Semiclassical quantization and index maps

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Smoothing operators

- On a compact manifold, $\Psi^{-\infty}(Z) = \mathcal{C}^\infty(Z^2; \Omega_R)$ is an algebra of operators on $\mathcal{C}^\infty(Z)$ with product

$$AB(z,z') = \int_Z A(z,z'')B(z'',z'). \tag{1}$$

- The group

$$G^{-\infty}(Z) = \{A \in \Psi^{-\infty}(Z);\ \exists\ (\mathrm{Id} + A)^{-1}\} \tag{2}$$

is open in $\Psi^{-\infty}(Z)$ (hence a nice smooth group) where invertibility means injectivity on $\mathcal{C}^\infty(Z)$ or equivalently the existence of a two sided inverse in $\mathrm{Id} + \Psi^{-\infty}(Z)$.

Odd K-theory

- If X is a compact space, for instance a manifold, then

  $$\mathrm{K}^1(X) = [X; G^{-\infty}(Z)] (\text{ homotopy classes of smooth maps} \tag{3}$$

  provided $\dim Z > 0$. That is, $G^{-\infty}(Z)$ is a classifying space for odd K-theory.
- Given a fibration of compact manifolds

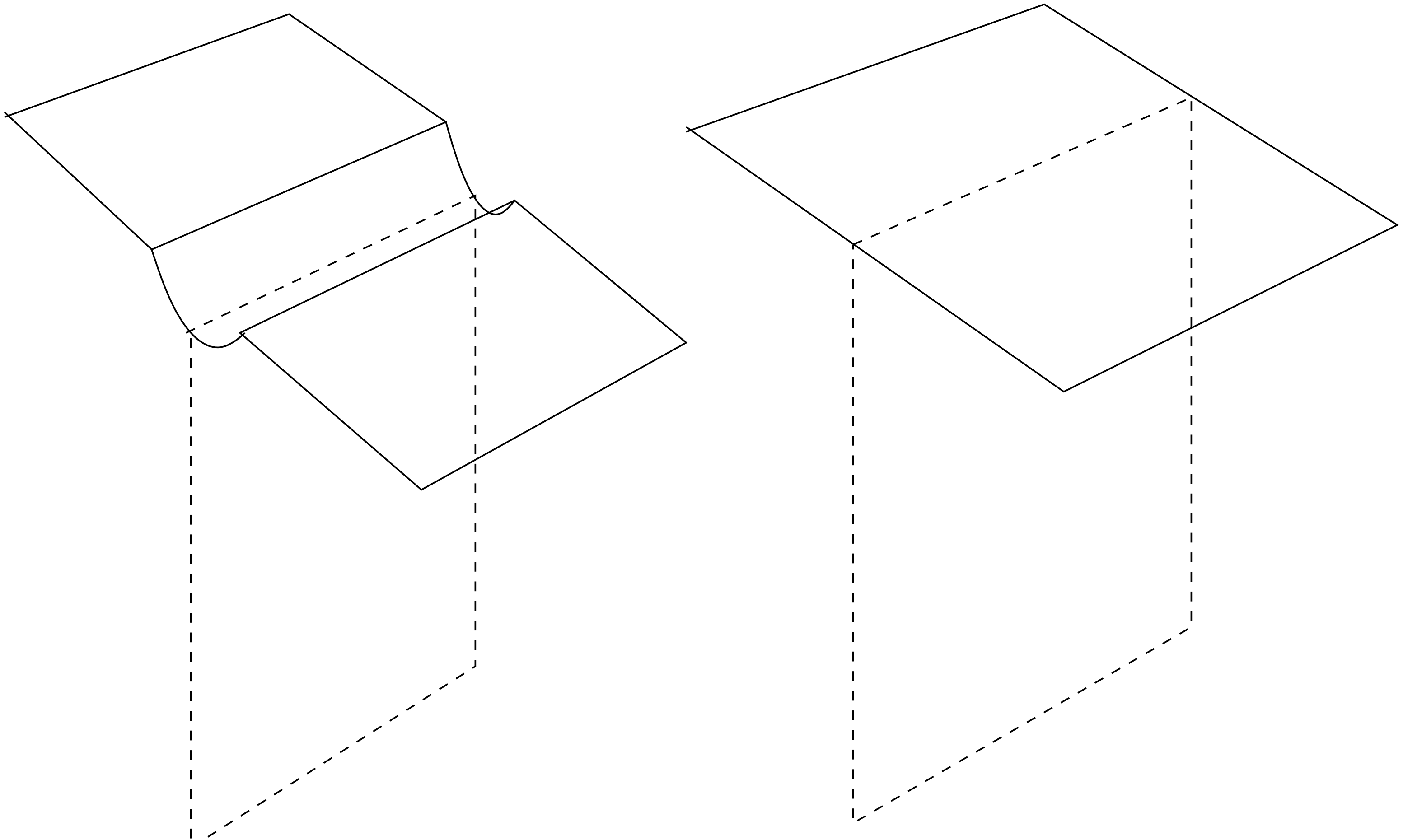
  $$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[r, "\phi"] & B \end{tikzcd}$$
  $$\Psi^{-\infty}(M/B) = \mathcal{C}^\infty(M^2_\phi; \Omega_R) \tag{4}$$

