CHAPTER 8

Elliptic boundary problems

Summary

Elliptic boundary problems are discussed, especially for operators of Dirac type. We start with a discussion of manifolds with boundary, including functions spaces and distributions. This leads to the 'jumps formula' for the relationship of the action of a differential operator to the operation of cutting off at the boundary; this is really Green's formula. The idea behind Calderòn's approach to boundary problems is introduced in the restricted context of a dividing hypersurface in a manifold without boundary. This includes the fundamental result on the boundary behaviour of a pseudodifferential operator with a rational symbol. These ideas are then extended to the case of an operator of Dirac type on a compact manifold with boundary with the use of left and right parametrices to define the Calderòn projector. General boundary problems defined by pseudodifferential projections are discussed by reference to the 'Calderòn realization' of the operator. Local boundary conditions, and the corresponding ellipticity conditions, are then discussed and the special case of Hodge theory on a compact manifold with boundary is analysed in detail for absolute and relative boundary conditions.

Introduction

Elliptic boundary problems arise from the fact that elliptic differential operators on compact manifolds with boundary have infinite dimensional null spaces. The main task we carry out below is the parameterization of this null space, in terms of boundary values, of an elliptic differential operator on a manifold with boundary. For simplicity of presentation the discussion of elliptic boundary problems here will be largely confined to the case of first order systems of differential operators of Dirac type. This has the virtue that the principal results can be readily stated.

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Read through Section 8.1–Section 8.2: It is pretty terse in places! Several vital sections are still missing.

8.1. Manifolds with boundary

Smooth manifolds with boundary can be defined in very much the same was as manifolds without boundary. Thus we start with a paracompact Hausdorff space X and assume that it is covered by 'appropriate' coordinate patches with corresponding transition maps. In this case the 'model space' is $\mathbb{R}^{n,1} = [0,\infty) \times \mathbb{R}^{n-1}$, a Euclidean half-space of fixed dimension, n. As usual it is more convenient to use as models all open subsets of $\mathbb{R}^{n,1}$; of course this means *relatively open*, not open as subsets of $\mathbb{R}^n$. Thus we allow any

$$O = O' \cap \mathbb{R}^{n,1}, \quad O' \subset \mathbb{R}^n \text{ open},$$

as local models.

By a smooth map between open sets in this sense we mean a map with a smooth extension. Thus if O_i for $i = 1, 2$ are open in $\mathbb{R}^{n,1}$ then smoothness of a map F means that

$$\text{(8.1)} \quad F : O_1 \to O_2, \ \exists\, O_i' \subset \mathbb{R}^n, \ i = 1, 2, \text{ open and } \tilde{F} : O_1' \to O_2' \\ \text{which is } \mathcal{C}^\infty \text{ with } O_i = O_i' \cap \mathbb{R}^{n,1} \text{ and } F = F'|O_1.$$

It is important to note that the smoothness condition is *much* stronger than just smoothness of F on $O \cap (0, \infty) \times \mathbb{R}^{n-1}$.

By a diffeomorphism between such open sets we mean an invertible smooth map with a smooth inverse. Various ways of restating the condition that a map be a diffeomorphism are discussed below.

With this notion of local model we define a *coordinate system (in the sense of manifolds with boundary)* as a homeomorphism of open sets

$$X \supset U \xrightarrow{\Phi} O \subset \mathbb{R}^{n,1}, \quad O, U \text{ open}.$$

Thus Φ^{-1} is assumed to exist and both Φ and Φ^{-1} are assumed to be continuous. The *compatibility* of two such coordinate systems (U_1, Φ_1, O_1) and (U_2, Φ_2, O_2) is the requirement that *either* $U_1 \cap U_2 = \phi$ *or* if $U_1 \cap U_2 \neq \phi$ then

$$\Phi_{1,2} = \Phi_2 \circ \Phi_1^{-1} : \Phi_1(U_1 \cap U_2) \to \Phi_2(U_1 \cap U_2)$$

is a diffeomorphism in the sense described above. Notice that both $\Phi_1(U_1 \cap U_2)$ and $\Phi_2(U_1 \cap U_2)$ are open in $\mathbb{R}^{n,1}$. The inverse $\Phi_{1,2}$ is defined analogously.

A $\mathcal{C}^\infty$ manifold with boundary can then be formally defined as a paracompact Hausdorff topological space which has a maximal covering by mutually compatible coordinate systems.

