Y;\mathbb{C}^2\otimes E)$ where γ_1 is the usual 2×2 matrix.

One of the more serious generalization of the isotropic picture that we need is

Proposition 45. *If Y is compact and* (29.21) *is a fibration with compact fibres (of positive dimension) then for any bundle E over M there are natural identifications*

$$\begin{aligned}\Pi_0(G^{-\infty}(M/Y;E))&=\mathrm{K}^1(Y),\\ \Pi_0(\mathcal{H}^{-\infty}(M/Y;E))&=\mathrm{K}^0(Y).\end{aligned}\tag{29.24}$$

To prove this I will rely on a construction that I will not give in the lectures. Not that it is hard, just that it is not so amusing.

Proposition 46. *For any fibration,* (29.21)*, with compact total space and any complex vector bundle E there is a sequence of elements $\Pi_j\in\Psi^{-\infty}(M/Y;E)$ which are projections, $\Pi_j^2=\Pi_j$ and are such that $A\Pi_j\to A$ and $\Pi_jA\to A$, as $j\to\infty$, for each $A\in\Psi^{-\infty}(M/Y;E)$.*

Proof of Proposition 45. We can retract onto operators acting on sections of the range of the bundle Π_j, for j large enough, and then stabilize to get elements of $\mathrm{K}^1(Y)$ or $\mathrm{K}^0(Y)$ and conversely. I should do this properly, but it is similar to the corresponding proof for a symplectic bundle, once we have the Π_j's. □

This again means that the twisting by the fibration does not matter and extends the claims made above that $G^{-\infty}(Z;E)$ is classifying for odd K-theory.

Finally then after wading through all this stuff, we get a theorem which should be almost self-proving at this stage. Let $\mathrm{GL}(E)$ be the bundle of invertible linear maps on the fibres of E and $\mathcal{H}(E)$ be the bundle with fibre the involutions on the fibres of $\mathbb{C}^2\otimes E$.

Theorem 11. *In case the base and fibre of the fibration* (29.21) *is compact, the semiclassical symbol restricts to give surjective maps with connected fibres*

$$\begin{aligned}G^{-\infty}_{\mathrm{sl}}(M/Y;E)&\longrightarrow\mathcal{S}(T^*(M/Y);\pi^*\mathrm{GL}(E))\\ \mathcal{H}^{-\infty}_{\mathrm{sl}}(M/Y;E)&\longrightarrow\mathcal{S}(T^*(M/Y);\pi^*\mathcal{H}(E)).\end{aligned}\tag{29.25}$$

Complementing E to a trivial bundle and using the standard stabilizations maps gives

$$\begin{aligned}\Pi_0(\mathcal{S}(T^*(M/Y);\pi^*\mathrm{GL}(E)))&\longrightarrow\mathrm{K}^1_c(T^*(M/Y))\ \textit{and}\\ \Pi_0(\mathcal{S}(T^*(M/Y);\pi^*\mathcal{H}(E)))&\longrightarrow\mathrm{K}^0_c(T^*(M/Y))\end{aligned}\tag{29.26}$$

which cover the images as E varies and then, using (29.24)*, define push-forward maps*

$$\begin{aligned}p_{\mathrm{sl}}:\mathrm{K}^1_c(T^*(M/Y))\ni[\sigma_{\mathrm{sl}}(A)]&\longmapsto[R_{\epsilon=1}(A)]\in\mathrm{K}^1(Y),\\ p_{\mathrm{sl}}:\mathrm{K}^0_c(T^*(M/Y))\ni[\sigma_{\mathrm{sl}}(A)]&\longmapsto[R_{\epsilon=1}(A)]\in\mathrm{K}^0(Y).\end{aligned}\tag{29.27}$$

30. Lecture 27: Analytic index of Atiyah and Singer Wednesday, 5 November, 2008

Reminder. *(And partly correction and revision) Last time, despite getting myself pretty seriously in knots on the board – and even in the notes to some extent, I more or less succeeded in defining the* semiclassical push-forward maps in K-theory. *Let me recall how this goes – see Theorem 11 – or at least should have gone. In fact I will add a little bit of stabilization. For a fibration with compact fibres,* (29.21), *(the compactness of the base is not needed at all here) we can define the stabilized semiclassical algebra on the fibres. As a space of 'functions' this is, as defined in Digression 1,*

$$\begin{aligned}(30.1)\quad \big\{F\in\mathcal{C}^\infty([[0,1]\times M^2_\phi;\{0\}\times\mathrm{Diag}];\mathbb{S}(\mathbb{R}^2)\otimes\pi_R^*\Omega);\\ F\equiv 0 \text{ at the old boundary.}\big\}\end{aligned}$$

where I am using illegal blow-up notation and M^2_ϕ is the fibre diagonal – the submanifold of M^2 consisting of the pairs of points in the same fibre. I have also dispensed with the vector bundle E over M, because it is supposed to be embedded in $\mathcal{S}(\mathbb{R})$ by a projection-valued map $\pi_E: M\longrightarrow\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R})$. This globalized definition means that the kernel of $A\in\mathcal{C}^\infty((0,1];\Psi^{-\infty}_{\mathrm{iso}}(M/Y\times\mathbb{R})$ is following form in local coordinates in a patch U in the base and $V\subset\phi^{-1}(U)$ in the fibre

$$\begin{aligned}(30.2)\quad A(\epsilon,y,z,z',y,y')=\\ \epsilon^{-d}F'(\epsilon,y,z,\frac{z-z'}{\epsilon},y,y')|dy'|,\ F\in\mathcal{C}^\infty([0,1]\times U\times V;\mathcal{S}(\mathbb{R}^d_Z\times\mathbb{R}^2))\end{aligned}$$

Here d is the dimension of the fibre, Z, and Ω has been trivialized over the coordinate patch, which is the $|dy'|$ and F' is the local representative of the function in (30.1). *Under composition these form an algebra of operators*

$$(30.3)\quad \Psi^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\ni A:\mathcal{C}^\infty([0,1]\times M;\mathcal{S}(\mathbb{R}))\longrightarrow\mathcal{C}^\infty([0,1]\times M;\mathcal{S}(\mathbb{R})).$$

Moreover they have a symbol map which captures the behaviour at $\epsilon=0$ obtained by restricting F to $\epsilon=0$ and taking the Fourier transform in the 'adiabatic variable' Z which pieces together globally:

$$(30.4)\quad \epsilon\Psi^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\longrightarrow\Psi^{-\infty}_{\mathrm{sl}(),\mathrm{iso}}(M/Y\times\mathbb{R})\xrightarrow{\sigma_{\mathrm{sl}}}\Psi^{-\infty}_{\mathrm{sus}(T^*(M/Y),\mathrm{iso}}(\mathbb{R}).$$

The image of the symbol map is just a family of isotropic smoothing operators on $\mathbb{R}$, i.e. elements of $\mathcal{S}(\mathbb{R}^2)$ depending smoothly, and in a Schwartz manner, on parameters in $T^*(M/Y)=T^*M/\phi^*(T^*Y)$ the bundle of 'fibre differentials'.

As usual $\Psi^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})$ is a Neumann-Fréchet algebra and we can define our usual group and space of involutives:-

$$\begin{aligned}(30.5)\quad \mathcal{G}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})&=\big\{A\in\Psi^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R});\\ &\qquad\exists\ B\in\Psi^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R}),\ (\mathrm{Id}+B)=(\mathrm{Id}+A)^{-1}\big\}\\ \mathcal{H}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})&=\big\{A\in\Psi^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\otimes M(2,\mathbb{C});(\gamma_1+A)^2=\mathrm{Id}\,\big\}.\end{aligned}$$

Now, the 'usual symbolic construction and correction' shows that the adiabatic symbol maps

$$\begin{aligned}\mathcal{G}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R}) &\xrightarrow{\sigma_{\mathrm{ad}}} \mathcal{G}^{-\infty}_{\mathrm{sus}(T^*(M/Y),\mathrm{iso}}(\mathbb{R})\\ \mathcal{H}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R}) &\xrightarrow{\sigma_{\mathrm{ad}}} \mathcal{H}^{-\infty}_{\mathrm{sus}(T^*(M/Y),\mathrm{iso}}(\mathbb{R})\end{aligned}\tag{30.6}$$

are 'homotopy equivalences' in the sense that they induce isomorphisms
(30.7)

$$\begin{aligned}\Pi_0\left(\mathcal{G}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\right) &\xrightarrow{\sigma_{\mathrm{ad}}} \Pi_0\left(\mathcal{G}^{-\infty}_{\mathrm{sus}(T^*(M/Y),\mathrm{iso}}(\mathbb{R})\right)=\mathrm{K}^1_c(T^*(M/Y))\\ \Pi_0\left(\mathcal{H}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\right) &\xrightarrow{\sigma_{\mathrm{ad}}} \Pi_0\left(\mathcal{H}^{-\infty}_{\mathrm{sus}(T^*(M/Y),\mathrm{iso}}(\mathbb{R})\right)=\mathrm{K}^0_c(T^*(M/Y)).\end{aligned}$$

Finally, I 'explained' but did not prove that the corresponding restriction to $\epsilon=1$

$$\begin{aligned}\mathcal{G}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R}) &\xrightarrow{R_{\epsilon=1}} \mathcal{G}^{-\infty}_{\mathrm{iso}}(M/Y\times\mathbb{R})\\ \mathcal{H}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R}) &\xrightarrow{R_{\epsilon=1}} \mathcal{H}^{-\infty}_{\mathrm{iso}}(M/Y\times\mathbb{R})\end{aligned}\tag{30.8}$$

lead to

$$\begin{aligned}\Pi_0\left(\mathcal{G}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\right) &\xrightarrow{R_{\epsilon=1}} \Pi_0\left(\mathcal{G}^{-\infty}_{\mathrm{iso}}(M/Y\times\mathbb{R})\right)=\mathrm{K}^1_c(Y)\\ \Pi_0\left(\mathcal{H}^{-\infty}_{\mathrm{ad}(\phi),\mathrm{iso}}(M/Y\times\mathbb{R})\right) &\xrightarrow{R_{\epsilon=1}} \Pi_0\left(\mathcal{H}^{-\infty}_{\mathrm{iso}}(M/Y\times\mathbb{R})\right)=\mathrm{K}^0_c(Y)\end{aligned}\tag{30.9}$$

where it is these last identifications which are not obvious, because of the twisting in the bundles.

Combining these two maps, the 'invertible' one for the symbol and the second one for the restriction to $\epsilon=1$ *gives the (semiclassical) push-forward, or index, maps*

$$\begin{aligned}(\phi\pi)_!:\mathrm{K}^1_c(T^*(M/Y)) &\longrightarrow \mathrm{K}^1_c(Y)\\ (\phi\pi)_!:\mathrm{K}^1_c(T^*(M/Y)) &\longrightarrow \mathrm{K}^1_c(Y)\end{aligned}\tag{30.10}$$

where $\pi:T^*(M/Y)$ *is the projection onto* M *and* $\phi:M\longmapsto Y$ *so the composite is* $\phi\pi:T^*(M/Y)$.

This is a 'generalization' (it isn't really nbecause of the non-compactness of the fibres) of the Thom isomorphism(s) – where for a real vector bundle, $V\longrightarrow Y$, $T^*(V/Y)\equiv V\times_Y V^*=V\oplus V^*=W$ *is naturally symplectic, and ultimately the Bott periodicity maps, where "*$M=Y\times\mathbb{R}^k$*" and "*$T^*(M/Y)=Y\times\mathbb{R}^{2k}$*". In contrast to these cases the maps* (30.10) *need not be isomorphisms.*

So, either I will go throught that or I will explain the content of the Atiyah-Singer index theorem in K-theory.

In the same setting of the fibration (29.21) we can consider *differential operators* or *pseudodifferential operators* on the fibres of M. These are just operators 'between sections of bundles on Z' but twisted by the diffeomorphisms involved in the transition maps for ϕ. In fact they can be defined perfectly directly. I will not go through the definition here, unless there is a desire for me to do so. Let me just 'remind you' further.

Pseudodifferential operators of order K on a fixed compact manifold Z can be defined either globally through their kernels or locally through coordinate patches – naturally I prefer the first definition but I will briefly describe the second one. First on $\mathbb{R}^n$ we define operators which *are not the isotropic pseudodifferential operators* *discussed above*. They are very closely related but they are not the same. To see the difference in terms of kernels, recall that the kernel of an isotropic pseudodifferential operator of order k (recall that there is an issue of a $\frac{1}{2}$ with the orders) is a distribution on $\mathbb{R}^{2n}$ which can be written in Weyl form as

$$(30.11)\quad A(\frac{z+z}{2},z-z')=\\ (2\pi)^{-n}\int_{\mathbb{R}^n}a(\frac{z+z}{2},\tau)e^{i(z-z')\cdot\tau}d\tau,\ a\in\rho_q^{-k/2}\mathcal{C}^\infty({}^q\overline{\mathbb{R}^{2n}})\\ \Longleftrightarrow A\in\Psi^k_{\text{qiso}}(\mathbb{R}^n).$$

The integral here is not convergent if $k\ge -2n$ but is then to be interpreted as the inverse Fourier transform on tempered distibutions. The 'ordinary' as opposed to 'isotropic' pseudodifferential operators are given by the same formula but with a different class of amplitudes:-

$$(30.12)\quad A(\frac{z+z}{2},z-z')=\\ (2\pi)^{-n}\int_{\mathbb{R}^n}a(\frac{z+z}{2},\tau)e^{i(z-z')\cdot\tau}d\tau,\ a\in\mathcal{S}(\mathbb{R}^n;\mathcal{C}^\infty(\overline{\mathbb{R}^2})\Longleftrightarrow A\in\Psi^k_{\mathcal{S}}(\mathbb{R}^n).$$

This is not quite standard notation, but the $\mathcal{S}$ denotes the rapid vanishing of the coefficients. This is again an algebra of operators on $\mathcal{S}(\mathbb{R}^n)$, but neither of these algebras is contained in the other,

$$(30.13)\quad \begin{gathered}\Psi^k_{\mathcal{S}}(\mathbb{R}^n)\circ\Psi^l_{\mathcal{S}}(\mathbb{R}^n)\subset\Psi^{k+l}_{\mathcal{S}}(\mathbb{R}^n),\\ \Psi^{k-1}_{\mathcal{S}}(\mathbb{R}^n)\longrightarrow\Psi^k_{\mathcal{S}}(\mathbb{R}^n)\xrightarrow{\sigma_k}\mathcal{S}(\mathbb{R}^n;\mathcal{C}^\infty(\mathbb{R}^{n-1};N_k)).\end{gathered}$$

Here N_k is the line bundle over the sphere at infinity generated by the $-k$th power of the defining function – it just hides homogeneity. The symbol sequence here is exact and multiplicative.

If the 'full symbol' a in (30.12) is supported in a set of the form $K\times\overline{\mathbb{R}^n}$ where K is compact (this is essentially impossible for isotropic operators) then the kernel $A(z,z')$ is smooth in $\Omega\times\Omega$ for any open set $\Omega\subset\mathbb{R}^n$ with $\Omega\cap K=\emptyset$. In particular there are plenty of pseudodifferential operators with kernels having compact support in sets of the form $U\times U$ where $U\subset\mathbb{R}^n$ is open. Furthermore, taking such an operator and 'changing coordinates' by making a diffeomorphism to U' again gives a kernel of the same form, with a different amplitude. This coordinate invariance allows pseudodifferential operators to be defined by coordinate covering on any compact manifold Z and acting between sections of any two vector bundles E_+ and E_- over Z. Thus

$$(30.14)\quad \Psi^k(Z;\mathbb{E})\ni A:\mathcal{C}^\infty(Z;E_+)\longrightarrow\mathcal{C}^\infty(Z;E_-)$$

can now be taken to be well-defined. There is a multiplicative symbol map which is invariantly defined and gives

$$(30.15)\quad \Psi^{k-1}(Z;\mathbb{E})\longrightarrow\Psi^k(Z;\mathbb{E})\xrightarrow{\sigma_k}\mathcal{C}^\infty(S^*Z;\pi^*\hom(\mathbb{E})\otimes N_k).$$

Here $S^*Z=\partial\overline{T^*Z}$ is the boundary of the radial compactification of the cotangent bundle, $\pi:S^*Z\longrightarrow Z$ (and I usuall drop the π^* from the notation) and N_k is the same bundle as before. Multiplicativity means

$$(30.16)\quad \Psi^k(Z;F,E_-)\circ\Psi^l(Z;E_+,F)\subset(=)\Psi^{k+l}(Z;E_+,E_-),\ \sigma_{k+l}(AB)=\sigma_k(A)\sigma_l(B).$$

Of course there is a lot more to be said, but I am assuming this is all 'well-known'. Now for the fibration (29.21) we can define the 'pseudodifferential operators acting on the fibres' because of coordinate invariance and we get a similar space of operators 'depending on Y as parameters':

(30.17)
$$\Psi^k(M/Y;\mathbb{E}) \ni A : \mathcal{C}^\infty(M;E_+) \longrightarrow \mathcal{C}^\infty(M;E_-),\ \psi^{-\infty} = \bigcap_k \Psi^k(M/Y;\mathbb{E}).$$

Finally then we arrive at the Atiyah-Singer setting where we have elliptic operators, meaning the symbol has an inverse.

$$\operatorname{Ell}^k(M/Y;\mathbb{E}) = \{A \in \Psi^k(M/Y;\mathbb{E});\ \exists\, b \in \mathcal{C}^\infty(S^*(M/Y);\mathbb{E}^- \otimes N_{-k},\ \sigma_k(A)b = b\sigma_k(A) = \operatorname{Id}\}. \tag{30.18}$$

Then we can construct parametrices and set

$$\mathcal{P}^k(M/Y;\mathbb{E}) = \{(A,B) \in \Psi^k(M/Y;\mathbb{E}) \oplus \Psi^{-k}(M/Y;\mathbb{E});\ R_R = \operatorname{Id} - A \circ B \in \Psi^{-\infty}(M/Y;E_-),\ R_L = \operatorname{Id} - BA \in \Psi^{-\infty}(M/Y;E_+)\}. \tag{30.19}$$

Theorem 12. *For any fibration* (29.21) *with compact total space there are natural maps inducing the* analytic index map

(30.20)
$$\begin{tikzcd}
\mathcal{P}^k(M/Y;\mathbb{E}) \arrow[r] \arrow[dd] & \operatorname{Ell}^k(M/Y;\mathbb{E}) \arrow[d] & \\
& \{a \in \mathcal{C}^\infty(S^*(M/Y);\hom(\mathbb{E}) \otimes N_k) \ \textit{invertible}\} \arrow[r] & \mathrm{K}^0_c(T^*(M/Y)) \arrow[d, "\mathrm{ind}_a"] \\
\mathcal{H}^{-\infty}(M/Y;\mathbb{E}) \arrow[r] & \mathcal{H}^{-\infty}_{\mathrm{iso}}(M/Y \times \mathbb{R}) \arrow[r] & \mathrm{K}^0(Y)
\end{tikzcd}$$

which is equal to the semiclassical push-forward map.