  is the algebra of fibrewise smoothing operators and again $\mathrm{K}^1(B) = [\mathcal{C}^\infty(B; G^{-\infty}(M/B))]$ is the (abelian group) of homotopy classes of smooth sections.

Semiclassical resolution

- Define the manifold with corners by blow up

$$\beta : Z^2_{\mathrm{sl}} = [Z^2 \times [0,1]; \mathrm{Diag} \times \{0\}] \longrightarrow Z^2 \times [0,1]. \qquad (5)$$



Semiclassical smoothing operators

- Now set

$$\Psi_{\mathrm{sl}}^{-\infty}(Z)=\{A\in\epsilon^{-n}\mathcal{C}^{\infty}(Z_{\mathrm{sl}}^{2};\Omega_R);A\equiv 0 \text{ at } \beta^{\#}(\epsilon=0)\}. \tag{6}$$

where $n=\dim Z$.

- This means that locally near the diagonal these kernels take the form

$$\epsilon^{-n}\tilde{A}(\epsilon,z,\frac{z-z'}{\epsilon}),\ \tilde{A}\in\mathcal{C}^{\infty}(U;\mathcal{S}(\mathbb{R}^n)). \tag{7}$$

- This is again an algebra with an exact, multiplicative, sequence given by the fibre Fourier transform of $\tilde{A}$

$$0\longrightarrow\epsilon\Psi_{\mathrm{sl}}^{-\infty}(Z)\longrightarrow\Psi_{\mathrm{sl}}^{-\infty}(Z)\xrightarrow{\sigma_{\mathrm{sl}}}\mathcal{S}(T^*Z)\longrightarrow 0. \tag{8}$$

Semiclassical quantization

- If $a \in \mathcal{S}(T^*Z; M(N, \mathbb{C}))$ is such that $\mathrm{Id}_{N\times N} + a$ is invertible then $A \in \Psi^{-\infty}_{\mathrm{sl}}(Z; \mathbb{C}^N)$ with $\sigma_{\mathrm{sl}}(A) = a$ is such that $\mathrm{Id} + A_\epsilon \in G^{-\infty}(X; \mathbb{C}^N)$ for small N.
- This 'quantization' is well-defined up to homotopy so for a fibration gives a 'push-forward' map

$$\mathrm{ind}_{\mathrm{sl}} : \mathrm{K}^1_{\mathsf{c}}(T^*(M/B)) \longrightarrow K^1(B) \tag{9}$$

where $T^*Z \longrightarrow T^*(M/B) \longrightarrow B$ is the induced fibration.