An alternative definition, i.e. characterization, of a manifold with boundary is that there exists a $\mathcal{C}^\infty$ manifold $\tilde{X}$ *without boundary* and a function $f \in \mathcal{C}^\infty(\tilde{X})$ such that $df \neq 0$ on $\{f = 0\} \subset \tilde{X}$ and

$$X = \left\{ p \in \tilde{X}; f(p) \geq 0 \right\},$$

with coordinate systems obtained by restriction from $\tilde{X}$. The doubling construction described below shows that this is in fact an equivalent notion.

8.2. Smooth functions

As in the boundaryless case, the space of functions on a compact manifold with boundary is the primary object of interest. There are two basic approaches to defining local smoothness, the one intrinsic and the other extrinsic, in the style of the two definitions of a manifold with boundary above. Thus if $O \subset \mathbb{R}^{n,1}$ is open we can simply set

$$\mathcal{C}^\infty(O) = \{u : O \to \mathbb{C}; \exists\, \tilde{u} \in \mathcal{C}^\infty(O'), \\ O' \subset \mathbb{R}^n \text{ open}, \quad O = O' \cap \mathbb{R}^{n,1}, u = \tilde{u}|_O \}.$$

Here the open set in the definition might depend on u. The derivatives of $\tilde{u} \in \mathcal{C}^\infty(O')$ are bounded on all compact subsets, $K \Subset 0$. Thus

$$\sup_{K\cap O^\circ} |D^\alpha u| < \infty, \quad O^\circ = O \cap ((0,\infty) \times \mathbb{R}^{n-1}). \tag{8.2}$$

The second approach is to use (8.2) as a definition, i.e.
to set

$$\mathcal{C}^\infty(O) = \{u : O^\circ \to \mathbb{C}; (8.2) \text{ holds } \forall\ K \Subset O \text{ and all } \alpha\}. \tag{8.3}$$

In particular this implies the continuity of $u \in \mathcal{C}^\infty(O)$ up to any point $p \in O \cap (\{0\} \times \mathbb{R}^{n-1})$, the boundary of O as a manifold with boundary.

As the notation here asserts, these two approaches are equivalent. This follows (as does much more) from a result of Seeley:

Proposition 8.1. *If $\mathcal{C}^\infty(O)$ is defined by* (8.3) *and $O' \subset \mathbb{R}^n$ is open with $O = O' \cap \mathbb{R}^{n,1}$ then there is a linear extension map*

$$E : \mathcal{C}^\infty(O) \to \mathcal{C}^\infty(O'), \quad Eu|_{O'} = u$$

which is continuous in the sense that for each $K' \Subset O'$, compact, there is some $K \Subset O$ such that for each α

$$\sup_{K'} |D^\alpha Eu| \le C_{\alpha,K'} \sup_{K\cap O} |D^\alpha u|.$$

The existence of such an extension map shows that the definition of a diffeomorphism of open sets O_1, O_2, given above, is equivalent to the condition that the map be invertible and that it, and its inverse, have components which are in $\mathcal{C}^\infty(O_1)$ and $\mathcal{C}^\infty(O_2)$ respectively.

Given the local definition of smoothness, the global definition should be evident. Namely, if X is a $\mathcal{C}^\infty$ manifold with boundary then

$$\mathcal{C}^\infty(X) = \left\{u : X \to \mathbb{C}; (\Phi^{-1})^*(u|_U) \in \mathcal{C}^\infty(O) \ \forall \text{ coordinate systems}\right\}.$$

This is also equivalent to demanding that local regularity, i.e.
the regularity of $(\Phi^{-1})^*(u|_O)$, hold for any one covering by admissible coordinate systems.

As is the case of manifolds without boundary, $\mathcal{C}^\infty(X)$ admits partitions of unity. In fact the proof of Lemma 6.3 applies verbatim; see also Problem 6.3.

The topology of $\mathcal{C}^\infty(X)$ is given by the supremum norms of the derivatives in local coordinates. Thus a seminorm

$$\sup_{K\Subset O} \left|D^\alpha (\Phi^{-1})^*(u|_U)\right|$$

arises for each compact subset of each coordinate patch. In fact there is a countable set of norms giving the same topology. If X is compact, $\mathcal{C}^\infty(X)$ is a Fréchet space, if it is not compact it is an inductive limit of Fréchet spaces (an LF space).