31. Lecture 28: Relative and compactly-supported K-theory Friday, 7 November, 2008

Reminder. *There are still some gaps in the definition of the analytic index of Atiyah and Singer which I wish to fill today – and try to give a little more background as well.*

Let me start by considering a compact manifold with boundary, X. In the main case of initial interest here it is the radially fibre compactified cotangent bundle of a fibration, $X = \overline{T^*(M/Y)}$. Given our basic odd and even classifying spaces $G^{-\infty}$ and $\mathcal{H}^{-\infty}$ there are six 'obvious' K-groups although with several possible, but equivalent definitions:

$$\begin{aligned}\mathrm{K}^0_{\mathrm{c}}(X \setminus \partial X) =&[X \setminus \partial X; \mathcal{H}^{-\infty}]_{\mathrm{c}} \xrightarrow{=} \\ &\Pi_0\left(\{f : X \longrightarrow \mathcal{H}^{-\infty}; f\big|_{\partial X} = \gamma_1\}\right) = \mathrm{K}^0_{\mathrm{c}}(X, \partial X)\end{aligned} \tag{31.1}$$

$$\mathrm{K}^0(X) = [X; \mathcal{H}^{-\infty}] \tag{31.2}$$

$$\mathrm{K}^0(\partial X) = [\partial X; \mathcal{H}^{-\infty}] \tag{31.3}$$

$$\begin{aligned}\mathrm{K}^1_{\mathrm{c}}(X \setminus \partial X) =&[X \setminus \partial X; \mathcal{G}^{-\infty}]_{\mathrm{c}} \xrightarrow{=} \\ &\Pi_0\left(\{f : X \longrightarrow \mathcal{G}^{-\infty}; f\big|_{\partial X} = \mathrm{Id}\}\right) = \mathrm{K}^1_{\mathrm{c}}(X, \partial X)\end{aligned} \tag{31.4}$$

$$\mathrm{K}^1(X) = [X; \mathcal{G}^{-\infty}] \tag{31.5}$$

$$\mathrm{K}^1(\partial X) = [\partial X; \mathcal{G}^{-\infty}]. \tag{31.6}$$

Note the natural equality between the K-spaces 'with compact supports in the interior' and the K-spaces 'relative to the boundary'. The maps are induced by inclusions and are isomorphism because we can make small perturbations.

Proposition 47. *For any compact manifold with boundary there is a* 6*-term exact sequence involving the K-spaces above:*

$$\begin{tikzcd} \mathrm{K}^0_c(X, \partial X) \arrow[r] & \mathrm{K}^0(X) \arrow[r] & \mathrm{K}^0(\partial X) \arrow[d, "\mathrm{cl}_{\mathrm{eo}}"] \\ \mathrm{K}^1(\partial X) \arrow[u, "\mathrm{cl}_{\mathrm{oe}}"] & \mathrm{K}^1(X) \arrow[l] & \mathrm{K}^1_c(X, \partial X) \arrow[l] \end{tikzcd} \tag{31.7}$$

in which the horizontal arrows are induced by inclusions or pull-backs and the vertical, connecting, maps involve identification of a collar neighbourhood of the boundary with $\overline{\mathbb{R}} \times \partial X$.

Exercise 30. I will probably not have time to go through the proof in class – of course this is a standard topolical argument, it is *just the details* that require checking! The spaces have been defined, so the definitions of the six maps need to be checked, and then the 12 statements corresponding to exactness at each space need to be checked. The horizontal maps are clear enough – inclusion of maps which are trivial near the boundary and restriction to the boundary respectively and these project to the homotopy classes. Make sure to check that the vertical maps are well-defined – really they involve retraction to finite rank, followed by 'suspension' from odd to even or even to odd by adding a real parameter and then this 'suspended' object can be converted into a compactly-supported map which is trivial outside a little collar neighbourhood $(0, 1) \times \partial X$ of the boundary.

Then to exactness. In the middle this is clear enough – a map trivial on the boundary comes from one which is compactly supported in the interior. Exactness at the relative spaces can be seen by observing that a class that is mapped to zero in the absolute, central, spaces generates a homotopy on the boundary from which it comes by the map from the boundary space. The exactness at the boundary spaces is really Bott periodicity, at least in the sense that it corresponds to the fact that $\mathrm{cl}_{\mathrm{eo}}$ and $\mathrm{cl}_{\mathrm{oe}}$ induce isomorphism – it is therefore perhaps the trickiest. Roughly said, a class on the boundary, say in $\mathrm{K}^1(\partial X)$, is represented by a map into $g \in \mathcal{C}^\infty(\partial X; \mathcal{G}^{-\infty})$. This is mapped into $\mathrm{cl}_{\mathrm{oe}}(g)$ which is a family of projections with an additional parameter, interpreted as the normal variable near the boundary (sse the sketch). If this is mapped to zero in the interior then this generates an homotopy. Twisting the neck of the boundary around – again see sketch – and using the essential surjectivity of $\mathrm{cl}_{\mathrm{oe}}$ this can be used to construct an absolute class on the manifold which restricts to the original class on the boundary.

Exercise 31. Since we do have Bott periodicity at our disposal there is a rather clearer way to look at the maps in (31.7). Namely we can work with the suspended classifying spaces $\mathcal{G}^{-\infty}_{\mathrm{sus}(p)}$ and consider the spiral of groups

(31.8)
$$\begin{tikzcd}
{[(X,\partial X),(G^{-\infty}_{\mathrm{sus}(p)},\{\mathrm{Id}\})]} \arrow[r] & {[X,G^{-\infty}_{\mathrm{sus}(p)}]} \arrow[r] & {[\partial X,G^{-\infty}_{\mathrm{sus}(p)}]} \arrow[d] \\
{[\partial X,G^{-\infty}_{\mathrm{sus}(p-1)}]} \arrow[u, dotted, "\text{to level } p-2"] & {[X,G^{-\infty}_{\mathrm{sus}(p-1)}]} \arrow[l] & {[(X,\partial X),(G^{-\infty}_{\mathrm{sus}(p-1)},\{\mathrm{Id}\})]} \arrow[l]
\end{tikzcd}$$

In the top left and bottom right the maps and homotopies are required to preserve the pairs, i.. the boundary is mapped to the identity. The horizontal maps are the same as before but the vertical maps involve 'using up' one of the suspension variables and turning it into the normal variable in the collar. See what it takes to show that (31.8) is a long exact sequence as a semi-infinite spiral – starting at $p = \infty$. Show that the maps commute with Bott periodicity and hence it collapses to a 6-term sequence and this is the same as in (31.7).

I have included this boundary sequence, both because it is important (and I plan/planned to include some discussion of analysis for operators on manifolds with boundary) and because it motivates the 'mixed' characterization of the K-theory with compact supports in the interior – however the discussion above is not actually needed for this.

Lemma 36. *For a compact manifold with boundary there is a natural identification*

$$\mathrm{K}^0(X,\partial X) = \Pi_0\left(\mathcal{R}^{-\infty}(X,\partial X)\right) \tag{31.9}$$
$$\mathcal{R}^{-\infty}(X,\partial X) =$$
$$\{(\gamma,g) \in \mathcal{C}^\infty(X;\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R})) \times \mathcal{C}^\infty(\partial X;\tilde{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R};\mathbb{C}^2);(Rg)^{-1}\gamma\big|_{\partial X}(Rg) = \gamma_1\}.$$

Recall that $\tilde{G}$ is the 'half-open loop group'. Meaning that the elements are smooth maps $G : \mathbb{R} \longrightarrow G^{-\infty}$ which approach the identity rapidly at $-\infty$ and approach some element Rg rapidly at $+\infty$.

Proof. To get the map from the space on the right into $\mathrm{K}^0(X,\partial X)$ first observe that there is alway a diffeomorphism from X onto X with an extra boundary strip

$[-1,1]\times\partial X$ glued on and this diffeomorphism well defined up to homotopy through such. Then just 'glue' the curve $g(t)^{-\infty}\gamma\big|_{\partial X}g(t)$ onto the end of γ – which can be assumed to be flat to its limit at the boundary, by identifying $\overline{\mathbb{R}}$ with $[-1,1]$. The map the other way can be taken to be inclusion where γ is flat to γ_1 at the boundary and so can be mapped to (γ,Id). Of course we need to check that these maps induce isomorphism at the level of homotopy but that is clear enough if one recalls Lemma 30 to use in a strip near the boundary. □

In this is the way we can see not only that the symbol data $(a,\mathbb{E})$ of an elliptic operator generates a class in $\mathrm{K}^0_{\mathrm{c}}(T^*(M/Y))$ but also where the identification of the Atiyah-Singer index and the semiclassical push-forward map comes from.

First the K-class of the symbol. Here we use a function $\Theta:\mathbb{R}\longrightarrow\mathbb{R}$ which is flat to 0 at $-\infty$ and to $\pi/2$ at $+\infty$.

Lemma 37. *The symbol data $a\in\mathcal{C}^\infty(S^*(M/Y);\pi^*\hom(\mathbb{E}))$ of an elliptic family $A\in\mathrm{Ell}^0(M/Y;\mathbb{E})$ (so a is invertible) generates a K-class through* (31.9)*, namely if E_+ and $E_=$ are identified with the ranges of projections $\pi_\pm\in\mathcal{C}^\infty(M;\mathbb{C}^N)$ for some N then*

$$\mathrm{Ell}^0(M/Y;\mathbb{E})\longmapsto[(\gamma_E,g)]\in\mathrm{K}_c(\overline{T^*(M/Y)},S^*(M/Y))=\mathrm{K}^0_c(T^*(M/Y),$$

$$\tag{31.10}\begin{gathered}\gamma_E=\begin{pmatrix}\mathrm{Id}_N-\pi_-&0&0&0\\0&-\pi_-&0&0\\0&0&\pi_+&0\\0&0&0&-(\mathrm{Id}_N-\pi_+)\end{pmatrix},\\ g(t)=\begin{pmatrix}\mathrm{Id}_N-\pi_-&0&0&0\\0&\cos(\Theta(t))\pi_-&\sin(\Theta(t))a&0\\0&\sin(\Theta(t))a^{-1}&\cos(\Theta(t))\pi_+&0\\0&0&0&\mathrm{Id}_N-\pi_+\end{pmatrix}.\end{gathered}$$

Here both γ_E and g should be stabilized; γ_E is defined on the whole of M but should be lifted to $\overline{T^*(M/Y)}$ by the projection but g is only defined over the boundary $S^*(M/Y)$, but depends on a parameter and has the property that the value at $+\infty$

$$\tag{31.11} Rg=\begin{pmatrix}\mathrm{Id}_N-\pi_-&0&0&0\\0&0&a&0\\0&a^{-1}&0&0\\0&0&0&\mathrm{Id}_N-\pi_+\end{pmatrix}$$

conjugates γ_E on the boundary to γ_1. There is an issue of orientations here which, as usual, I have not checked.

Proof. It is only necessary to show that this does what it is supposed to in the sense that it defines an element of the space in (31.9). □

So, the point here is that the symbol of the elliptic operator allows us to identify the two bundles E_+ and E_- over the boundary of $\overline{T^*(M/Y)}$ and hence to deform them back into a family of involutions which has compact support in the interior.

Easy part of Theorem 12. The step I have not discussed in the diagram (30.20) is the surjectivity of the map on the right in the middle row, onto $\mathrm{K}^0_{\mathrm{c}}(T^*(M/Y))$ – which is what we have just been discussing. In fact we know from the discussion above that every compactly supported class on $T^*(M/Y)$ can be represented by a pair (γ,g) in the space on the right in (31.9) and that the class is invariant under homotopies in this space. Now, $T^*(M/Y)$ is a real vector bundle and hence $\overline{T^*(M/Y)}$ is a bundle of balls. If γ is restricted to the zero section of $T^*(M/Y)$ it defines a family $\tilde{\gamma} \in \mathcal{C}^\infty(M; \mathcal{H}^{-\infty})$. Moreover the ball bundle can be 'retracted' onto the zero section. This means that γ is homotopic in $\mathcal{C}^\infty(\overline{T^*(M/Y)}; \mathcal{H}^{-\infty})$ to the pull-back of $\tilde{\gamma}$. One the other hand we know that $\tilde{\gamma}$ is homotopic to a family of the form γ_E in (31.9) (in principle the ranks might be different but we already know that they are equal since the projection is homotopic to γ_1 at the boudnary. Thus γ is homotopic to a γ_E in $\mathcal{C}^\infty(\overline{T^*(M/Y)}; \mathcal{H}^{-\infty})$. Under this homotopy, 'information' is streaming out across the boundary, in particular there is an homotpy $\gamma' \in \mathcal{C}^\infty([0,1] \times S^*(M/Y); \mathcal{H}^{-\infty})$ starting at γ_1 and finishing at γ_E lifted under π. In fact we know from Proposition ? that such an homotopy can be realized as a curve under cojugation, that is there exists

$$
\begin{gathered}
\tilde{g} \in \mathcal{C}^\infty(S^*(M/Y); \tilde{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R}; \mathbb{C}^2)) \text{ s.t. } \gamma'(s(t)) = \tilde{g}^{-1}(t)\gamma_1\tilde{g}(t), \\
s : [0,1] \longrightarrow \overline{\mathbb{R}},\ s(0) = \infty,\ s(1) = -\infty.
\end{gathered} \tag{31.12}
$$

It follows from this that the map $R(g)$ conjugates γ_E to γ_1 and I claim that it is, after stabilization, homotopic to the image of an elliptic symbol.

***

Hence the map to $\mathrm{K}^0_{\mathrm{c}}(T^*(M/Y))$ is surjective. □

32. Lecture 29: Toeplitz operators and the semiclassical limit Monday, 10 November, 2008

33. Lecture 30: Topological index Wednesday, 12 November, 2008

Today I want to go through the definition of the topolical index map, $\mathrm{ind}_{\mathrm{t}}$, for a fibration and then start the proof of the equality

$$\mathrm{ind}_{\mathrm{t}} = (\phi\pi)_!^{(\mathrm{sl})} \tag{33.1}$$

where the notation on the right indicates that this is the (direct) push-forward map in K-theory produced by semiclassical quantization.

The topological index is defined following a construction of Gysin. The basic idea here is to 'trivialize' the topology of a given fibration of compact manifolds

$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & Y \end{tikzcd} \tag{33.2}$$

by embedding into a trivial fibration.

Proposition 48. *For any fibration of compact manifolds,* (33.2)*, there is an embedding as a subfibration of a product* $i : M \longrightarrow M' = \mathbb{R}^N \times Y \xrightarrow{\pi_R} Y$ *giving a commutative diagram*

$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[rr, hook, "i"] \arrow[dr, "\phi"'] & & M' \arrow[r, "\pi_L"] \arrow[dl, "\pi_R"] & \mathbb{R}^N \\ & & Y. & & \end{tikzcd} \tag{33.3}$$

After stabilization, taking the product with some $\mathbb{R}^M$*, any two such embeddings are homotopic.*

Proof. Any compact manifold, such as M, can be embedded in a Euclidean space of sufficiently high dimension – indeed the dimension can be estimated quite well. Here we do not care about the codimension of the embedding. To do this, take a finite covering of M by coordinate neighbourhoods U_i, $i = 1, \dots, k$, on each of which the coordinate map is $F_i : U_i \longrightarrow \mathbb{R}^n$, $n = \dim M$. Then take a partition of unity, χ_i, subordinate to the cover and consider the smooth map

(33.4)
$$i' : M \longrightarrow \mathbb{R}^{nk},\ i(p) = \sum_i e_i \chi_i F_i(p),\ e_i : \mathbb{R}^n \longrightarrow \mathbb{R}^{nk} \text{ being the } i\text{th embedding.}$$

This is a globally smooth map which is injective and has everywhere injective differential. It is therefore a global embedding. To get the embedding i giving (33.3) take $i = i' \times \phi : M \longrightarrow \mathbb{R}^N \times Y$ where $N = nk$.