Excision

- This semiclassical index map is additive (i.e. multiplicative) and consistent under open embeddings (in particular for non-compact fibrations).

$$\begin{tikzcd}
M \arrow[rr, hook, "\iota"] \arrow[dr, "\phi"'] & & \tilde{M} \arrow[dl, "\tilde{\phi}"] \\
& B &
\end{tikzcd}$$

implies the commutativity of

$$\begin{tikzcd}
\mathrm{K}^1_{\mathsf{c}}(T^*(M/B)) \arrow[rr, "\iota_!"] \arrow[dr, "\mathrm{ind}_{\mathrm{sl}}"'] & & \mathrm{K}^1_{\mathsf{c}}(T^*(M/B)) \arrow[dl, "\mathrm{ind}_{\mathrm{sl}}"] \\
& \mathrm{K}^1(B). &
\end{tikzcd} \tag{10}$$

Multiplicativity

- For a tower of fibrations

$$\begin{tikzcd} Z' \arrow[r, dash] & M' \arrow[d, "\phi'"] \\ Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & B \end{tikzcd} \tag{11}$$

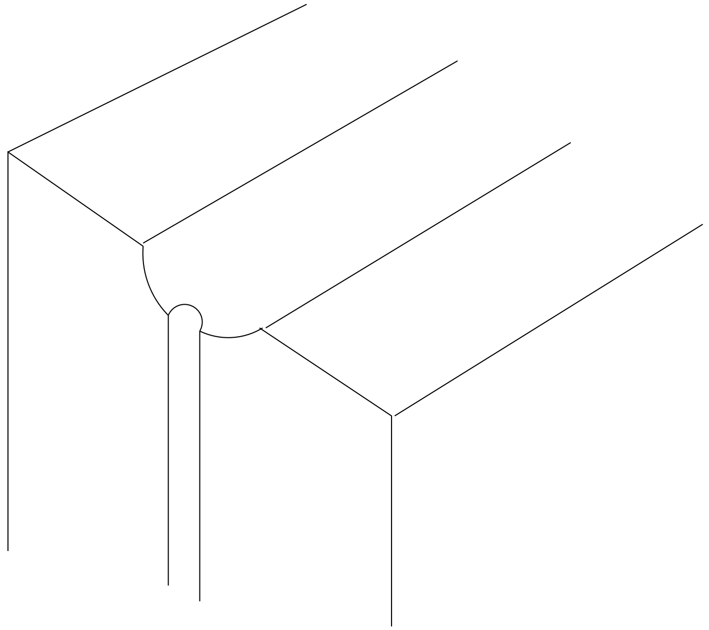
the index composes ('commutes')

$$\begin{tikzcd} \mathrm{K}^1_{\mathsf{c}}(T^*(M'/B)) \arrow[r, "\mathrm{ind}_{\mathrm{sl}}"'] \arrow[rr, bend left, "\mathrm{ind}_{\mathrm{sl}}"] & \mathrm{K}^1_{\mathsf{c}}(T^*(M/B)) \arrow[r, "\mathrm{ind}_{\mathrm{sl}}"'] & \mathrm{K}^1(B). \end{tikzcd} \tag{12}$$

Iterated resolution

- Define the double blow up for a fibration $M \xrightarrow{\phi} B$

$$M^2_{\phi\,\mathrm{sl}} = [M^2 \times [0,1] \times [0,1]; \mathrm{Diag}_\phi \times [0,1] \times \{0\}; \mathrm{Diag} \times [0,1] \times \{0\}]. \tag{13}$$



Iterated semiclassical calculus

- If $n = \dim B$, $p = \dim Z$, set

$$\Psi^{-\infty}_{\phi\,\mathrm{sl}}(M) = \{A \in \epsilon^{-n}\delta^{-p}\mathcal{C}^\infty(M^2_{\phi\,\mathrm{sl}};\Omega_R); \\ A \equiv 0 \text{ at } \beta^{\#}(\epsilon = 0) \cup \beta^{\#}(\delta = 0)\}. \quad (14)$$

- This produces an algebra with kernels of the form

$$\epsilon^{-n}\delta^{-p}A'(\epsilon, \delta, b, z, \frac{b-b'}{\epsilon\delta}, \frac{z-z'}{\epsilon}). \quad (15)$$

- This algebra has two ‘symbol maps’ one valued in the semiclassical algebra on the fibres, the other fully commutative. Both are multiplicative and using them easily gives naturality of the index map for multiple fibrations.

Thom isomorphism

- Thom isomorphism. For the fibration corresponding to a real vector bundle

$$\begin{tikzcd}\mathbb{R}^k \arrow[r, dash] & V \arrow[d, "\pi"] \\ & B\end{tikzcd} \tag{16}$$

$\mathrm{ind}_{\mathrm{sl}} : \mathrm{K}^1_{\mathsf{c}}(T^*(V/B)) \longleftrightarrow \mathrm{K}^1_{\mathsf{c}}(B)$ is an isomorphism.

