

From Microlocal to Global Analysis

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Preface

This semester, Spring 2008, I am trying to get these lectures notes close to a finished form. They represent accumulated notes from various different ‘Microlocal Analysis’ courses and seminars at MIT. In particular in the seminar this semester, which is a continuation of a course (also run as a seminar) last semester, we hope to complete a proof of the families index theorem of Atiyah and Singer and some version of Weyl asymptotics for self-adjoint elliptic pseudodifferential operators; maybe we will also get to Fourier integral operators.

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Introduction

I shall assume some familiarity with distribution theory, with basic analysis and functional analysis and a passing knowledge of the theory of manifolds. Any one or two of these prerequisites can be easily picked up along the way, but the prospective student with none of them should perhaps do some preliminary reading:

Distributions: A good introduction is Friedlander's book [6]. For a more exhaustive treatment see Volume I of Hörmander's treatise [10].

Analysis on manifolds: Most of what we need can be picked up from Munkres' book [11] or Spivak's little book [14].