The stable homotopy equivalence we really do not need, but let me indicate how to do it anyway. First, we can always increase the dimension N by adding an extra factor of $\mathbb{R}^p$ to M' and extending the map i by mapping M to 0 in this factor. Given two embeddings, stabilize them to have image spaces of the same dimension, and then stabilize further by adding an extra factor of the stabilized fibre dimension, $\mathbb{R}^N$, to each map, interpreting the first as mapping into the first factor and the second into the second factor. Now simply use the standard rotation between the factors of $\mathbb{R}^N$ to deform one map into the other – checking of course that the conditions persist. $\square$

Having embedded ϕ as a subfibration of a trivial fibration we now use the collar neighbourhood theorem.

Proposition 49. *The image $i(M) \subset \mathbb{R}^N \times Y$ of an embedding (33.3) has an open neighbourhood Ω with closure $\overline{\Omega} \subset \mathbb{R}^N \times Y$ a compact manifold with boundary which fibres over $i(M)$ as a radially compactified (real) vector bundle and gives a commutative diagram*

$$\begin{tikzcd} \overline{\mathbb{R}^q} \arrow[r, dash] & \overline{\Omega} \arrow[r, hook] \arrow[d, "\pi"', shift right] & \mathbb{R}^N \times Y \arrow[ddl, "\pi_R"] \\ & i(M) \arrow[u, bend right, "0_\pi"'] \arrow[d, "\phi"] & \\ & Y. & \end{tikzcd} \tag{33.5}$$

Proof. This really is just the collar neighbourhood theorem, perhaps with a little smoothness in parameters. Namely, an embedded submanifold, such as $i(M) \subset \mathbb{R}^N \times Y$ has an open neighbourhood which fibres over the manifold and in such a way that the resulting bundle is diffeomorphic to an open neighbourhood of the zero section of a vector bundle over $i(M)$ with the fibration being the bundle projection. Given this we can easily shrink the neighbourhood a little so that it is the image of a closed ball bundle, and call this $\overline{\Omega}$ and the projection π. The only thing to check is that we can make this bundle structure over $i(M)$ compatible with π_R, meaning that π_R factors through it. This is just the requirement that the vector bundle structure π projects the intersection of $\overline{\varpi}$ with $\mathbb{R}^N \times \{y\}$ to the fibre Z_y above y. To ensure that each of the fibres of $\overline{\Omega}$ over $i(M)$ is contained in one of the fibres $\mathbb{R}^N$ it is enough to recall that one proof of the collar neighbourhood theorem proceeds through the exponential maps of *any* Riemannian metric, in the normal directions to the embedded submanifold. In fact it is enough to use a bundle of directions complementary to the tangent bundle. If the metric is taken to be the product metric on $\mathbb{R}^N \times Y$ and the bundle of initial points for the exponential map to be the normals to Z_y within the fibre then the resulting map π respects the fibres. $\square$

Note that the vector bundle structure defined by π on a neighbourhood is necessarily isomorphic to the normal bundle to $i(M)$ in $\mathbb{R}^N \times Y$ and hence to the bundle of normals to the fibres Z_y in $\mathbb{R}^N$. This is later important when we look at the Chern character of the index, i.e. the Atiyah-Singer index formula.

Consider what (33.5) shows for the fibre cotangent bundle $T^*(M/Y)$ for the original fibration ϕ. In the construction above, $\overline{\Omega}$ has been identified smoothly with the total space of a radially compactified vector bundle $U \longrightarrow M$, if we use i to identify M with $i(M)$. This means that the fibre cotangent bundle of $\overline{\Omega}$, as a fibration over $i(M)$ is identified with

$$T^*(\Omega/Y) \simeq T^*(M/Y) \oplus (U \oplus U^*) = T^*(M/Y) \oplus W \tag{33.6}$$

as vector bundles over M. Here $W = U \oplus U^*$ has a natural symplectic structure given in terms of the pairing of U and its dual U^*. Thus, by the Thom isomorphism

$$\mathrm{K}^0_{\mathrm{c}}(T^*(M/Y)) \simeq \mathrm{K}^0_{\mathrm{c}}(T^*(\overline{\Omega}/Y)) \tag{33.7}$$

where we regard W as a symplectic vector bundle over $T^*(M/Y)$.

Now, $\Omega \hookrightarrow \mathbb{R}^N \times Y$ is an open subset, with consistent fibration. Thus the fibre cotangent bundle also embeds as an open subset

$$T^*(\Omega/Y) \hookrightarrow T^*((\mathbb{R}^N \times Y)/Y) = T^*(\mathbb{R}^N) \times Y = \mathbb{R}^{2N} \times Y. \tag{33.8}$$

Thus compactly supported K-theory on the open subset is mapped into compactly supported K-theory of the larger open set

$$\iota : \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/Y)) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2N} \times Y). \tag{33.9}$$

Finally we can apply Bott periodicity to get a composite map which can be written out in steps:

$$\begin{tikzcd}
\mathrm{K}^0_{\mathrm{c}}(T^*(M/Y)) \arrow[r, "\mathrm{Thom}"] \arrow[d, "\mathrm{ind}_{\mathrm{t}}"'] & \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/Y)) \arrow[d, "\iota"] \\
\mathrm{K}^0(Y) & \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2N} \times Y) \arrow[l, "\mathrm{Bott}"]
\end{tikzcd} \tag{33.10}$$

but in principle might depend on the embedding.

Exercise 32. Show that this topological index map does not change under stabilization by additional Euclidean factors in the embedding and also under homotopies of the embedding. Hence conclude that it is in fact well-defined.

I do not feel the need to show the independence of the choice of embedding in the definition of $\mathrm{ind}_{\mathrm{t}}$ because we can show (33.1) without using this. Since the map on the right, given by semiclassical quantization, knows nothing of the embedding this will show the naturality of the topological index as well.

Theorem 13. *The identity* (33.1) *between semiclassical push-forward and topological index maps holds for any embedding of M as a subfibration of a trivial fibration as in* (33.3)*.*

Proof. The strategy is to follow semiclassical quantization around the diagram (33.10) where we have to be a bit careful of some of the identifications that have been made. So we need to check that all the maps in the following diagram are well-defined and all the triangles commute:

$$\begin{tikzcd}
\mathrm{K}^0_{\mathrm{c}}(T^*(M/Y)) \arrow[dd, "q_{\mathrm{sl}}"'] & \mathrm{K}^0_{\mathrm{c}}(\tilde{W}/T^*(M/Y)) \arrow[l, "q_{\mathrm{sl}}"'] \arrow[d, "i"] \arrow[ddl, "q_{\mathrm{sl}}"'] \\
 & \mathrm{K}^0_{\mathrm{c}}(T^*(\overline{\Omega}/Y)) \arrow[d, "\iota"] \arrow[dl, "q_{\mathrm{sl}}"'] \\
\mathrm{K}^0(Y) & \mathrm{K}^0_{\mathrm{c}}(\mathbb{R}^{2N} \times Y) \arrow[l, "q_{\mathrm{sl}}"']
\end{tikzcd} \tag{33.11}$$

Perhaps unhelpfully there are five maps labelled q_{sl} and I have added an extra step compared to (33.10) corresponding to the identification of the normal neighbourhood of $i(M)$ with the normal bundle. Thus the map on the left is semiclassical quantization (of involutions) on the fibres of a fibration. The top map is isotropic quantization (in the same general sense) for a symplectic vector bundle over a base – in this case the base is $T^*(M/Y)$. This we know gives the Thom isomorphism so gives the top arrow in (33.10) after reversal. The top sloping map is the combination of these –

isotropic on the fibres of a fibration over a fibre-bundle of manifolds. We need to check that this is well-defined and gives a commutative first triangle. As you can imagine at this point, the commutativity is some double-adiabatic argument but slightly different to what we did before since one part is a compact manifold and the other is a symplectic vector bundle – before they were both bundles. There is a more significant difference in that this is not the fibre product of two fibrations but a double fibration, one is above the other. This means the double-adiabatic algebra needs to be a little different. The second sloping arrow is somewhat new. I mentioned this at some point, but this is the 'same' as the left arrow except that now we have a fibration where the fibres are compact manifolds with boundary. Once this quantization map is defined we need to show commutativity of the triangle above it, meaning that the 'isotropic' quantization can be replaced by the 'manifold' quantization (hence coordinate invariant) in this case. Again I mentioned earlier that this was pretty obvious, but it does need to be done. Finally the bottom q_{sl} is again adiabatic quantization. So once again the commutativity here is the key with the Bott periodicity map isotropic but the one above it not defined precisely this way. □

34. Lecture 31: Iterated fibrations and Multiplicativity Friday, 14 November, 2008

Reminder. *We need to complete the proof of the equality of the topological index, introduced last time, and the semiclassical push-forward map in K-theory.*

First for the new construction for today, although it is not really so new. Namely extending the smoothing algebra and semiclassical and adiabatic constructions to a compact manifold with boundary. A $\mathcal{C}^\infty$ manifold with boundary is a Hausdorff topological space with a covering by open sets on each of which is homeomorphism is given to a (relatively) open subset of $[0,\infty)\times\mathbb{R}^{n-1}$ such that the transition maps, on intersections, are smooth. Note smoothness for a map on $U\subset[0,\infty)\times\mathbb{R}^{n-1}$ means boundedness of all derivatives including up to the boundary.

Given such a manifold Z there are two competing candidates for smooth functions. Namely the 'obvious' $\mathcal{C}^\infty(Z)$ which consists of the functions smooth in local coordinates and $\dot{\mathcal{C}}^\infty(Z)\subset\mathcal{C}^\infty(Z)$ consisting of the smooth functions which also vanish to infinite order at the boundary. The same sorts of definitions make sense on a manifold with corners, but for the moment we only need the case of the product Z^2. Just as in the case of a manifold without boundary, the density bundle Ω is well defined and its sections can be invariantly integrated over compact sets. This means that there are two clases of smoothing operators on Z; those with kernels in $\mathcal{C}^\infty(Z^2;\pi_R^*\Omega)$ and the smaller class with kernels in $\dot{\mathcal{C}}^\infty(Z^2;\pi_R^*\Omega)$. These spaces can be conveniently interpreted as $\mathcal{C}^\infty(Z;\mathcal{C}^\infty(Z;\Omega))$ and $\dot{\mathcal{C}}^\infty(Z;\dot{\mathcal{C}}^\infty(Z;\Omega))$ respectively.

Both spaces are closed under operator composition, essentially by Fubini's theorem with the composition looking the same as in the boundaryless case

$$A\circ B(z,z')=\int_Z A(z,z'')B(z'',z'). \tag{34.1}$$

The two algebras of smoothing operators will be denoted $\Psi^{-\infty}(Z)$ and $\dot{\Psi}^{-\infty}(Z)$, with the 'dot' denoting the infinite vanishing at the boundary.

Similarly there is no difficulty in extending the construction of the semiclassical algebra to this setting, I leave the details to you. However there is one useful thing to note about a compact manifold with boundary. Namely it is always possible to 'double' a compact manifold with boundary Z to a compact manifold without boundary, $2Z$ which as a set is two copies of Z with boundaries identified. In fact $2Z$ is not really well-defined in the sense that there is no natural $\mathcal{C}^\infty$ structure on this double, by there is a choice so that $Z\longrightarrow 2Z$ is a diffeomorphism onto its range, which is one of the copies of Z.

Lemma 38. *Suppose $Z\longrightarrow X$ is an embedding of a compact manifold with boundary (or corners for that matter) as the closure of an open subset of a compact manifold without boundary (which is always possible) then the algebra $\dot{\Psi}^{-\infty}(Z)$ is naturally identified with the subalgebra of $\Psi^{-\infty}(Z)$ corresponding to the kernels with support in $Z\times Z\subset X\times X$.*

Proof. The basic observation is that $\dot{\mathcal{C}}^\infty(Z)$ is identified with

$$\{u\in\mathcal{C}^\infty(X);\operatorname{supp}(u)\subset Z\}. \tag{34.2}$$

Applying this in both factors gives the result, provided densities are taken care of. □

In particular, irrespective of the choice of $\mathcal{C}^\infty$ structure on $2Z$, $\dot\Psi^{-\infty}(Z)$ is the subalgebra of $\Psi^{-\infty}(2Z)$ with kernels supported in $Z\times Z$.

This is important for our proof and also allows us to *define* the adiabatic algebra for $Z\times\mathbb{R}^n$ for instance as the subalgebra

$$\begin{aligned}(34.3)\quad \dot\Psi^{-\infty}_{\mathrm{ad,iso}}(Z;\mathbb{R}^n)=\big\{A\in\Psi^{-\infty}_{\mathrm{ad,iso}}(2Z;\mathbb{R}^n);\\ \text{the kernel has }\operatorname{supp}(A)\subset(0,1)\times Z\times Z\times\mathbb{R}^n\times\mathbb{R}^n\big\}.\end{aligned}$$

This saves quite a bit of work and allows everything to be extended to fibrations etc although there are still some things to check. Let me just restate the basic result we have used in the compact boundaryless case in this context.

Proposition 50. *For a fibration of compact manifolds where the total space M has boundary, but the base Y does not,*
(34.4)
$$\dot{\mathcal{H}}^{-\infty}_{\mathrm{ad,iso}}(M/Y;\mathbb{R}^n)=\{I=\gamma_1+a;\ a\in\dot\Psi^{-\infty}_{\mathrm{ad,iso}}(M/Y;\mathbb{R}^n)\otimes M(2,\mathbb{C}),\ I^2=\mathrm{Id}\}$$
has a semiclassical symbol map which induces an 'homotopy equivalence' (identity on components)

$$(34.5)\qquad \dot{\mathcal{H}}^{-\infty}_{\mathrm{ad,iso}}(M/Y;\mathbb{R}^n)\xrightarrow{\sigma_{\mathrm{ad}}}\mathcal{S}(T^*(M/Y);\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n))$$

which via restriciton to $\epsilon=1$ induces the push-forward map

$$\begin{aligned}(34.6)\quad \mathrm{K}^0_c(T^*((M\setminus\partial M)/Y))\simeq\Pi_0(\mathcal{S}(T^*(M/Y);\mathcal{H}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n))\xrightarrow{R_{\epsilon=1}}\\ \dot{\mathcal{H}}^{-\infty}_{\mathrm{iso}}(M/Y;\mathbb{R}^n)\simeq\mathrm{K}^0(Y).\end{aligned}$$

Proof. Everything here is pretty much as before but I should really go through it step by step. In particular the last part, which is the fact that the homotopy classes of sections of the bundle over Y of involutions which are fibre-smoothing perturbations of γ_1 reduces to the K-theory of Y – again this uses the existence of finite-rank exhausting families of projections. □

Now, having extended the semiclassical quantization, or push-forward, map to fibrations where the fibres are compact manifolds with boundary it is important to note that this is related to the isotropic case.

Proposition 51. *Under the compactification map $\mathbb{R}^n\hookrightarrow\overline{\mathbb{R}^n}$ the algebras $\dot\Psi^{-\infty}(\overline{\mathbb{R}^n})$ and $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ are identified.*

Proof. This is basically the identification of $\mathcal{S}(\mathbb{R}^n)$ with $\dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^n})$. □

Now the same thing is almost true of the adiabatic versions of these algebras. The only difference is the (by some accounts weird) scaling in the isotropic case. Indeed the kernel in the isotropic case can be written

$$(34.7)\quad \epsilon^{-n}F(\epsilon,\frac{\epsilon(z+z')}{2},\frac{z-z'}{\epsilon},Z,Z')=T_\epsilon\epsilon^{-2n}F(\epsilon,\frac{z+z'}{2},\frac{z-z'}{\epsilon^2},Z,Z')T_\epsilon^{-1}$$

where T_ϵ is the coordinate change $z\longmapsto z/\epsilon$.

Proposition 52. *The parameter-dependent coordinate transformation, T_ϵ, reduces an isotropic-adiabatic family of operators on $\mathbb{R}^n$ to an adiabatic family on the manifold with boundary $\overline{\mathbb{R}^n}$ with parameter ϵ^2.*

This is enough to take care of almost all of the commutativity results we need, except for the most important one. Namely we need to show the commutativity of the top triangle in (33.11).

Proposition 53. *Let $\pi_U : U \longrightarrow M$ be a real vector bundle over the total space of a fibration* (33.2) *then the semiclassical push-forward maps give a commutative diagram*

$$\begin{tikzcd} \mathrm{K}^0_c(T^*(M/Y) \oplus (U \oplus U')) \arrow[d, "q_{\mathrm{sl}}"'] \arrow[dr, "q_{\mathrm{sl}}"] & \\ \mathrm{K}^0_c(T^*(M/Y)) \arrow[r, "(\phi\pi)_!"'] & \mathrm{K}^0(Y) \end{tikzcd} \tag{34.8}$$

where the sloping map is given by semiclassical quantization on the fibres, which compact are manifolds with boundary, of $\phi\overline{\pi_U} : \overline{U} \longrightarrow Y$, the vertical map is given by isotropic quantization on the fibres of U and the horizontal map is given by semiclassical quantization on the fibres of ϕ.