- This has a rather straightforward proof (due to Atiyah) once one shows surjectivity in the case $V = B \times \mathbb{R}$.

Embedding

- A fibration can always be embedded in a trivial fibration

$$\begin{tikzcd}
M \arrow[rr, "e"] \arrow[dr, "\phi"'] & & \mathbb{R}^N \times B \arrow[dl, "\pi_B"] \\
& B. &
\end{tikzcd} \tag{17}$$

- This embedding factors into an iterated fibration and an open embedding

$$\begin{tikzcd}
N(M/B) \arrow[r, "\tilde{e}"] \arrow[d, shift right, "\pi"'] & \mathbb{R}^N \times B \arrow[ddl, "\pi_B"] \\
M \arrow[u, shift right, "o"'] \arrow[d, "\phi"'] & \\
B. &
\end{tikzcd} \tag{18}$$

Semiclassical equals topological index

- Combining these results we deduce a form of the Atiyah-Singer families index theorem

$$\begin{tikzcd}
\mathrm{K}^1_{\mathsf{c}}(T^*(M/B)) \arrow[r, "\mathrm{ind}_{\mathrm{sl}}^{-1}"] \arrow[d, "\mathrm{ind}_{\mathrm{sl}}"'] \arrow[d, bend left=60, "\mathrm{ind}_{\mathsf{t}}"] & \mathrm{K}^1_{\mathsf{c}}(T^*(N(M/B)/B)) \arrow[d, "i_!"] \\
\mathrm{K}^1(B) & \mathrm{K}^1_{\mathsf{c}}((T^*\mathbb{R}^N) \times B) \arrow[l, "\mathrm{ind}_{\mathrm{sl}}"]
\end{tikzcd} \tag{19}$$

since the composite around the right is, by definition, the topological index.

Even K-theory

- A (complex) vector bundle (over a compact space) can always be embedded in a trivial bundle and so be identified with a smooth family of projections $p : X \longrightarrow M(N, \mathbb{C}), p^2 = p.$
- For a compact space $\mathrm{K}^0(X)$ is $\{(p_+, p_-)\}$ modulo equivalence under conjugation and stability.
- For a noncompact space $\mathrm{K}^0_{\mathsf{c}}(X)$ is the set of projections constant outside a compact set, modulo equivalence, stability and addition of constant projections.
- In either case there is an easily verified natural isomorphism

$$\mathrm{K}^1_{\mathsf{c}}(X) = \mathrm{K}^0_{\mathsf{c}}(X \times \mathbb{R}). \qquad (20)$$

$\mathrm{K}^0_{\mathsf{c}}$ for vector bundles

- For a real vector bundle, V, $\mathrm{K}^0_{\mathsf{c}}(V)$ is the given by an equivalence relation on pairs of projections (p, p_∞) where p_∞ is constant and $p = p_\infty$ outside a compact set.
- In this case $\mathrm{K}^0_{\mathsf{c}}(V)$ can also be realized via an equivalence relation on pairs of projections $\{(p_+, p_-)\}$ smooth on $\overline{V}$, with a smooth isomorphism a between their ranges over $\mathbb{S}V$, the boundary of $\overline{V}$.
- $\mathrm{K}^0_{\mathsf{c}}(V)$ is exhausted by the classes of such pairs where $p_\pm$ are projections over the base.
- Any representative of a class in $\mathrm{K}^0_{\mathsf{c}}(V)$ in the third sense can be smoothly deformed through representatives in the second sense to one in the first sense (and vice versa).