The boundary of X, ∂X, is the union of the $\Phi^{-1}(O \cap (\{0\} \times \mathbb{R}^{n-1}))$ over coordinate systems. It is a manifold without boundary. It is compact if X is compact. Furthermore, ∂X has a global defining function $\rho \in \mathcal{C}^\infty(X)$; that is $\rho \ge 0$, $\partial X = \{\rho = 0\}$ and $d\rho \ne 0$ at ∂X. Moreover if ∂X is compact then any such boundary defining function can be extended to a product decomposition of X near ∂X:

$$\exists C \ \supset \partial X, \text{open in } X \ \epsilon > 0 \text{ and a diffeomorphism} \varphi : C \simeq [0,\epsilon)_\rho \times \partial X. \tag{8.4}$$

If ∂X is not compact this is still possible for an appropriate choice of ρ. For an outline of proofs see Problem 8.1.

LEMMA 8.1. *If X is a manifold with compact boundary then for any boundary defining function $\rho \in \mathcal{C}^\infty(X)$ there exists $\epsilon > 0$ and a diffeomorphism* (8.4).

PROBLEM 8.1.

The existence of such a product decomposition near the boundary (which might have several components) allows the doubling construction mentioned above to be carried through. Namely, let

$$\tilde{X} = (X \cup X)/\partial X \tag{8.5}$$

be the disjoint union of two copies of X with boundary points identified. Then consider

$$\begin{aligned} \mathcal{C}^\infty(\tilde{X}) = \{(u_1, u_2) \in \mathcal{C}^\infty(X) \oplus \mathcal{C}^\infty(X);& \\ (\varphi^{-1})^*(u_1|_C) = f(\rho, \cdot), (\varphi^{-1})^*(u_2|_C) = f(-\rho, \cdot),& \\ f \in \mathcal{C}^\infty((-1,1) \times \partial X)\}&. \end{aligned} \tag{8.6}$$

This is a $\mathcal{C}^\infty$ structure on $\tilde{X}$ such that $X \hookrightarrow \tilde{X}$, as the first term in (8.5), is an embedding as a submanifold with boundary, so

$$\mathcal{C}^\infty(X) = \mathcal{C}^\infty(\tilde{X})|_X.$$

In view of this possibility of extending X to $\tilde{X}$, we shall not pause to discuss all the usual 'natural' constructions of tensor bundles, density bundles, bundles of differential operators, etc. They can simply be realized by restriction from $\tilde{X}$. In practice it is probably preferable to use intrinsic definitions.

The definition of $\mathcal{C}^\infty(X)$ implies that there is a well-defined restriction map

$$\mathcal{C}^\infty(X) \ni u \longmapsto u|_{\partial X} \in \mathcal{C}^\infty(\partial X).$$

It is always surjective. Indeed the existence of a product decomposition shows that any smooth function on ∂X can be extended locally to be independent of the chosen normal variable, and then cut off near the boundary.

There are important points to observe in the description of functions near the boundary. We may think of $\mathcal{C}^\infty(X) \subset \mathcal{C}^\infty(X^\circ)$ as a subspace of the smooth functions on the interior of X which describes the 'completion' (compactification if X is compact!) of the interior to a manifold with boundary. It is in this sense that the action of a differential operator $P \in \operatorname{Diff}^m(X)$

$$P : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

should be understood. Thus P is just a differential operator on the interior of X with 'coefficients smooth up to the boundary.'

Once this action is understood, there is an obvious definition of the space of $\mathcal{C}^\infty$ functions which vanish to all orders at the boundary,

$$\dot{\mathcal{C}}^\infty(X) = \{u \in \mathcal{C}^\infty(X); Pu|_{\partial X} = 0 \ \forall\ P \in \operatorname{Diff}^*(X)\}.$$

Having chosen a product decomposition near the boundary, Taylor's theorem gives us an isomorphism

$$\mathcal{C}^\infty(X)/\dot{\mathcal{C}}^\infty(X) \cong \bigoplus_{k \ge 0} \mathcal{C}^\infty(\partial X) \cdot [d\rho|_{\partial X}]^k.$$

8.3. Distributions

It is somewhat confusing that there are *three* (though really only *two*) spaces of distributions immediately apparent on a compact manifold with boundary. Understanding the relationship between them is important to the approach to boundary problems used here.