Proof. The proof is very close to the similar commutation result for the direct sum of two symplectic bundles. There are two differences, first of course one of the fibrations has fibres which are compact manifolds and the second difference is that U is a bundle over M, not over Y, so this is not a fibre product of bundles over Y. In particular there is only one form of (34.8) – it does not make sense to try to quantize the ϕ fibration *before* the quantization on the fibres of U since the fibres vary along the fibres of M. Still, pretty much the approach works.

Thus, we wish to construct and use a double-adiabatic algebra of smoothing operators. Consider what the kernels should be. There are two parameters, ϵ and δ and in terms of local coordinates y in the base, z on the fibres of ϕ and u linear coordinates on U_m, locally trivialized, the kernels should be of the form

$$\epsilon^{-n}\delta^{-p}F(\epsilon,\delta,y,z,\frac{z-z'}{\epsilon},\frac{\epsilon^{\frac{1}{2}}\delta(u+u')}{2},\frac{u-u'}{\epsilon^{\frac{1}{2}}\delta}) \tag{34.9}$$

where F is smooth in all variables and Schwartz in the last three collections of variables. Note the difference with the double isotropic case, the ϵ semiclassical parameter (de-) quantizes in both variables, whereas the δ parameter does so only in the fibre variables.

So the kernels are specified locally near the *fibre diagonal* which is $z = z'$ by (34.9) and away from $z = z'$ the kernels are supposed to be smooth in the z and z' variables (the difference does not make sense since they are generally in different coordinate patches) and rapidly vanishing with all derivatives as $\epsilon \downarrow 0$. The behaviour in u and u' is already specified globally on the fibres of U since they are linear.

Of course the main thing to show is that these operators form an algebra. However this is not significantly different from the earlier discussions. Certainly for $\epsilon >$ this is just an adiabatic family in the isotropic smoothing operators on the fibres of U so it is only necessary to check what happesn as $\epsilon \downarrow 0$. The rapid vanishing in the off-diagonal part in the z, z' variables quelches all other behaviour as is easily seen. Thus it suffices to look at the composition of two kernels of the form (34.9) with compact support in the one coordinate patch Ω in the local fibres

Z and with $U = \mathbb{R}^p$ locally. The composite is then

$$(34.10)\quad \epsilon^{-2n}\delta^{-2p}\int_\Omega\int_{\mathbb{R}^p} F(\epsilon,\delta,y,z,\frac{z-z''}{\epsilon},\frac{\epsilon^{\frac12}\delta(u+u'')}{2},\frac{u-u''}{\epsilon^{\frac12}\delta})$$
$$G(\epsilon,\delta,y,z'',\frac{z''-z'}{\epsilon},\frac{\epsilon^{\frac12}\delta(u''+u')}{2},\frac{u''-u'}{\epsilon^{\frac12}\delta})$$
$$=\epsilon^{-n}\delta^{-p}H(\epsilon,\delta,y,z,\frac{z-z'}{\epsilon},\frac{\epsilon^{\frac12}\delta(u+u')}{2},\frac{u-u'}{\epsilon^{\frac12}\delta})$$

where

$$(34.11)\quad H(\epsilon,\delta,y,z,Z,t,s),\ =\int_\Omega\int_{\mathbb{R}^p} F(\epsilon,\delta,y,z,Z',t+\frac{\epsilon\delta^2(r+s)}{4},\frac{s-r}{2})$$
$$G(\epsilon,\delta,y,z-\epsilon Z',Z-Z',t+\frac{\epsilon\delta^2(r-s)}{4},\frac{s+r}{2}).$$

Secondly we need to understand the symbolic properties in the two, or in some sense three, adiabatic limits. These follow directly from (34.11). □

35. Topic 4: Thom isomorphism and the Todd class
In place of lecture for Monday, 18 November, 2008

For any complex/symplectic vector bundle, $W \longrightarrow Y$, the diagram

$$\begin{tikzcd} \mathrm{K}^0_{\mathrm{c}}(W) \arrow[r, "\mathrm{Ch}\wedge\mathrm{Td}(W)"] \arrow[d, "\mathrm{Thom}"'] & \mathrm{H}^{\mathrm{even}}_{\mathrm{c}}(W) \arrow[d, "\mathrm{Thom}"] \\ \mathrm{K}^0(Y) \arrow[r, "\mathrm{Ch}"'] & \mathrm{H}^{\mathrm{even}}(Y) \end{tikzcd} \tag{35.1}$$

commutes.

36. Topic 5: Atiyah-Hirzebruch theorem
In place of lecture for Wednesay, 20 November, 2008

Theorem 14. *For any smooth manifold, X, the total Chern character gives an isomorphsim*

$$\mathrm{Ch}^* : \mathrm{K}^*_c(X) \otimes \mathbb{C} \xrightarrow{\simeq} \mathrm{H}^*_c(X;\mathbb{C}). \tag{36.1}$$

This was originally proved by Atiyah and Hirzebruch using a spectral sequence argument coming from a filtration of K-theory and cohomology (which indeed works for any generalized cohomology theory) based on the 'skeleton' of the manifold as a CW complex. I will outline here a more pedestrian argument, which is essentially sheaf theory, corresponding to the Mayer-Vietoris complex. In fact it is really a Čech-theoretic version of the argument of Atiyah and Hirzebruch.

First recall the long exact sequence for K-theory for a manifold relative to its boundary – Proposition 47. Although I did not go through the proof in detail, any reasonable proof extends to the non-compact case to give the analogous sequence with compact supports:-

$$\begin{tikzcd} \mathrm{K}^0_{\mathrm{c}}(X,\partial X) \arrow[r] & \mathrm{K}^0_{\mathrm{c}}(X) \arrow[r] & \mathrm{K}^0_{\mathrm{c}}(\partial X) \arrow[d, "\mathrm{cl}_{\mathrm{eo}}"] \\ \mathrm{K}^1_{\mathrm{c}}(\partial X) \arrow[u, "\mathrm{cl}_{\mathrm{oe}}"] & \mathrm{K}^1_{\mathrm{c}}(X) \arrow[l] & \mathrm{K}^1_{\mathrm{c}}(X,\partial X) \arrow[l] \end{tikzcd} \tag{36.2}$$

From this we can pass to the Mayer-Vietoris sequence for a decomposition into manifolds with boundary in the following sense. Let X be a (generally non-compact) manifold. Let $\rho_i \in \mathcal{C}^\infty(X)$ be two real functions such that $H_i = \{\rho_i\}$ are smooth disjoint hypersurfaces on which $d\rho_i \neq 0$ and in addition

$$H_1 \subset X_2 = \{\rho_2 \geq 0\},\ H_2 \subset X_1 = \{\rho_1 \geq 0\},\ X = X_1 \cup X_2. \tag{36.3}$$

Since they do not intersect, these hypersurfaces lie in the interior of the 'other' manifold with boundary. Thus $Y = X_1 \cap X_2$ is also a manifold with boundary.

Picture.

Proposition 54. *There is a long exact (Mayer-Vietoris) complex in K-theory*

$$\begin{tikzcd} \mathrm{K}^0_c(Y\setminus\partial Y) \arrow[r] & \mathrm{K}^0_c(X_1\setminus H_1)\oplus \mathrm{K}^0_c(X_2\setminus H_2) \arrow[r] & \mathrm{K}^0_c(X) \arrow[d] \\ \mathrm{K}^1_c(X) \arrow[u] & \mathrm{K}^1_c(X_1\setminus \partial X_1)\oplus \mathrm{K}^1_c(X_2\setminus \partial X_2) \arrow[l] & \mathrm{K}^1_c(Y\setminus\partial Y). \arrow[l] \end{tikzcd} \tag{36.4}$$

Here the top right and bottom left horizontal maps are the sums of 'inclusions' given by extending maps trivial to the boundary to be trivial across it. The other two horizontal maps are also given by the two inclusion maps, with appropriately chosen signs. The vertical, connecting, homomorphisms are the sums, with orientations, of the restrictions to two H_i composed with $\mathrm{cl}_{\mathrm{oe}}$ or $\mathrm{cl}_{\mathrm{eo}}$ and then embedded in the interior of Y.

Proof. I leave the proof that this is a complex and exactness – which is clear except on the sides – as an extended exercise, at least for the moment. ☐

Proposition 55. *If the sequence* (36.4)*, and the corresponding sequence for cohomology with compact supports, are wrapped up then the combined Chern characters give a commutative diagram (i.e a natural transformation)*

(36.5)
$$\begin{tikzcd}
 & & \mathrm{H}^*_c(X) \arrow[dddd, bend left=60] \\
 & & \mathrm{K}^*_c(X) \arrow[u, "\mathrm{Ch}^*"'] \arrow[dd] \\
\mathrm{H}^*_c(\mathring{X}_1) \oplus \mathrm{H}^*_c(\mathring{X}_2) \arrow[uurr] & \mathrm{K}^*_c(\mathring{X}_1) \oplus \mathrm{K}^*_c(\mathring{X}_2) \arrow[l, "\mathrm{Ch}^*"'] \arrow[ur] & \\
 & & \mathrm{K}^*_c(Y \setminus \partial Y) \arrow[ul] \arrow[d, "\mathrm{Ch}^*"'] \\
 & & \mathrm{H}^*_c(Y \setminus \partial Y) \arrow[uull]
\end{tikzcd}$$

Proof. This mainly involves the earlier discussion of the behaviour of the Chern character under $\mathrm{cl}_{\mathrm{oe}}$ and $\mathrm{cl}_{\mathrm{eo}}$. □

Any compact manifold can be reconstructed from combinations of this type:-

Proposition 56. *For any compact manifold X there are two sequences of open submanifolds $X'_j \subset X$, with $X'_0 = \emptyset$ and $X'_N = X$, and $B'_j \subset X$ such that $X'_j = X'_{j-1} \cup B_j$, the closures $X_{j-1}\overline{X'_{j-1}}$ and $B_j = \overline{B'_j}$ in X'_j are smooth manifolds with boundary with X'_j decomposed in terms of them as in* (36.3) *for each j and such that the intersection $B'_j \cap X'_{j-1}$ is a finite union of disjoint balls.*

Proof. This can be accomplished by covering X with finitely many sufficiently small balls with respect to some Riemann metric and then slightly adjusting the radii to avoid unpleasant intersections. In particular this gives a good open cover. □

Discuss tensor products of abelian groups briefly and that

$$\mathrm{Ch}^* : \mathrm{K}^*_c(X) \otimes \mathbb{C} \longrightarrow \mathrm{H}^*_c(X). \tag{36.6}$$

Proof of Theorem 14. First check that for an open ball the combined Chern character does indeed give an isomorphism

$$\mathrm{Ch}^* : \mathrm{K}^*_c(B) \otimes \mathbb{C} \longrightarrow \mathrm{H}^*_c(B; \mathbb{C}). \tag{36.7}$$

It follows that this is equally true for a finite union of disjoint open balls.

Now, proceeding inductively we may assume that the same is true for X'_j for $j < k$. Then in the (36.5), after tensoring with $\mathbb{C}$, for X'_j relative to X_{j-1} and B_j, two if the Ch^* arrows are known to be isomorphisms. It follows, from diagram chasing often called the 'Five Lemma' that the third is also an isomorphism, proving the desired result. □

37. Topic 6: The Atiyah-Singer index formula
In place of lecture for Friday, 22 November, 2008

$$\mathrm{Ch}(\mathrm{ind}(P)) = \int T^*(M/B)\,\mathrm{Ch}\,\sigma(P)) \wedge \mathrm{Td}(\phi) \text{ in } \mathrm{H}^{\mathrm{even}}(B). \tag{37.1}$$

38. Topic 7: Product-type pseudodifferential operators In place of lecture for Monday, 25 November, 2008

Algebras of psuedodifferential operators associated to products of manifolds and fibrations and why they can be useful. The will be used in the next 'Exercise', can be used to give a smoothed-out version of the original embedding proof of the index theorem by Atiyah and Singer and will be used below in the disussion of smooth K-homology.

39. Topic 8: More on the determinant bundle
In place of lecture for Wednesday, 27 November, 2008

These are as yet very crude notes.

Consider the 3×3 commutative block in which the groups are only roughly identified:-

(39.1)
$$
\begin{matrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{matrix}
\quad = \quad
\begin{tikzcd}
G^{-\infty}_{\mathrm{sus}} \arrow[r] \arrow[d] & \dot G^0_{\mathrm{sus}} \arrow[r, "\sigma"] \arrow[d] & G^{-\infty}_{\mathrm{sus}(2)} \arrow[d] \\
\tilde G^{-\infty}_{\mathrm{sus}} \arrow[r] \arrow[d, "R"] & \dot G^0_{\mathrm{sus\,pt}\,\infty} \arrow[r, "\sigma"] \arrow[d, "R"] & \tilde G_{\mathrm{sus}(2)} \arrow[d, "R"] \\
G^{-\infty} \arrow[r] & \dot G^0 \arrow[r, "\sigma"] & G^{-\infty}_{\mathrm{sus,ind}=0}
\end{tikzcd}
$$

In more detail:-

G_{11} : This is the classifying group for even K-theory $G^{-\infty}_{\mathrm{sus,iso}}(\mathbb{R}^2)$ consisting of the elements $a \in \mathcal{S}(\mathbb{R}^5)$ where the first variable is a parameter, so the product is pointwise in this variable and in the last four variables is as smoothing operators on $\mathcal{S}(\mathbb{R}^2)$ and $\mathrm{Id} + a(t)$ is required to be invertible for all t.

G_{21} : This is the contractible, half-free version of the preceeding group - it consists of smooth loops in $G^{-\infty}(\mathbb{R}^2)$ which have Schwartz derivative and tend to Id as $t \to -\infty$.

G_{31} : This is the classifying group for odd K-theory $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^2)$.

G_{*1} : Is therefore the (flat) delooping sequence for $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^2)$.

G_{12} : This is the symbolically suspended group of invertible isotropic pseudodifferential operators on $\mathbb{R}$ with values in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R})$ and normalization condition. As functions the kernels can be identified with functions on $\overline{\mathbb{R}^3} \times \mathbb{R}^2$ which are $\mathcal{C}^\infty$ and Schwartz in the last two variables. The first variable, t, is a parameter and the functions are required to vanish to infinite order at C which is a great half circle in the t direction. The are quantized to operators by Weyl quantization in the second two variables and then we require $\mathrm{Id} + a(t)$ to be invertible for all t. This group is contractible.

G_{22} : This group is supposed to be similar to the previous one except it is now of product type. As functions the elements are smooth on $[\overline{\mathbb{R}^3}, \{t = \infty\}] \times \mathbb{R}^2$ and vanish to infinite order at the lift of C to the blow up – which means the closure of the complement of $t = \infty$. The product extends to these more general functions and we look at the group of invertible perturbations as before. This is also a contractible group.

G_{32} : This is really a $*$-extended version of the usual group $\dot G^0_{\mathrm{iso}}(\mathbb{R}; \mathbb{R})$. The latter consists of the smooth functions on $\overline{\mathbb{R}^2} \times \mathbb{R}^2$ which are Scwartz in the last two variables, flat at a point C' at the on the bounding sphere and such that $\mathrm{Id} + a$ is invertible. The $*$-extension adds arbitrary lower order terms in $\dot\Psi^0_{\mathrm{iso}}(\mathbb{R}; \mathbb{R})$ which do not affect invertibility.

G_{*2} : This is an exact sequence of contractible groups!

G_{13} : This is the image of the full symbol map from G_{12}. It consists of a $*$-algebra where, after some reorganization, all terms are Schwartz maps from $\mathbb{R}^2$ into Schwartz operators on $\mathbb{R}$ and the leading term is such that $\mathrm{Id} + b$ is invertible. This is a classifying space for odd K-theory.

G_{23} : This is a half-open version of the preceeding group. That is the individual terms are not Schwartz but are (I think after rearrangement) Schwartz in one variable with values in the half-open flat loops in the other; it has a $*$-product. It is again a contractible group.

G_{33} : This is a $*$-extension of $G^{-\infty}_{\mathrm{sus,ind}=0}(\mathbb{R})$.

G_{i*} : For each i this is quantization sequence.

Thus the operators in the top left block of four groups all correspond to certain functions on $\mathbb{R}^5$. The top two of the right column and the left two on the bottom row correspond to functions on $\mathbb{R}^4$ and the bottom right group to functions on $\mathbb{R}^3$. In all cases the last two variables are Schwartz. So we can really imagine the functions as being on $\mathbb{R}^3$, $\mathbb{R}^2$ and $\mathbb{R}$ respectively.

Log-multiplicative functionals:

(1) $\mathrm{ind} : G_{11} \longrightarrow \mathbb{Z}$, $\mathrm{ind}(g) = \frac{1}{2\pi i}\int_{\mathbb{R}} \mathrm{tr}(g^{-1}\dot{g}(t))dt$.
(2) $\eta : G_{12} \longrightarrow \mathbb{C}$, $\eta(g) = \overline{Tr}(g^{-1}\dot{g})$ where $\overline{Tr}$ is the regularized trace-integral which is a trace on the algebra.
(3) $\tilde{\eta} : G_{21} \longrightarrow \mathbb{C}$, $\tilde{\eta}(g) = \frac{1}{2\pi i}\int_{\mathbb{R}} \mathrm{tr}(g^{-1}\dot{g}(t))dt$ which makes sense because of the flatness of the loops.
(4) $\tilde{\eta} : G_{22} \longrightarrow \mathbb{C}$, $\tilde{\eta}(g) = \overline{Tr}(g^{-1}\dot{g})$ where $\overline{Tr}$ is the regularized trace-integral which is a trace on the algebra, since the parameter is the 'good' variable in product suspension.