Even semiclassical quantization

- If $p : \overline{T^*Z} \longrightarrow M(N, \mathbb{C})$ is a family of projections smooth on the radial compactification then it is the semiclassical symbol of a family of projections $P \in \Psi^0_{\mathrm{sl}}(X; \mathbb{C}^n)$, $P^2 = P$.
- If $a \in \mathcal{C}^\infty(\mathbb{S}^*Z; M(N, \mathbb{C}))$ is an isomorphism between the ranges of $p_\pm$, smooth projection-valued maps on $\overline{T^*Z}$ then it can be semi-classically quantized to $A \in \Psi^0_{\mathrm{sl}}(Z; \mathbb{C}^N)$ such that

$$AP_+ = A = P_- A. \tag{21}$$

- This can be done smoothly in parameters and such that A has constant rank null space in the range of P_+.
- This proves the identity

$$\mathrm{ind}^0_{\mathrm{sl}} = \mathrm{ind}_{\mathsf{AS}} \,. \tag{22}$$

Compact operators

- To indicate how ‘robust’ semiclassical quantization is, I will show how to 3-twist the index.
- The compact operators, $\mathcal{K}$, on a separable, infinite dimensional, Hilbert space, $\mathcal{H}$, form a C^*-algebra.
- The only linear isomorphism of $\mathcal{K}$ preserving the product and $*$ are given by conjugation by a unitary operator

$$\mathcal{K} \ni K \longmapsto UKU^*. \tag{23}$$

- So the automorphism group of $\mathcal{K}(\mathcal{H})$ is $\mathcal{PU}(\mathcal{H})$ (not $\mathcal{U}(\mathcal{H})$ since the multiples of the identity act trivially).

Azumaya bundles

- Consider an ‘Azumaya bundle’ $\mathcal{A} \longrightarrow Z$, over a compact manifold Z. This means that the fibres of $\mathcal{A}$ are $*$-algebras and the bundle has trivializations over an open cover U_i of Z by isomorphisms to $U_i \times \mathcal{K}$.
- Identify two Azumaya bundles if they are *stably* isomorphic
$$\mathcal{A} \hat{\otimes} \mathcal{K} \equiv \mathcal{A}' \hat{\otimes} \mathcal{K}. \tag{24}$$
- The equivalence classes form an abelian group (sometimes called the infinite Brauer group) which is isomorphic to $H^3(Z;\mathbb{Z})$. The isomorphism is via the Dixmier-Douady class $\alpha \in H^3(Z;\mathbb{Z})$ of $\mathcal{A}$.
- Finite dimensional bundles correspond precisely to torsion classes in $H^3(Z;\mathbb{Z})$.

Twisted K-theory

- The group $\mathcal{G}$ of operators of the form $\mathrm{Id} + K$, $K \in \mathcal{K}(\mathcal{H})$, is also a classifying group for K^1 . It is the norm completion of $G^{-\infty}$.
- An Azumaya bundle $\mathcal{A}$ over Z generates a bundle of groups $\mathcal{G}(\mathcal{A})$ and

$$\mathrm{K}^1(Z;\mathcal{A}) = [Z;\mathcal{G}(\mathcal{A}\hat{\otimes}\mathcal{K})] \tag{25}$$

  the set of homotopy classes of continuous sections is the twisted K-theory of Z defined by $\mathcal{A}$.
- Up to isomorphism this twisted K-group (and the corresponding even one) depends only on $\alpha(\mathcal{A}) \in H^3(Z;\mathbb{Z})$ but there is no natural isomorphism.
- For a manifold Z. $\mathcal{C}^\infty(Z^2;\mathcal{K} \otimes \Omega_R)$ is the algebra of ‘smoothing operators with values in $\mathcal{K}$.’

Twisted semiclassical quantization

- If $\mathcal{A}$ is an Azumaya bundle over the base B of a fibration it is straightforward to generalize the discussion above to give an index map

$$\mathrm{ind}^{\mathcal{A}}_{\mathrm{sl}} : \mathrm{K}^1_{\mathsf{c}}(T^*(M/B); (\pi\phi)^*\mathcal{A}) \longrightarrow \mathrm{K}^1(B; \mathcal{A}). \qquad (26)$$

- In joint work with Varghese Mathai and Is Singer we show that $\mathrm{ind}^{\mathcal{A}}_{\mathrm{sl}} = \mathrm{ind}^{\mathcal{A}}_{\mathsf{t}}$ for the corresponding topological index.
- Here I have discussed the odd index, the reduction to the even case outline above can also be carried through and allows us to discuss the index of a family of Dirac operators on the fibres of M twisted by a projective Hilbert bundle associated to $\mathcal{A}$.