These four maps are consistent under inclusion – i.e. they are all restrictions of the last map. Thus, restricting to the null spaces of these maps we get a commutative square in the top left corner. The exponential, $\exp(2\pi i\tilde{\eta})$ on G_{21} descends to G_{31} where it is the multiplicative Fredholm determinant. The exponential $\exp(2\pi i\eta)$ on G_{12} again descends to 'our' multiplicative determinant on the doubly suspended group. Again we can restrict to the subgroup where $\det = 1$ in these two cases and get short exact sequences on the top row and left column.

Exercise 33. Extend this commutative diagram to the whole 3×3 square. In particular show (I believe Frédéric Rochon has already done this) that the image groups under R and σ respectively in G_{32} and G_{23} are the full groups as before – the same as without the $\tilde{\eta} = 0$ restriction. This shows how we can kill the determinant line bundle since the resulting group in the 33 slot is the central extension of $G^{-\infty}_{\mathrm{sus,ind}=0}$ by the determinant bundle.

Proposition 57. *The fact that the determinant bundle is 'primitive' as in* (24.21) *is equivalent to the fact that the non-zero elements give a* $\mathbb{C}^*$ *central extension:*

$$\mathbb{C}^* \longrightarrow \mathcal{L}^* \longrightarrow G^{-\infty}_{\mathrm{sus,ind}=0}. \tag{39.2}$$

Exercise 34. Check it. Also, while you are at it, define an Hermitian inner product on the determinant line bundle which reduces this to a U(1) extension. In the geometric case this was done by Bismut and Freed.

I will use this central extension to define and discuss the (reduced) K-theory 2-gerbe later.

Remark 2. The right hand column, in the unreduced picture, constructs the determinant bundle via the $*$-extended, suspended delooping sequence.

40. Topic 9: K-homology
In place of lecture for Friday, 29 November, 2008

I believe this was a holiday even in Berkeley! In any case I was recovering from a surfeit of Heritage turkey. Still, let me pretend that I was diligently working – I have been meaning to write an account of 'smooth' K-homology for some time.

41. Lecture 32: The K-theory gerbe Monday, 1 December, 2008

First let me apologize for not having been able to keep up with the notes while I was away. With any luck I will catch up a bit with what I had meant to put in about the Chern character etc.

Today I want to describe the K-theory gerbe in one of its forms. Rather than define what a gerbe is – in the widest sense the term is used for any geometrical object which is classified by, or at least realizes all, integral 3-cohomology – I will describe it and then try to explain the salient features. In brief the *universal* K-theory gerbe is a 'geometric invariant' associated with, in the first instance a bundle with some structure over, a (reduced) classifying space for odd K-theory which 'captures' the primitive three-dimensional cohomology class.

However, first let me recall the 'geometric invariants' – in degrees 0, 1 and 2, that we have already introduced, since the gerbe is analogous to these:-

1. The index.
2. The determinant.
3. The determinant line bundle.

Of course the first two of these don't look very geometric but that is what happens in low degree.

The index. We have two basic 'series' of classifying spaces the loop groups of (a) $G^{-\infty}$ and the loop spaces of the space of involutions $\mathcal{H}^{-\infty}$. The index is most easily seen as the map

$$\mathcal{H}^{-\infty} \ni \{I = \gamma_\infty + \gamma; \gamma \in \Psi^{-\infty} \otimes M(2,\mathbb{C}); I^2 = \mathrm{Id}\} \ni I \longmapsto \frac{1}{2}\operatorname{tr}(\gamma) \in \mathbb{Z}. \tag{41.1}$$

We have usually taken $\gamma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The index labels the components, i.e. induces an isomorphism

$$\mathrm{ind} : \Pi_0(\mathcal{H}^{-\infty}) \longrightarrow \mathbb{Z} \tag{41.2}$$

which is additive (under compression to finite rank and direct sum) and as such is unique up to sign (which needs to be worried about).

The (flat-pointed) loop group on $G^{-\infty}$, $G^{-\infty}_{\mathrm{sus}}$ is also a classifying space for even K-theory and we showed that the index can be transferred to it. Namely the map

$$\mathrm{cl}_{\mathrm{eo}} : \mathcal{H}^{-\infty} \longrightarrow G^{-\infty}_{\mathrm{sus}}(\cdot;\mathbb{C}^2) \tag{41.3}$$

is an homotopy equivalence and under it

$$\mathrm{ind} = \frac{1}{2}\operatorname{tr} = {\mathrm{cl}_{\mathrm{eo}}}^* \mathrm{ind}_{\mathrm{sus}},\ \mathrm{ind}_{\mathrm{sus}}(g) = \frac{1}{2\pi 1}\int_{\mathbb{R}} \operatorname{tr}\left(g^{-1}(t)\frac{dg(t)}{dt}\right) dt. \tag{41.4}$$

So, it is reasonable just to write $\mathrm{ind}_{\mathrm{sus}} : G^{-\infty}_{\mathrm{sus}} \longmapsto \mathbb{Z}$ as 'ind' and take (41.4) as a natural identification; however I will still use the notation $\mathrm{ind}_{\mathrm{sus}}$ where this seems helpful.[6]

Now, the index on $G^{-\infty}_{\mathrm{sus}}$ can be recognized as the functional induced by the 1-form

$$\mathrm{Ch}^{\mathrm{odd}}_1 = \frac{1}{2\pi i}\operatorname{tr}\left(g^{-1}dg\right) \text{ on } G^{-\infty}. \tag{41.5}$$

[6] The index functional on the higher loop groups $G^{-\infty}_{\mathrm{sus}(2k+1)}$ was supposed to have been discussed in the write-ups while I was away – this may still appear.

Namely, under the evaluation map and projection

$$\begin{tikzcd} \mathbb{R}\times G^{\infty}_{\mathrm{sus}} \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi_2"] & G^{-\infty} \\ G^{-\infty}_{\mathrm{sus}} \end{tikzcd}, \ \mathrm{ind}_{\mathrm{sus}} = (\pi_2)_*(\mathrm{ev}^*\,\mathrm{Ch}^{\mathrm{odd}}_1). \tag{41.6}$$

The determinant. This, meand $\mathrm{ind}_{\mathrm{sus}}$, was the basis of the (second) construction of the Fredholm determinant. Recall the delooping sequence:-

$$G^{-\infty}_{\mathrm{sus}} \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus}} \xrightarrow{R} G^{-\infty}. \tag{41.7}$$

Here, the middle group is the half-open flat loops,

$$g:\mathbb{R}\longrightarrow\Psi^{-\infty}\ \text{s.t.}\ \frac{dg}{dt}\in\mathcal{S}(\mathbb{R};\Psi^{-\infty}),\ g(t)\in G^{-\infty}\ \forall\, t\in\mathbb{R}\ \text{and}\ \lim_{t\to-\infty} g(t)=0. \tag{41.8}$$

This central group is contractible and the construction (41.6) extends to it to define

$$\begin{tikzcd} \mathbb{R}\times \tilde{G}^{\infty}_{\mathrm{sus}} \arrow[r, "\mathrm{ev}"] \arrow[d, "\pi_2"] & G^{-\infty} \\ \tilde{G}^{-\infty}_{\mathrm{sus}} \end{tikzcd}, \ \tilde{\eta} = (\pi_2)_*(\mathrm{ev}^*\,\mathrm{Ch}^{\mathrm{odd}}_1) : \tilde{G}^{-\infty}_{\mathrm{sus}} \longrightarrow \mathbb{C}. \tag{41.9}$$

This 'eta function' has the properties that it restricts to $\mathrm{ind}_{\mathrm{sus}}$ on the subgroup $G^{-\infty}_{\mathrm{sus}}$ and is log-additive, so the exponentiated function

$$\det = \exp(2\pi i\tilde{\eta}) : G^{-\infty} \longrightarrow \mathbb{C}^* \tag{41.10}$$

is well-defined, multiplicative and restricts to the usual determinant on $\mathrm{GL}(N,\mathbb{C})$ included into $G^{-\infty}$ by stabilization. Moreover, we know that

$$d\tilde{\eta} = R^*\,\mathrm{Ch}^{\mathrm{odd}}_1 . \tag{41.11}$$

It follows that as a map (41.10), det represents a generating class for $H^1(G^{-\infty},\mathbb{Z})$.

Exercise 35. If this is not 'geometric' enough for you, the picture can be expanded a little. Namely consider the possible values of '$\log\det$' at a point of $G^{-\infty}$ – there should be a $\mathbb{Z}$ of them at each point. To do this explicitly, take $\tilde{G}^{-\infty}\times\mathbb{C}$ and then identify all pairs $(\tilde{g}_1,z_1)$ and $(\tilde{g}_2,z_2)$ if $R(\tilde{g}_1)=R(\tilde{g}_2)$ and $z_1-z_2 = 2\pi i\,\mathrm{ind}(\tilde{g}_2\circ(\tilde{g}_1)^{-1})$. Show that this results in a principal bundle

$$\begin{tikzcd} \mathbb{Z} \arrow[r, dash] & Z \arrow[r, "\tilde{\eta}"] \arrow[d] & \mathbb{C} \\ & G^{-\infty}. \end{tikzcd} \tag{41.12}$$

over $G^{-\infty}$ with structure group $\mathbb{Z}$ on which $\tilde{\eta}$ is a 'connection' in the sense that it is a well-defined function on the total space of the bundle which shifts by n under the action of $n\in\mathbb{Z}$.

Determinant line bundle. The determinant bundle was constructed over the group $G^{-\infty}_{\mathrm{sus,ind}=0}[[\rho]]$, the component of the identity in $G^{-\infty}_{\mathrm{sus}}[[\rho]]$ using the quantization sequence. Here $G^{-\infty}_{\mathrm{sus}}[[\rho]]$ is a $*$ extension of the group $G^{-\infty}_{\mathrm{sus}}$. Namely as a space it is consists of formal power series in ρ – which is just another way of saying sequences –

$$h=\sum_{j=0}^{\infty}h_j\rho^j,\ h_0\in G^{-\infty}_{\rm sus},\ h_j\in\Psi^{-\infty}_{\rm sus},\ p=h\circ k=\sum_j^{\infty}B_j(h,k)\rho^j \tag{41.13}$$

where the product is associative and the B_j are differential operators (acting only in the suspension variable):

$$B_0(h,k)=hk,\ B_j(h,k)=\sum_{l+l'+p+p'=j}c_{l,l',p,p'}\frac{d^ph_l}{dt^p}\frac{d^{p'}k_{l'}}{dt^{p'}} \tag{41.14}$$

where the product on the right is in $\Psi^{-\infty}$.

For the quantization sequence, the product just comes from the formula for the composition of isotropic pseudodifferential operators on $\mathbb{R}$ – sometimes called the Moyal product.

Now, the subject of today's lecture is the next step, the K-theory gerbe. To construct this again consider the delooping sequence, but now it nees to be both restricted and expanded. The basic delooping sequence is (41.7) above. The restriction is to kill off the determinant – so consider the subgroups

$$\begin{tikzcd} G^{-\infty}_{\rm sus,ind=0} \arrow[r] \arrow[d, hook] & \tilde G^{-\infty}_{\rm sus,\tilde\eta=0} \arrow[r, "R"] \arrow[d, hook] & G^{-\infty}_{\rm det=1} \arrow[d, hook] \\ G^{-\infty}_{\rm sus} \arrow[r] & \tilde G^{-\infty}_{\rm sus} \arrow[r, "R"] & G^{-\infty}. \end{tikzcd} \tag{41.15}$$

From the earlier discussion, the top row is exact.

The expansion is to consider the $*$ product. Thus, consider just the case $\epsilon^2=0$, meaing pairs

$$h=h_0+\epsilon h_1,\ h_0\in G^{-\infty}_{\rm det=1},\ h_1\in\Psi^{-\infty} \tag{41.16}$$

with the projected $*$ product

(41.17)

$$h\circ k=(h_0k_0)+\epsilon\big(h_0k_1+h_1k_0+B(h_0,k_0)\big),\ B(h_0,k_0)=\frac{1}{2i}\left(\frac{dh_0}{dt}k_0-h_0\frac{dk_0}{dt}\right).$$

Then if we take the restricted groups

$$\begin{gathered} G^{-\infty}_{\rm sus,ind=0}[\epsilon/\epsilon^2]=G^{-\infty}_{\rm sus,ind=0}+\epsilon\Psi^{-\infty}_{\rm sus},\\ \tilde G^{-\infty}_{\rm sus,\tilde\eta=0}[\epsilon/\epsilon^2]=\tilde G^{-\infty}_{\rm sus,\tilde\eta=0}+\epsilon\Psi^{-\infty}_{\rm sus} \end{gathered} \tag{41.18}$$

where there are no restrictions on the lower order terms, we get a new short exact sequence in place of (41.7):

$$\begin{tikzcd} \mathcal{L} \arrow[d] & & \\ G^{-\infty}_{\rm sus,ind=0}[\epsilon/\epsilon^2] \arrow[r] & \tilde G^{-\infty}_{\rm sus,\tilde\eta=0}[\epsilon/\epsilon^2] \arrow[r, "R"] & G^{-\infty}_{\rm det=1}. \end{tikzcd} \tag{41.19}$$

Here I have included the fact that deterimant bundle is well-defined over the 'dressed' group $[\epsilon/\epsilon^2]$ – it also comes equipped with a connection.

The basic 'bundle gerbe' construction (the idea is due to Michael Murray) is to take the fibre product of this thought of as a fibration. That is, consider

$$\mathcal{G} = \left\{(g, g') \in \tilde{G}^{-\infty}_{\mathrm{sus},\tilde{\eta}=0}[\epsilon/\epsilon^2]; R(g) = R(g')\right\} = \left(\tilde{G}^{-\infty}_{\mathrm{sus},\tilde{\eta}=0}[\epsilon/\epsilon^2]\right)^{[2]} \tag{41.20}$$

which is the 'fibre diagonal' in the full product of the central (contractible) group with itself. This is, by construction, a bundle over $G^{-\infty}_{\det=1}$. Moreover, there is a map back to the (dressed) flat-pointed loop group:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{S} & G^{-\infty}_{\mathrm{sus},\mathrm{ind}=0}[\epsilon/\epsilon^2] \\ \downarrow & & \\ G^{-\infty}_{\det=1} & & \end{array} \tag{41.21}$$

Here $S(g, g') = h$ if and only if $g' = hg$ – since $R(g') = R(g)$ the composite $g^{-1}g' = h$ is flat to the identity at both ends, and hence is an element of $G^{-\infty}_{\mathrm{sus},\mathrm{ind}=0}[\epsilon/\epsilon^2]$. We can use S to pull back the determinant line bundle and so get a tower

$$\begin{array}{ccc} \tilde{\mathcal{L}} = S^*\mathcal{L} & & \mathcal{L} \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow{S} & G^{-\infty}_{\mathrm{sus},\mathrm{ind}=0}[\epsilon/\epsilon^2] \\ \downarrow & & \\ G^{-\infty}_{\det=1} & & \end{array} \tag{41.22}$$

In fact, as recalled above, we constructed a connection on $\mathcal{L}$ which therefore pulls back to a connection on $\tilde{\mathcal{L}}$.

So what is a gerbe? Well, as I said above, there are different points of view on this. In all cases one is supposed to be able to extract a class in $H^3(X, \mathbb{Z})$, where X is the base, from the gerbe. I would distinguish between several different, but closely related objects.

42. Lecture 33: The B-field Wednesday, 3 December, 2008

Reminder. *Last time I described the K-theory gerbe, without defining the class of objects of which this is an example, namely the notion of a* bundle gerbe. *Today I will finish the discussion of the B-field on the K-gerbe, then quickly show how the K-theory gerbe defines gerbe data – which I wrote down in the notes for yesterday but did not discuss – and use this to motivate the, or at least* a, *general bundle gerbe.*

The determinant bundle in (41.19) has connection given by (21.15). The curvature of this connection was computed in (24.20):-

$$\omega = -c\int_{\mathbb{R}} \operatorname{tr}(a^{-1}\frac{da}{dt}(a^{-1}da)^2)dt \text{ at } a \in G^{-\infty}_{\mathrm{sus,ind}=0}[\rho/\rho^2]. \tag{42.1}$$

In the discussion of the transgression of the Chern forms for the delooping sequence this form was 'lifted' to $\tilde{G}^{-\infty}_{\mathrm{sus}}$ simply by observing that the integral is still convergent – because one term has been differentiated with respect to t. This leads to

$$\tilde{\eta}_2 = \int_{\mathbb{R}} \operatorname{tr}(a^{-1}\frac{da}{dt}(a^{-1}da)^2)dt \text{ at } a \in \tilde{G}^{-\infty}_{\mathrm{sus}} \tag{42.2}$$

which therefore defines a form on the top part of the $*$-extended group and restricts to the subgroup defined by $\tilde{\eta} = \tilde{\eta}_0 = 0$.

The curvature of the pull-back of the determinant line bundle from $G^{-\infty}_{\mathrm{sus,ind}=0}$ to $\mathcal{G}$ in (41.22) is the pull-back of the curvature, so it is – up to a constant which I have lost but which is important – equal to

$$S^*\omega = \int_{\mathbb{R}} \operatorname{tr}(a^{-1}\frac{da}{dt}(a^{-1}da)^2)dt,\ a = h^{-1}g,\ (h,g) \in \mathcal{G}. \tag{42.3}$$

To compute this we need to expand out the differential, $d(h^{-1}g) = -h^{-1}dhh^{-1}g + h^{-1}dg$ and similarly for the derivative with respect to t. This gives a total of eight terms.

Lemma 39. *The pull-back of the curvature of the determinant line bundle is*

$$\begin{gathered} S^*\omega = \pi_R^*\tilde{\eta}_2 - \pi_L^*\tilde{\eta}_2 + d\alpha,\ \textit{where} \\ \alpha = \int_{\mathbb{R}} \operatorname{tr}\left(\frac{dg}{dt}g^{-1}(dh)h^{-1} - \frac{dh}{dt}h^{-1}(dg)g^{-1}\right)dt \textit{ and} \\ \pi_L,\ \pi_R : \mathcal{G} \longrightarrow \tilde{G}^{-\infty}_{\mathrm{sus},\tilde{\eta}=0} \end{gathered} \tag{42.4}$$

are the two projections.

Proof. After expanding out (42.3) as indicated above, the two 'pure' terms in which only one of h or g is differentiated are the two terms obtained by pull-back of $\tilde{\eta}_2$.

The other six can be combined to give

$$\begin{aligned}
&\int_{\mathbb{R}} \operatorname{tr}((h^{-1}g)^{-1}\frac{d(h^{-1}g)}{dt}((h^{-1}g)^{-1}d(h^{-1}g))^2)dt \\
&= \int_{\mathbb{R}} \operatorname{tr}(g^{-1}\frac{dg}{dt}((g^{-1}dg)^2)dt - \int_{\mathbb{R}} \operatorname{tr}(\frac{dh}{dt}h^{-1}((dh)h^{-1}))^2)dt \\
&+ \int_{\mathbb{R}} \operatorname{tr}(g^{-1}\frac{dg}{dt}((h^{-1}g)^{-1}d(h^{-1}g))^2)dt - \int_{\mathbb{R}} \operatorname{tr}(g^{-1}\frac{dh}{dt}h^{-1}gd(h^{-1}g))^2)dt \\
&= \pi_R^*\tilde\eta_2 - \pi_L^*\tilde\eta_2 + d\left(\int_{\mathbb{R}} \operatorname{tr}\left(\frac{dg}{dt}g^{-1}(dh)h^{-1} - \frac{dh}{dt}h^{-1}(dg)g^{-1}\right)dt\right) \\
&\qquad + \int_{\mathbb{R}} \frac{d}{dt}\operatorname{tr}((dh)h^{-1}(dg)g^{-1}).
\end{aligned} \tag{42.5}$$

The last term evaluates to $\operatorname{tr}((da)a^{-1}(da)a^{-1}) = 0$ by symmetry, where $a = R_\infty(g) = R_\infty(h)$ is the common base-, or end-, point. Thus we arrive at (42.4). $\square$

Theorem 15. *For the K-theory (principal) bundle gerbe*

$$\begin{tikzcd}
 & \tilde{\mathcal{L}} = S^*\mathcal{L} \arrow[d] & \mathcal{L} \arrow[d] \\
\tilde G^{-\infty}_{\mathrm{sus},\tilde\eta=0} \arrow[dr, "R_\infty"'] & \mathcal{G} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "S"] \arrow[d] & G^{-\infty}_{\mathrm{sus,ind}=0}[\epsilon/\epsilon^2] \\
 & G^{-\infty}_{\det=1} &
\end{tikzcd} \tag{42.6}$$

the pulled-back determinant line bundle has a connection $\nabla_{\mathcal{G}}$ over $\mathcal{G}$ with curvature

$$\omega_{\mathcal{G}} = \pi_R^*\tilde\eta_2 - \pi_L^*\tilde\eta_2 \tag{42.7}$$

where the 'B-field' $\tilde\eta_2$ is a 2-form on $\tilde G^{-\infty}_{\mathrm{sus},\tilde\eta=0}$ with basic differential

$$d\tilde\eta_2 = R_\infty^* c\operatorname{tr}((a^{-1}da)^3). \tag{42.8}$$

It is important to track the constant, which I have not (yet) done.

Proof. This has all been done! The connection is obtained by adding α to the pulled-back connection and the formula for the differential of $\tilde\eta_2$ was worked out earlier. $\square$

What does all this buy us? Or asked another way, are there any interesting examples? In fact there are plenty of examples!

One such is to consider the group $\mathrm{SU}(N)$ of unitary $N\times N$ matrices of determinant one. This Lie group is connected and simply connected – this we have really already used. Now, we can certainly embed it into the stabilized group

$$i_N : \mathrm{SU}(N) \longrightarrow G^{-\infty}_{\det=1}, \tag{42.9}$$

say in the isotropic model by making it act on the first N eigenfunctions of the harmonic oscillator (and stabilizing by the identity of course). Here we use the consistency of the usual and the Fredholm determinant. Thus, we can pull the K-theory gerbe back to $\mathrm{SU}(N)$ and we have an induced 'gerbe'

(42.10)
$$\begin{tikzcd}
 & & \tilde{\mathcal{L}} \arrow[d] \\
i_N^*\tilde{G}^{-\infty}_{\mathrm{sus},\tilde{\eta}=0} \arrow[r, equal] & \mathcal{E} \arrow[dr] & \mathcal{E}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[d] \\
 & & \mathrm{SU}(N).
\end{tikzcd}$$

Here $\mathcal{E}^{[2]}$ is the fibre-product of $\mathcal{E}$ with itself – which is to say it is the pull-back of $\mathcal{G}$ which is just the fibre-product of $\tilde{G}^{-\infty}_{\mathrm{sus},\tilde{\eta}=0}$ with itself over $G^{-\infty}_{\mathrm{det}=1}$. Moreover, the set up (42.10), with $\mathcal{L}$ the pulled-back line bundle over $\mathcal{E}^{[2]}$, comes equipped with a connection on $\mathcal{L}$ and a B-field on the total space of the bundle with 'curvature' a multiple of the 3-form $\mathrm{tr}((g^{-1}dg)^3)$ on $\mathrm{SU}(N)$. How cool is that? Is this the gerbe of Meinrenken – [6] or is the 'curvature' a multiple of the minimal integral form $\frac{1}{24}\,\mathrm{tr}((g^{-1}dg)^3)$. This needs to be checked!

Other 'obvious' examples come more directly from index theory and I will describe these below. First let me try to abstract from the K-theory gerbe to get the notion of a 'bundle gerbe' which is due to Michael Murray [8].

So, abstractly, consider a setup as in (42.10)

(42.11)
$$\begin{tikzcd}
 & \mathcal{L} \arrow[d, "p"] \\
\mathcal{E} \arrow[dr, "\pi"'] & \mathcal{E}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[d, "\pi^{[2]}"] \\
 & X
\end{tikzcd}$$

where X can be a finite dimensional (say compact) smooth manifold and we are no longer assuming that $\mathcal{E}$ is pulled back from somewhere else. We will need to specify what sort of bundle $\mathcal{E}$ is over X. Since we can expect, in general, that $\mathcal{E}$ will be infinite dimensional we will need to specify the structure group. Let me just gloss over this for the moment to get the formal set up clear first. So, just pretend everything is finite-dimensional (which it could be) and then what makes the discussion above, relating the K-theory gerbe to Čech gerbe data, work? What we have used is:-

(1) $\mathcal{E}$ is a fibre bundle over X.
(2) $\mathcal{E}^{[2]}$ is the fibre product of $\mathcal{E}$ with itself – meaning it is the fibre diagonal in $\mathcal{E}\times\mathcal{E}$.
(3) $\mathcal{L}$ is a line bundle over $\mathcal{E}^{[2]}$.
(4) $\mathcal{L}$ has a *primitivity property* – if we consider $\mathcal{E}^{[3]}$, the triple fibre product and the three projections

(42.12)
$$\pi_O:\mathcal{E}^{[3]}\longrightarrow\mathcal{E}^{[2]},\ O=F,S,C$$

where π_F is projection onto the rightmost two factors, π_S onto the leftmost two factors and π_C onto the outer two factors[7] then there is a given trivialization

$$T : \pi_S^*\mathcal{L} \otimes \pi_F^*\mathcal{L} \xrightarrow{\simeq} \pi_C^*\mathcal{L} \text{ over } \mathcal{E}^{[3]} \text{ or}$$
$$\tilde{T} : \pi_S^*\mathcal{L} \otimes \pi_F^*\mathcal{L} \otimes \pi_C^*\mathcal{L}' \xrightarrow{\simeq} \mathbb{C}. \tag{42.13}$$

(5) Finally we need this trivialization to be natural, in an appropriate sense. Namely if we go up to $\mathcal{E}^{[4]}$ then there are four versions of T from the four ways of mapping from $\mathcal{E}^{[4]}$ back to $\mathcal{E}^{[3]}$ by dropping one of the factors. Then the tensor product of the four pulled-back line bundles as in the second version of (42.13) is canonically trivial and we require that the product of the four $\tilde{T}$'s should reduce to the identity.

What is triviality for such a bundle gerbe? It is the condition that there is a line bundle K over E such that there is an isomorphism of line bundles

$$\mathcal{L} \xrightarrow{\simeq} \pi_R^*K \otimes \pi_L^*K'. \tag{42.14}$$

Definition 10. A bundle gerbe with connection is a bundle as in (42.11) satisfying 1–5 where $\mathcal{E} \longrightarrow X$ is a smooth Fréchet fibre bunle, $\mathcal{L}$ is a smooth line bundle over $\mathcal{E}^{[2]}$ with smooth connection ∇, the diffeomorphism T is smooth and under (42.13) the connection pulls back to the product connection.

Exercise 36. Show, if only formally, that under the triviality condition the B-field can be taken to be the curvature of K and hence the 3-form which is its derivative vanishes. Going a little further, show that the Dixmier-Douady invariant, in $H^3(X;\mathbb{Z})$ vanishes in this case.

There are finite dimensional examples. Recall that in $\mathrm{SU}(N)$ there are still non-trivial multiples of the identity, at least if $N > 1$. Namely $\tau\,\mathrm{Id} \in \mathrm{SU}(N)$ if $\tau^N = 1$. These Nth roots of unity form a normal subgroup and the quotient is the smaller group $\mathrm{PU}(N)$:

$$\{\tau \in \mathbb{C}; \tau^N = 1\} \longrightarrow \mathrm{SU}(N) \longrightarrow \mathrm{PU}(N). \tag{42.15}$$

Proposition 58 (At least mainly due to Serre.). *Let E be a principal $\mathrm{PU}(N)$ bundle over a compact manifold X then the central extension (42.15) induces a primitive, flat, line bundle, L_N, over $\mathrm{PU}(N)$ which defines a bundle gerbe*

$$\begin{tikzcd}
 & \mathcal{L}_N \arrow[d, "p"] & L_N \arrow[d] \\
E & E^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "S"] \arrow[d, "\pi^{[2]}"] & \mathrm{PU}(N) \\
 & X \arrow[from=2-1, "\pi"'] &
\end{tikzcd} \tag{42.16}$$

the Dixmier-Douady invariant for which is a torsion element of $H^3(X;\mathbb{Z})$ and all such elements arise this way.

[7] These letter stand for **F**irst, **S**econd and **C**omposte, coming from the composition of operators.

Exercise 37. Suppose $E \longrightarrow X$ is a principal bundle for a group G where G has a central extension by the circle (or $\mathbb{C}^*$) – meaning there is a short exact sequence of groups

$$\mathrm{U}(1) \longrightarrow \hat{G} \longrightarrow G. \tag{42.17}$$

Show that E fixes a bundle gerbe over X (assuming appropriate regularity especially if the setup is infinite dimensional).

Here is another example taken from the recent preprint [5]. Let X be a compact manifold and suppose that L is a complex line bundle over X and $f : X \longrightarrow \mathbb{C}^*$ is a smooth function. The former defines an element of $H^2(X,\mathbb{Z})$ and the latter an element of $H^1(X;\mathbb{Z})$. Together this gives an integral 3-class, how can we construct a bundle gerbe out of this data? Choose an Hermitian inner product on the fibres of L, so that the circle bundle

$$\hat{L} = \{p \in L; \|p\| = 1\} \xrightarrow{p} X \tag{42.18}$$

is well-defined. It is indeed a principal $\mathrm{U}(1)$ bundle over X. Thus if we take the fibre product $\hat{L}^{[2]}$ over X then we have the usual S map

$$\hat{L}^{[2]} \xrightarrow{s} \mathrm{U}(1). \tag{42.19}$$

This map is itself 'primitive' (sometimes called a groupoid character), meaning that the three versions of it over $\hat{L}^{[3]}$ satisfy

$$\pi_S^* s \cdot \pi_F^* s = \pi_C^* s. \tag{42.20}$$

Next think about the map $f : X \longrightarrow \mathrm{U}(1)$. Together with (42.19) this leads to a map to the 2-torus:

$$s \times \pi^{[2]} * f : \hat{L}^{[2]} \longrightarrow \mathbb{T}^2. \tag{42.21}$$

Over the torus there is a line bundle, corresponding to the fundamental, volume, class in $H^2(\mathbb{T}^2;\mathbb{Z})$. This line bundle can be pulled back to $\hat{L}^{[2]}$ giving at least the basic setup of a bundle gerbe.

Exercise 38 (Maybe for me). Check that if L is equipped with an Hermitian connection then this defines a connection $d+\gamma$ on the (trivial) pull-back of L to $\hat{L}$. Then show that the structure above is a bundle gerbe in the sense of Definition 10 and with B-field on $\hat{L}$ $c d\log f \wedge \gamma$ (including working out the constant) and with curvature 3-form

$$\frac{c'}{2\pi i}\omega \wedge d\log f \text{ on } X. \tag{42.22}$$

Addenda to Lecture 33 The notion of equivalence of a bundle gerbes needs to be addressed, corresponding to a weakening of the notion of triviality.

First we can say that two gerbes over the same base, are isomorphic if there is a fibre-preserving Fréchét isomorphism between the corresponding bundles $\mathcal{E}_i$, $i = 1, 2$ such that under the induced isomorphisms of the $\mathcal{E}_i^{[2]}$ the bundles $\mathcal{L}_i$ become isomorphic and that under the induced isomorphism of the $\mathcal{E}_i^{[3]}$ the primitivity isomorphism T_i are intertwined.

This corresponds to the ability to pull back gerbes. Thus suppose Γ is a gerbe as in (42.11) and $\Phi : \mathcal{E}_1 \longrightarrow \mathcal{E}$ is a smooth bundle-preserving map, where $\mathcal{E}_1 \longrightarrow X$ is a locally trivial Fréchét fibre bundle. Then $\Phi^*\Gamma$ is the gerbe with line bundle $(\Phi^{[2]})^*\mathcal{L}$ over $\mathcal{E}_1$.

Exercise 39. You should check that all the conditions hold for the pull-back and that if ∇ is a primitive connection on $\mathcal{L}$ then the pull-back is a primitive connection on $\Phi^*\mathcal{L}$. Check that Γ and $\Phi^*\Gamma$ have the *same* Dixmier-Douady invariant.

Now we can say that one gerbe Γ_i *extends* another, Γ_2, if there is a fibre smooth map $\Phi : \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ such that $\Gamma_1 \equiv \Phi^*\Gamma_1$. Two gerbes are *equivalent* if each extends the other.

Next we can consider the 'tensor product' of two bundle gerbes Γ_i with Fréchét fibrations $\mathcal{E}_i$ and primitive line bundles with connection $\mathcal{L}_i$, over the same base X. The *tensor product* $\Gamma_1 \otimes \Gamma_2$ (maybe it should be written as an exterior tensor product, $\Gamma_1 \boxtimes \Gamma_2$) is just obtained by taking the fibre product of the bundles $\mathcal{E} = (\mathcal{E}_1) \times_X (\mathcal{E}_2)$ and the exterior tensor product of the primitive line bundles. Alternatively one can think of this in two steps. First define the tensor product when the fibrations are the same – just as the tensor product of the two line bundles and connections. Then define the general case as the tensor product in this sense of the two pull-backs – of Γ_i to $\mathcal{E}$ under the two projections $p_i : \mathcal{E} \longrightarrow \mathcal{E}_i$.

Exercise 40. Check it all – that the required conditions hold for these operations to be well-defined and most importantly that the Dixmier-Douady invariant of the tensor product is the sum of the Dixmier-Douady invariants.

Exercise 41. Make sure that you can see that duality also works – the dual of a gerbe is just the gerbe with the dual bundle and dual connection and that this process reverses the sign of the Dixmier-Douady invariant. Observer that the tensor product of a gerbe and its dual is isomorphic to a trivial gerbe.

Definition 11. Let $\mathcal{F} \longrightarrow X$ be a Fréchét bundle over a manifold X, then a bundle gerbe Γ over $\mathcal{F}^{[2]}$ (with Fréchét bundle $\mathcal{E}$ and primitive line bundle $\mathcal{L}$) is *primitive* if there is a smooth Fréchét bundle map (as bundles over $\mathcal{F}^{[3]}$)

$$\pi_S^*\mathcal{E} \times_{\mathcal{F}^{[3]}} \pi_F^*\mathcal{E} \longmapsto \pi_C^*\mathcal{E} \tag{42.23}$$

43. Lecture 34: Dixmier-Douady invariant Friday, 5 December, 2008

Even though it is wandering further into Čech theory than I really wanted to go, I will discuss the Brylinzki-Hitchin definition of a gerbe (calling it 'gerbe data'), the derivation of the Dixmier-Douady class and show how the K-theory gerbe (and more generally any bundle gerbe) defines such gerbe data. If there is a little more time I will go through, at least in outline, the construction of a principal PU-bundle from gerbe data.

Let me start with the notion of a Čech type gerbe of Brylinski and later modified by Hitchin. For orientation, start with the 0-gerbe, the line bundle.

Definition 12. *Line bundle data* (to be considered as one word) on a manifold X – for convenience taken to be compact here – consists of the following:-

(1) A (finite) covering of X by open sets, U_i, $i \in N$.
(2) A $\mathcal{C}^\infty$ complex line bundle $L_i \longrightarrow U_i$ over each U_i.
(3) An isomorphism of complex line bundles for each i, j such that $U_{ij} = U_i \cap U_j \neq \emptyset$,
$$T_{ij} : L_i\big|_{U_{ij}} \longrightarrow L_j\big|_{U_{ij}} \text{ with } T_{ji} = T_{ij}^{-1}.$$
(4) The compatibility (cocylce) condition for each i, j, k such that $U_{ijk} = U_i \cap U_j \cap U_k \neq \emptyset$,

$$T_{ki}T_{jk}T_{ij} = \mathrm{Id} \text{ on } L_i\big|_{U_{ijk}}. \tag{43.1}$$

There are extreme cases of such vector bundle data. One possibility is that there is only one element in the open cover, $U_1 = Z$, and then $L \longrightarrow Z$ is simply a complex line bundle. Alternatively, all the line bundles could be trivial, $L_i = U_i \times \mathbb{C}$ and then we get what is the usual notion of a trivialization of a line bundle. Namely, the T_{ij} become maps $t_{ij} : U_{ij} \longrightarrow \mathbb{C}^*$ and the cocycle condition becomes

$$t_{ki}t_{jk}t_{ij} = 1. \tag{43.2}$$

In fact such line bundle data *always* defines a complex line bundle. Simply define the 1-dimensional complex vector space at each point by

$$L_p = \{(z_i) \in \bigoplus_{p \in U_i} (L_i)_p ; T_{ij}z_i = z_j \ \forall\ i, j \text{ s.t. } p \in U_{ij}\}. \tag{43.3}$$

Then $L = \cup_p L_p$ is a complex line bundle and moreover there exist bundle isomorphisms

$$T_i : L\big|_{U_i} \longrightarrow L_i \text{ s.t. } T_{ij}T_i = T_j \text{ on } U_{ij}. \tag{43.4}$$

I do not want to follow all this Čech stuff to its logical conclusion, but observe that the converse of (43.4) is also true. If $\tilde{L}$ is a line bundle over X and there are bundle isomorphisms $\tilde{T}_i : L\big|_{U_i} \longrightarrow L_i$ for each i such that $T_{ij}\tilde{T}_i = \tilde{T}_j$ on U_{ij} then L and $\tilde{L}$ are globally isomorphic as vector bundles. Moreover, one can *refine* line bundle data given a refinement of the cover. That is, if U'_l, $l \in N'$, is another open cover with a map $I : N' \longrightarrow N$ such that $U'_l \subset U_{I(l)}$ for all $l \in N'$, then the $L'_l = L_{I(l)}\big|_{U'_l}$ carry 'obvious' induced line bundle data and the line bundle generated by this data is globally isomorphic to that generated by the original data.

Now, one can always find a *good open cover* which refines a given cover; it suffices to take a covering by sufficiently small balls with respect to some Riemannian structure on the manifold. The condition that an open cover be *good* is that all the non-trivial intersections of its elements be contractible. So, one can find a refinement to a good open cover still denoted U_i. In that case there is a trivialization of each $T_i : L_i \longrightarrow \tilde{L}_i = \mathbb{C} \times U_i$ over U_i. Then $\tilde{L}_i$ with $t_{ij} = \tilde{T}_j T_{ij} \tilde{T}_i^{-1} \in \mathcal{C}^\infty(U_i, \mathbb{C}^*)$ gives new line bundle data which also generates an isomorphic line bundle. Since U_{ij} is also contractible, one can choose logarithms

$$\gamma_{ij} \in \mathcal{C}^\infty(U_i, \mathbb{C}) \text{ s.t. } t_{ij} = \exp(2\pi i \gamma_{ij}),\ \gamma_{ji} = -\gamma_{ij}. \tag{43.5}$$

Now, on the triple overlaps

$$n_{ijk} = \gamma_{ij} + \gamma_{jk} + \gamma_{ki} \in \mathbb{Z} \text{ on } U_{ijk} \tag{43.6}$$

is constant and integral, since by the cocycle condition (43.2) $\exp(2\pi i n_{ijk}) = 1$ (and U_{ijk} is contractible). Moreover this satisfies the closure condition for Čech cocycles, that

$$n_{ijk} + n_{jkl} + n_{kli} + n_{lij} = 0 \text{ if } U_{ijkl} \neq \emptyset. \tag{43.7}$$

Thus the n_{ijk} fix a Čech 2-cocycle and hence a Čech cohomology class

$$\omega(L) \in \check{H}^2(X, \mathbb{Z}). \tag{43.8}$$

Of course, there is some work here to show that the Čech cohomology class is independent of the choice of good cover, etc.

Then one arrives at the well-known result:-

Theorem 16. *Two complex line bundles over a compact manifold are globally isomorphic if and only if they define the same class in $\check{H}^2(Z, \mathbb{Z})$ and every such class corresponds to an isomorphism class of line bundles.*

Proof. The main thing to see is the independence of the class $\omega(L)$ of the choice of good open cover – this really amounts to showing that the same class arises under refinement to another good open cover, since any two open covers have a common, good, refinement. The converse, that each class arises this way, follows by the fact that any Čech cocycle n_{ijk} with respect to some open cover, arises from γ_{ij}'s through (43.6). Namely one can just choose a partition of unity χ_i subordinate to the open cover and set

$$\gamma_{ij} = \sum_k \chi_k n_{ijk} \text{ on } U_{ij}. \tag{43.9}$$

Exponentiating the γ_{ij}'s gives line bundle data which in turn generates the original class n_{ijk}. □

So, why have I gone through all this standard Čechy stuff? Basically, I just wanted to prepare for the Čech version of a gerbe.

Definition 13. *Gerbe data* on a compact manifold X consists of

(1) A (finite) open cover U_i of Z.
(2) A $\mathcal{C}^\infty$ line bundle $L_{ij} \longrightarrow U_{ij}$ over each non-trivial $U_{ij} = U_i \cap U_j$ with $L_{ji} = L'_{ij}$ (the dual).
(3) For each non-trivial U_{ijk} a trivialization

$$T_{ijk} : L_{ij} \otimes L_{jk} \otimes L_{ki} \longrightarrow \mathbb{C} \text{ on } U_{ijk}. \tag{43.10}$$

(4) The cocycle condition that over each non-trivial U_{ijkl}

$$T_{ijk}T^{-1}_{jkl}T_{kli}T^{-1}_{lij} = 1 \tag{43.11}$$

where this makes sense because the tensor product of the four 3-fold tensor products, as in (43.10), is canonically trivial:

$$L_{ij} \otimes L_{jk} \otimes L_{ki} \otimes L'_{jk} \otimes L'_{kl} \otimes L'_{lj} \otimes L_{kl} \otimes L_{li} \otimes L_{ik} \otimes L'_{li} \otimes L'_{ij} \otimes L'_{jl} = \mathbb{C}. \tag{43.12}$$

Proposition 59. *Any gerbe data defines a class (the Dixmier-Douady class)* DD $\in H^3(X, \mathbb{Z})$ *which is constant under refinement and any two collections of gerbe data with the same Dixmier-Douady class are isomorphic after refiment (to a common good open cover).*

Proof. The definition of the Dixmier-Douady class follows the same idea as for line bundle data above. Namely, first refine to a good open cover (of course one has to define this process and check that it does indeed give new gerbe data). Then all the L_{ij} are trivial, with trivializations $\tilde{T}_{ij}$. The T_{ijk} now become maps $t_{ijk} : U_{ijk} \longrightarrow \mathbb{C}^*$ and so have logarithms, γ_{ijk}. These generate integers

$$n_{ijkl} = \gamma_{ijk} - \gamma_{jkl} + \gamma_{kli} - \gamma_{lij} \text{ on } U_{ijkl} \tag{43.13}$$

and these form a Čech 3-cocycle and hence class $[n] \in \check{H}^3(X; \mathbb{Z})$.

So, now the checking begins! I leave it to you (after consulting Brylinski's book, [2], if you prefer) to show that this class is well-defined, i.e. does not change under refinement and determines the gerbe data up to the natural notion of isomorphism after sufficient refinement. Moreover, every integral Čech 3-class arises this way. $\square$

Theorem 17. *Čech gerbe data in the sense of Definition 13 defines a principal* PU *bundle over* X, *where* PU = U / U(1) *is the quotient of the group of unitary operators on a separable, infinite-dimensional, Hilbert space by the multiples of the identity, all* PU *bundles (up to isomorphism) arise this way and two principal* PU *bundles are isomorphic if and only if the Dixmier-Douady invariants of their gerbe data are equal.*

Proof. Not very likely. $\square$

Now, let me check that we can extract 'gerbe data' from the K-theory gerbe as just described. To do this, consider the pull-back of the gerbe under some map from a finite dimensional manifold $X \longrightarrow G^{-\infty}_{\det=1} \longrightarrow G^{-\infty}$ which therefore represents an odd K-class on X. Let $\mathcal{E}$ be the pull-back of the bundle $\tilde{G}^{-\infty}_{\mathrm{sus},\tilde{\eta}=0}$. The first thing to note is that we can find local sections of $\mathcal{E}$, meaning it is locally trivial. Indeed, without the restriction to $\tilde{\eta} = 0$ this was discussed earlier. Since $\tilde{\eta}$ exponentiates to $\det \circ R_\infty$, it is enough to recall that R_∞ is surjective, since on a local section of $\tilde{G}^{-\infty}_{\mathrm{sus}}$ on which $\det \circ R_\infty = 0$ the funtion $\tilde{\eta}$ is necessarily constant. Thus, there is an open cover U_i of X on the elements of which $\mathcal{E}$ has a section (and as a principal bundle is then trivial). On the overlaps U_{ij} there are two sections, and hence a section of $\mathcal{E}^{[2]}$. Using this the determinant line bundle may be pulled back to define a line bundle L_{ij} over U_{ij}. It only remains to check the properties required of gerbe data in Definition 13. That L_{ji} is the dual of L_{ij} follows from the primitivity of the determinant line bundle and the fact that it is canonically trivial over the diagonal. Similarly the existence of a trivialization of the triple tensor product in (43.10) over any U_{ijk} follows from the primitivity of $\mathcal{L}$, as does the naturality (43.11).

Thus the K-theory gerbe does define Čech gerbe data.

Exercise 42 (I will do this eventually). Show that the Dixmier Douady invariant of the pull-back of the K-theory gerbe to X is (a multiple of) the second odd Chern class of the element $\mathrm{K}^1(X)$ which the map defines.

44. Lecture 35: The K-theory 2-gerbe
Monday, 8 December, 2008

Frédéric Rochon and I were talking about this, so I thought I would put in what I know here. I am not at all sure that this is the right way to go – the usual theory of 2-gerbes is categorical. This is for good reasons to do with the non-commutativity of the groups. However, in this case it seems we can construct genuine bundles which reproduce, as a kind of curvature, the 4-class which is the second Chern class for reduced K-theory – after we have killed off the index and the determinant line bundle. The following gerbal discussion is still quite preliminary.

So, start with the bottom row of (39.1). We do not need the rest of the diagram (although it might be better to start with the right column instead). So this is just the quantization sequence with the initial group shrunk, by killing the determinant and as a result the image group is larger:

$$\begin{tikzcd} G^{-\infty}_{\det=1}(\mathbb{R};\mathbb{R}) \arrow[r] & \dot{G}^0(\mathbb{R};\mathbb{R}) = \mathcal{F} \arrow[d] \\ & G^{-\infty}_{\mathrm{sus,ind}=0,\mathcal{L}}(\mathbb{R}). \end{tikzcd} \tag{44.1}$$

Now, take the self-fibre product of this fibration to get the same picture as in the construction of the gerbe

$$\begin{tikzcd} & S^*\mathcal{L} \arrow[d] & \mathcal{L} \arrow[d] & \\ S^*\mathcal{E} \arrow[dr, "p_{S^*\mathcal{E}}"'] & S^*\mathcal{E}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[d, "\pi^{[2]}_{S^*\mathcal{E}}"] & \mathcal{E}^{[2]} \arrow[r, shift left, "\pi_R"] \arrow[r, shift right, "\pi_L"'] \arrow[d, "\pi^{[2]}_{\mathcal{E}}"'] & \mathcal{E} \arrow[dl, "p_{\mathcal{E}}"] \\ \mathcal{F} \arrow[dr, "p_{\mathcal{F}}"'] & \mathcal{F}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "S"] \arrow[d, "\pi^{[2]}_{\mathcal{F}}"] & G^{-\infty}_{\det=1}(\mathbb{R};\mathbb{R}) & \\ & G^{-\infty}_{\mathrm{sus,ind}=0,\mathcal{L}}(\mathbb{R}) & & \end{tikzcd} \tag{44.2}$$

except that now we have the K-theory gerbe over the structure group. The diagram in (44.2) is completed by pulling this back.

So, the sense in which this is supposed to be a 2-gerbe is:

Proposition 60. *There is a connection on $S^*\mathcal{L}$ such that its curvature, a two-form on $S^*\mathcal{E}^{[2]}$, splits as the difference of the two pull-backs of a 2-form on $S^*\mathcal{E}$ under the maps π_L and π_R. The differential of this 2-form is basic, i.e. is a 3-form on $\mathcal{F}^{[2]}$, which in turn splits as a difference of the two pull-backs of a 3-form on $\mathcal{F}$. The differential of this is basic and is the desired 4-form 'curvature' on the base $G^{-\infty}_{\mathrm{sus,ind}=0,\mathcal{L}}(\mathbb{R})$.*

Okay, so let's see if this really works! According to my computation the same miraculous cancellation does indeed occur. In brief, we take the previously described connection on $\mathcal{L}$ over $\mathcal{E}^{[2]}$. We already know that the curvature of this connection splits, as the difference of the pull-back from the two factors of $\mathcal{E}$ of a two form the differential of which is basic and is a multiple of the three form

$$\operatorname{Tr}((a^{-1}da)^3) \text{ on } G^{-\infty}. \tag{44.3}$$

Thus, pulled back to a bundle gerbe over $\mathcal{F}^{[2]}$ this all proceeds the same for the pulled-back connection and pulled-back forms. It follows that what we get on $\mathcal{F}^{[2]}$ is the pull-back of (44.3) to $\mathcal{F}^{[2]}$. Writing out the pull-back map, S, as $a = h^{-1}g$ the pulled-back 3-form is

$$\overline{\operatorname{Tr}}\left(g^{-1}h\cdot d(h^{-1}g)\cdot g^{-1}h\cdot d(h^{-1}g)\cdot g^{-1}h\cdot d(h^{-1}g)\right). \tag{44.4}$$

Here I have written the extended trace functional on $\dot{G}^0$ even though all the composed factors are in $G^{-\infty}$ since that is what we need to work it out – of course it reduces to the trace in this case. As a first step, commute the g^{-1} to the right to make it more 'symmetric'. This in principle produces a trace-defect term. In fact it does not, since any one of the middle $d(g^{-1}h)$ terms is smoothing so the commutation is justified so (44.4) is equal to

$$\operatorname{Ch}_3^{\mathrm{odd}} = \overline{\operatorname{Tr}}\left((h\cdot d(h^{-1}g)\cdot g^{-1})^3\right) = \overline{\operatorname{Tr}}\left(((dg)g^{-1} - (dh)h^{-1})^3\right). \tag{44.5}$$

Expanding this out into eight terms gives

$$\begin{gathered}\operatorname{Ch}_3^{\mathrm{odd}} = \\ \overline{\operatorname{Tr}}\left(((dg)g^{-1})^3\right) - \overline{\operatorname{Tr}}\left(((dg)g^{-1})^2(dh)h^{-1}\right) \\ -\overline{\operatorname{Tr}}\left((dg)g^{-1}(dh)h^{-1}(dg)g^{-1}\right) + \overline{\operatorname{Tr}}\left((dg)g^{-1}((dh)h^{-1})^2\right) \\ -\overline{\operatorname{Tr}}\left((dh)h^{-1}((dg)g^{-1})^2\right) + \overline{\operatorname{Tr}}\left((dh)h^{-1}(dg)g^{-1}(dh)h^{-1}\right) \\ +\overline{\operatorname{Tr}}\left(((dh)h^{-1})^2(dg)g^{-1}\right) - \overline{\operatorname{Tr}}\left(((dh)h^{-1})^3\right).\end{gathered} \tag{44.6}$$

The first and last terms here are the 'pure' terms pulled back from the factors. Now, consider the third and sixth terms on the right. By commutation, using the trace-defect formula these can be written

$$\begin{gathered}-\overline{\operatorname{Tr}}\left((dg)g^{-1}(dh)h^{-1}(dg)g^{-1}\right) = -\overline{\operatorname{Tr}}\left((dh)h^{-1}((dg)g^{-1})^2\right) - \alpha \\ \overline{\operatorname{Tr}}\left((dh)h^{-1}(dg)g^{-1}(dh)h^{-1}\right) = \overline{\operatorname{Tr}}\left((dg)g^{-1}((dh)h^{-1})^2\right) + \alpha \\ \alpha = c\int_{\mathbb{R}} \operatorname{Tr}\left(((dm)m^{-1})^2\frac{d}{dt}((dm)m^{-1})\right).\end{gathered} \tag{44.7}$$

Here m is the common 'full symbol' of h and g (which lie in the same fibre) over the symbol group. Thus these defect terms cancel and (44.6) can be rewritten as

$$\begin{gathered}\operatorname{Ch}_3^{\mathrm{odd}} = \overline{\operatorname{Tr}}\left(((dg)g^{-1})^3\right) - \overline{\operatorname{Tr}}\left(((dh)h^{-1})^3\right) \\ -\overline{\operatorname{Tr}}\left(((dg)g^{-1})^2(dh)h^{-1}\right) + 2\overline{\operatorname{Tr}}\left((dg)g^{-1}((dh)h^{-1})^2\right) \\ -2\overline{\operatorname{Tr}}\left((dh)h^{-1}((dg)g^{-1})^2\right) + \overline{\operatorname{Tr}}\left(((dh)h^{-1})^2(dg)g^{-1}\right).\end{gathered} \tag{44.8}$$

That is,

$$\begin{gathered}S^*\operatorname{Ch}_3^{\mathrm{odd}} = \eta_3(g) - \eta_3(h) + d\mu(g,h), \\ \eta_3(g) = \overline{\operatorname{Tr}}\left((dgg^{-1})^3\right),\ \mu(g,h) = \overline{\operatorname{Tr}}\big((dg)g^{-1}(dh)h^{-1}\big).\end{gathered} \tag{44.9}$$

So, in terms of the diagram (44.2), η_3 is a form on the total space of $\mathcal{F}$ whereas μ is a form on the total space of $\mathcal{F}^{[2]}$. Recall that $\operatorname{Ch}_3^{\mathrm{odd}}$ was obtained as the differential of the form η_2 on $\mathcal{E}$ so

$$\tilde{\eta}_2 = S^*\eta_2 - p_{\mathcal{E}}^*\mu \text{ on } \mathcal{E} \tag{44.10}$$

is a (different) B-field for the pulled-back gerbe over $\mathcal{F}^{[2]}$ (because the extra term comes from $\mathcal{F}^{[2]}$ so cancels out) so

$$\pi_L^*\tilde{\eta}_2 - \pi_R^*\tilde{\eta}_2 = \pi_L^* S^*\eta_2 - \pi_R^* S^*\eta_2 = \omega(S^*\mathcal{L}) \tag{44.11}$$

gives the curvature of the pulled-back connection on $S^*\mathcal{L}$. On the other hand

$$\begin{aligned} d\eta_3 &= -d\overline{\mathrm{Tr}}(dg \wedge d(g^{-1}) \wedge dg \cdot g^{-1}) \\ &= \overline{\mathrm{Tr}}(dg \wedge d(g^{-1}) \wedge dg \cdot d(g^{-1})) \\ &= \frac{1}{2}\overline{\mathrm{Tr}}([(dg_g^{-1}, (d(g)g^{-1})^3]_+) \\ = \frac{1}{2}c' \int_{\mathbb{R}} \mathrm{Tr}&\left(\frac{dg}{dt}g^{-1}(d(g)g^{-1})^4\right) + \frac{1}{2}c'd\int_{\mathbb{R}} \mathrm{Tr}\left(\frac{dg}{dt}g^{-1}(d(g)g^{-1})^3\right) \\ &\equiv c'' \, \mathrm{Ch}_4^{\mathrm{even}} \ \text{ on } G^{-\infty}_{\mathrm{sus,ind}=0,\mathcal{L}}. \end{aligned} \tag{44.12}$$

Question 4. What is the structure of this '2-gerbe' which makes the forms descend in this way. In particular can it be abstracted to produce a general class of objects which, collectively, produce all integral 4-forms on the base?

In answer to a question from Frédéric Rochon: Yes it is possible to construct a 2-gerbe in the sense of a diagram like the left part of (6.14) which reproduces the decomposed 4-form $\omega_1 \wedge \omega_2$ where the ω_i are the curvatures of line bundles, L_i, over a compact manifold X. Not surprisingly this follows rathe closely the discussion from [] which is sketched above at the end of Lecture 34.

In particular we may start as in (42.18). Suppose L_i, $i = 1, 2$ are Hermitian line bundles with connections $\nabla^{(i)}$ over X and consider the fibre self-product of the corresponding circle bundles $\hat{L}_i$. Each of these generates a character

$$\chi_i : \hat{L}_i^{[2]} \longrightarrow \mathrm{U}(1), \ \chi_i(u, v)v = u. \tag{44.13}$$

We can combine these two constructions. Set $p : \mathcal{F} = \hat{L}_2 \longrightarrow X$ and let $p^{[2]} : \mathcal{F}^{[2]} \longrightarrow X$ be the double projection. Then denote the pull $\hat{L}_2$ back to the total space of this bundle as

$$q : \mathcal{E} = (p^{[2]})^*\hat{L}_1 = \hat{L}_1 \times_X \hat{L}_2^{[2]} \longrightarrow \mathcal{F} = \hat{L}_2^{[2]}. \tag{44.14}$$

It follows that

$$q^{[2]} : \mathcal{E}^{[2]} = \hat{L}_1^{[2]} \times_X \hat{L}_2^{[2]} \longrightarrow \mathcal{F}^{[2]} \tag{44.15}$$

can be identified as the fibre product over X of the two fibre self-products. This constructs the desired little 2-gerbe:-

$$\begin{tikzcd} & \mathcal{L} = S^*\mathcal{T} \arrow[d] & \mathcal{T} \arrow[d] & \\ \mathcal{E} \arrow[dr, "q"'] & \mathcal{E}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "S"] \arrow[d, "q^{[2]}"] & \mathbb{T} \arrow[r, equal] & \mathrm{U}(1) \times \mathrm{U}(1) \\ \mathcal{F} \arrow[dr, "p"'] & \mathcal{F}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[d, "p^{[2]}"] & & \\ & X. & & \end{tikzcd} \tag{44.16}$$

Here $\mathbb{T}$ is the 2-torus with line bundle $\mathcal{T}$ over it, constructed to have curvature $-d\log\chi_1 \wedge d\log\chi_2/4\pi^2$. Namely, identify the 2-torus as

$$\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2 \tag{44.17}$$

with the additive action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ and then identify

(44.18)
$$\mathcal{T} = \mathbb{R}^2 \times \mathbb{C}/\sim,\ (t_1+n_1, t_2+n_2, z) \sim (t_1, t_2, z'),\ z' = \exp(\pi i(n_2t_1 - n_1t_2))z.$$

Note that the torus is recovered as the quotient of the square $[0,1]\times[0,1]$ in which opposite sides are identified. Thus any point in the interior of a side is identified with one other point and (44.18) gives an identification of the fibres above these two points – where either $n_1 = 1$ or $n_2 = 1$ – by multiplication by $\exp(\pi t_1)$ or $\exp(-\pi i t_2)$. You might think there should be a 2π here instead of a π but notice that all four corner points are identified. The four lines are identified by multiplication by $\pm 1 - -1$ if they are both on a side or 1 if the are opposite. This is consistent. Thus the fibres are well-defined and clearly there are consistent local smooth trivializations.

An invariant connection on the trivial bundle for this action is

$$d - \lambda,\ \lambda(t_1, t_2) = \pi i(t_2dt_1 - t_1dt_2). \tag{44.19}$$

This descends to a connection on $\mathcal{T}$ which has curvature $dt_1 \wedge dt_2$.

So this gives the line bundle over $\mathcal{E}^{[2]} = \hat{L}_1^{[2]} \times_X \hat{L}_2^{[2]}$; it is just the pull-back of the primitive line bundle on the 2-torus by the product of the two characters of the line bundles over $\hat{L}_i^{[2]}$. The curvature of this line bundle with pulled-back connection is of course the pull-back of the curvature, meaning it is

$$\omega = d\log\chi_1 \wedge d\log\chi_2/4\pi^2 \tag{44.20}$$

where I have not been keeping track of possible ± 1's.

Now, recall what happens when a line bundle with Hermitian connection is pulled back to its circle bundle. As already noted, it becomes trivial and its connection therefore takes the form $d+\gamma$ where γ is a (pure imaginary) 1-form on the total space of the circle bundle – it is indeed a connection form in the sense of principal bundles. Now, pulling back under the two maps from $\hat{L}^{[2]}$ to $\hat{L}$, the connection form satisfies the 'gerbe' condition

$$\pi_L^*\gamma - \pi_R^*\gamma = d\log\chi,\ \chi : \hat{L}^{[2]} \longrightarrow \mathrm{U}(1) \tag{44.21}$$

being the groupoid character of the line bundle.

So, applying this discussion first to $\mathcal{E} \longrightarrow \hat{L}_1^{[2]}$, being the pull-back of $\hat{L}_1$ to $\mathcal{F}^{[2]}$ we find the desired decomposition of the curvature of the line bundle $\mathcal{L}$ over $\mathcal{E}^{[2]}$ in terms of a B-field on $\mathcal{E}$:

$$\omega = d\log\chi_1 \wedge d\log\chi_2/4\pi^2 = \ \pi_L^*\beta - \pi_R^*\beta,\ \beta = \gamma_1 \wedge d\log\chi_2. \tag{44.22}$$

The exterior differential of the B-field, which is to say the curvature of the bundle gerbe, is

$$d\beta = q^*\delta,\ \delta = \omega_1 \wedge d\log\chi_2 \text{ on } \mathcal{F}^{[2]}. \tag{44.23}$$

Now we can proceed to the next step since again there is a decomposition in terms of the connection form for $\hat{L}_2$:

$$\delta = \pi_L^*\mu - \pi_R^*\mu,\ \mu = \omega_1 \wedge \gamma_2 \text{ on } \mathcal{F}. \tag{44.24}$$

Finally we recover the 'curvature' four form as

$$d\mu = p^*\rho,\ \rho = \omega_1 \wedge \omega_2 \text{ on } X. \tag{44.25}$$

So indeed, this is a second example of a bundle 2-gerbe.

For bonus grade on the course, fix a condition on a tower as in

$$\begin{tikzcd}
 & \mathcal{L} \arrow[d] \\
\mathcal{E} & \mathcal{E}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[d, "q^{[2]}"] \\
\mathcal{F} \arrow[from=2-1, ur, phantom] & \mathcal{F}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[d, "p^{[2]}"] \\
 & X
\arrow[from=2-1, to=3-2, "q"']
\arrow[from=3-1, to=4-2, "p"']
\end{tikzcd} \tag{44.26}$$

so that the curvature descends in this way. If you really want to think about this, and someone should, it might be wise to make one further 'step back'. Let $\mathcal{D}$ denote the circle bundle of the line bundle and χ the groupoid character as discussed above. Then the tower becomes

$$\begin{tikzcd}
\mathcal{D} & \mathcal{D}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "\chi"] \arrow[d, "l^{[2]}"] & \mathrm{U}(1) \\
\mathcal{E} & \mathcal{E}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[d, "q^{[2]}"] \\
\mathcal{F} & \mathcal{F}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[d, "p^{[2]}"] \\
 & X.
\arrow[from=1-1, to=2-2, "l"']
\arrow[from=2-1, to=3-2, "q"']
\arrow[from=3-1, to=4-2, "p"']
\end{tikzcd} \tag{44.27}$$

So you can even imagine what a general bundle k-gerbe is in this sense! Think of this as a Rube Goldberg marchine with a ball rolling down and a bell at each level.

So, let me review where we are with the succession of gerbes (which sounds like Kings and Queens of England). Note that the numbering convention is not mine. We can think of X as a compact manifold although as we have already seen the non-compact and infinite dimensional cases may be particularly interesting.

(1) A (-2)-gerbe is a map $X \longrightarrow \mathbb{Z}$ which is continuous or smooth. This generates the 0-cohomology over $\mathbb{Z}$; zut, it *is* the integral cohomology.

(2) A (-1)-gerbe is a smooth (for me, you might take continuous) map $f : X \longrightarrow \mathrm{U}(1)$ (or $\mathbb{C}^*$ would work as well). This generates a covering space X_f for X – namely above each point $x \in X$ take all the possible values of $\log f/2\pi i$. Thus $p_f : X_f \longrightarrow X$ is a principal $\mathbb{Z}$ bundle. We can take the

fibre product of X_f with itself to get the diagram

$$
\begin{tikzcd}
X_f \arrow[dr, "p_f"'] & X_f^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "\delta_f"] \arrow[d, "p_f^{[2]}"] & \mathbb{Z} \\
& X. &
\end{tikzcd}
\tag{44.28}
$$

In fact there is no particular reason to insist here that X_f be a principal $\mathbb{Z}$ bundle. Rather we can just take any smooth fibre bundle (even infinite dimensional if you want) $\mathcal{E}_{-1}$ over X with a map $\delta : \mathcal{E}_{-1}^{[2]} \longrightarrow \mathbb{Z}$ which is additive-multiplicative in the sense that $\delta(z_1, z_3) = \delta(z_1, z_2) + \delta(z_2, z_3)$ for any three points in the same fibre. Of course, this is not an enourmous generalization since the function has to be locally constant anyway.

(3) Now, a 0-gerbe is a(n Hermitian) line bundle, $p : L \longrightarrow X$. We take either L^*, the complement of the zero section, of $\hat{L}$, the circle bundle and proceed as above, giving:-

$$
\begin{tikzcd}
\hat{L} \arrow[dr, "p"'] & \hat{L}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "S"] \arrow[d, "p^{[2]}"] & \mathrm{U}(1) \\
& X. &
\end{tikzcd}
\tag{44.29}
$$

Here we use the fact that the pull-back of L to $\hat{L}$ is canonically trivial and S is the composite of the inverse of this trivialization on the right with the trivialization on the left. It is a 'groupoid character'

$$
S(z_1, z_3) = S(z_1, z_2)S(z_2, z_3) \ \forall \ z_1, z_2, z_3 \in \hat{L}_p. \tag{44.30}
$$

Now, if we want we can interpret this character as a (-1) gerbe and build the tower already here:-

$$
\begin{tikzcd}
X_S \arrow[dr, "p_S"'] & X_S^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "\delta_S"] \arrow[d, "p_S^{[2]}"] & \mathbb{Z} \\
\hat{L} \arrow[dr, "p"'] & \hat{L}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "S"] \arrow[d, "p^{[2]}"] & \mathrm{U}(1) \\
& X. &
\end{tikzcd}
\tag{44.31}
$$

(4) 1-gerbe as above
(5) 2-gerbe as above, but better written out a little more fully!

So, having seen that these cases are really uniform, we are just getting higher towers with more complicated 'primitivity conditions' as we go along. One lesson we can easily draw from this is that the construction above of the decomposed cases – the cup product of a 1- and a 2-class and the cup product of two 2-classes coming from line bundles – can be generalized.

Exercise 43. Work out the 'tensor product' of a 1-gerbe and a (-1)-gerbe. If the gerbe is as in (42.11) and the (-1)-gerbe is a smooth map $f : X \longrightarrow \mathrm{U}(1)$ first push the gerbe the extra step to get a bigger tower:-

(44.32)
$$
\begin{tikzcd}
\widehat{\mathcal{L}} \arrow[dr, "p"'] & \widehat{\mathcal{L}}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "S"] \arrow[d, "p^{[2]}"] & \mathrm{U}(1) \\
\mathcal{E} \arrow[dr, "\pi"'] & \mathcal{E}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[d, "\pi^{[2]}"] & \\
 & X. &
\end{tikzcd}
$$

Now just take the product with the map f to get a map to the torus and proceed to check (I mean that you should do so ...) that the pull-back does indeed give a 2-gerbe by adding an extra tower at the top:

(44.33)
$$
\begin{tikzcd}
 & (S\times f)^*\mathcal{T} \arrow[d] & \mathcal{T} \arrow[d] \\
\widehat{\mathcal{L}} \arrow[dr, "p"'] & \widehat{\mathcal{L}}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[r, "S\times f"] \arrow[d, "p^{[2]}"] & \mathrm{U}(1)\times\mathrm{U}(1) \\
\mathcal{E} \arrow[dr, "\pi"'] & \mathcal{E}^{[2]} \arrow[l, shift right, "\pi_L"'] \arrow[l, shift left, "\pi_R"] \arrow[d, "\pi^{[2]}"] & \\
 & X. &
\end{tikzcd}
$$

Show that the 4-form curvature of this bundle 2-gerbe is $\gamma \wedge d\log f/2\pi i$, where γ is the 3-form curvature of the gerbe.

Exercise 44. See if you can see exactly what happens when one takes the tensor product in this sense of a 2-gerbe and a (-1)-gerbe, to produce a 3-gerbe. My my, this numerology is dumb.

Exercise 45. Develop the same construction for the 'tensor product' of a gerbe and a line bundle, masquerading as a 0-gerbe; if you are brave enough even the product of a 2-gerbe and a 0-gerbe.

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