The *coarsest* (as it is a little dangerous to say *largest*) space is $\mathcal{C}^{-\infty}(X^\circ)$, the dual of $\mathcal{C}^\infty_c(X^\circ;\Omega)$, just the space of distributions on the interior of X. The elements of $\mathcal{C}^{-\infty}(X^\circ)$ may have unconstrained growth, and unconstrained order of singularity, approaching the boundary. They are not of much practical value here and appear for conceptual reasons.

Probably the most natural space of distributions to consider is the dual of $\mathcal{C}^\infty(X;\Omega)$ since this is arguably the direct analogue of the boundaryless case. We shall denote this space

$$\dot{\mathcal{C}}^{-\infty}(X) = (\mathcal{C}^\infty(X;\Omega))' \tag{8.7}$$

and call it the space of *supported distributions*. The 'dot' is supposed to indicate this support property, which we proceed to describe.

If $\tilde{X}$ is any compact extension of X (for example the double) then, as already noted, the restriction map $\mathcal{C}^\infty(\tilde{X};\Omega) \longrightarrow \mathcal{C}^\infty(X;\Omega)$ is continuous and surjective. Thus, by duality, we get an injective 'extension' map

$$\dot{\mathcal{C}}^{-\infty}(X) \ni u \longmapsto \tilde{u} \in \mathcal{C}^{-\infty}(\tilde{X}),\ \tilde{u}(\varphi) = u(\varphi|_X). \tag{8.8}$$

We shall regard this injection as an identification $\dot{\mathcal{C}}^{-\infty}(X) \hookrightarrow \mathcal{C}^{-\infty}(\tilde{X})$; its range is easily characterized.

Proposition 8.2. *The range of the map* (8.8) *is the subspace consisting of those* $\tilde{u} \in \mathcal{C}^{-\infty}(\tilde{X})$ *with* $\operatorname{supp}\tilde{u} \subset X$.

The proof is given below. This proposition is the justification for calling $\dot{\mathcal{C}}^{-\infty}(X)$ the space of supported distributions; the dot is support to indicate that this is the subspace of the 'same' space for $\tilde{X}$, i.e.
$\mathcal{C}^{-\infty}(\tilde{X})$, of elements with support in X.

This notation is consistent with $\dot{\mathcal{C}}^\infty(X) \subset \mathcal{C}^\infty(\tilde{X})$ being the subspace (by extension as zero) of elements with support in X. The same observation applies to sections of any vector bundle, so

$$\dot{\mathcal{C}}^\infty(X;\Omega) \subset \mathcal{C}^\infty(\tilde{X};\Omega)$$

is a well-defined closed subspace. We set

$$\mathcal{C}^{-\infty}(X) = (\dot{\mathcal{C}}^\infty(X;\Omega))' \tag{8.9}$$

and call this the space of *extendible distributions* on X. The inclusion map for the test functions gives by duality a restriction map:

$$\begin{gathered} R_X : \mathcal{C}^{-\infty}(\tilde{X}) \longrightarrow \mathcal{C}^{-\infty}(X), \\ R_X u(\varphi) = u(\varphi)\ \forall\ \varphi \in \dot{\mathcal{C}}^\infty(X;\Omega) \hookrightarrow \mathcal{C}^\infty(\tilde{X};\Omega). \end{gathered} \tag{8.10}$$

We write, at least sometimes, R_X for the map since it has a large null space so should *not* be regarded as an identification. In fact

$$\operatorname{Nul}(R_X) = \Big\{v \in \mathcal{C}^{-\infty}(\tilde{X}); \operatorname{supp}(v) \cap X^\circ = \phi\Big\} = \dot{\mathcal{C}}^{-\infty}(\tilde{X}\backslash X^\circ), \tag{8.11}$$

is just the space of distributions supported 'on the other side of the boundary'. The primary justification for calling $\mathcal{C}^{-\infty}(X)$ the space of extendible distributions is:

PROPOSITION 8.3. *If X is a compact manifold with boundary, then the space $\mathcal{C}_c^\infty(X^\circ;\Omega)$ is dense in $\dot{\mathcal{C}}^\infty(X;\Omega)$ and hence the restriction map*

$$\mathcal{C}^{-\infty}(X) \hookrightarrow \mathcal{C}^{-\infty}(X^\circ) \tag{8.12}$$

is injective, whereas the restriction map from (8.10)*, $R_X : \dot{\mathcal{C}}^{-\infty}(X) \longrightarrow \mathcal{C}^{-\infty}(X)$, is surjective.*

PROOF. If V is a real vector field on $\tilde{X}$ which is inward-pointing across the boundary then

$$\exp(sV) : \tilde{X} \to \tilde{X}$$

is a diffeomorphism with $F_s(X) \subset X^\circ$ for $s > 0$. Furthermore if $\varphi \in \mathcal{C}^\infty(\tilde{X})$ then $F_s^*\varphi \to \varphi$ in $\mathcal{C}^\infty(\tilde{X})$ as $s \to 0$. The support property shows that $F_s^*\varphi \in \mathcal{C}_c^\infty(X^\circ)$ if $s < 0$ and $\varphi \in \dot{\mathcal{C}}^\infty(X)$. This shows the density of $\mathcal{C}_c^\infty(X^\circ)$ in $\dot{\mathcal{C}}^\infty(X)$. Since all topologies are uniform convergence of all derivatives in open sets. The same argument applies to densities. The injectivity of (8.12) follows by duality.

On the other hand the surjectivity of (8.10) follows directly from the Hahn-Banach theorem. □

PROOF OF PROPOSITION 8.2. For $\tilde{u} \in \mathcal{C}^{-\infty}(\tilde{X})$ the condition that $\operatorname{supp}\tilde{u} \subset X$ is just

$$\tilde{u}(\varphi) = 0 \ \forall\ \varphi \in \mathcal{C}_c^\infty \subset (\tilde{X}\backslash X;\Omega) \subset \mathcal{C}^\infty(\tilde{X};\Omega). \tag{8.13}$$

Certainly (8.13) holds if $u \in \dot{\mathcal{C}}^{-\infty}(X)$ since $\varphi|_X = 0$. Conversely, if (8.13) holds, then by continuity and the density of $\mathcal{C}_c^\infty(\tilde{X}\backslash X;\Omega)$ in $\mathcal{C}^\infty(\tilde{X}\backslash X^\circ;\Omega)$, what follows from Proposition 8.3, $\tilde{u}$ vanishes on $\dot{\mathcal{C}}^\infty(X\backslash X^\circ)$. □

It is sometimes useful to consider topologies on the spaces of distributions $\mathcal{C}^{-\infty}(X)$ and $\dot{\mathcal{C}}^{-\infty}(X)$. For example we may consider the *weak topology.* This is given by all the seminorms $u \mapsto \|\langle u, \phi\rangle\|$, where ϕ is a test function.

LEMMA 8.2. *With respect to the weak topology, the subspace $\mathcal{C}_c^\infty(X^\circ)$ is dense in both $\dot{\mathcal{C}}^{-\infty}(X)$ and $\mathcal{C}^{-\infty}(X)$.*

8.4. Boundary Terms

To examine the precise relationship between the supported and extendible distributions consider the space of 'boundary terms'.

$$\dot{\mathcal{C}}_{\partial X}^{-\infty}(X) = \left\{u \in \dot{\mathcal{C}}^{-\infty}(X); \operatorname{supp}(u) \subset \partial X\right\}. \tag{8.14}$$

Here the support may be computed with respect to any extension, or intrinsically on X. We may also define a map 'cutting off' at the boundary:

$$\mathcal{C}^\infty(X) \ni u \mapsto u_c \in \dot{\mathcal{C}}^{-\infty}(X), u_c(\varphi) = \int_X u\varphi \ \forall\ \varphi \in \mathcal{C}^\infty(X;\Omega). \tag{8.15}$$

PROPOSITION 8.4. *If X is a compact manifold with boundary then there is a commutative diagram*

$$\begin{tikzcd} & & & \dot{\mathcal{C}}^\infty(X) \arrow[d] & \\ & & & \mathcal{C}^\infty(X) \arrow[dl, "()_c"'] \arrow[d] & \\ 0 \arrow[r] & \dot{\mathcal{C}}^{-\infty}_{\partial X}(X) \arrow[r] & \dot{\mathcal{C}}^{-\infty}(X) \arrow[r] & \mathcal{C}^{-\infty}(X) \arrow[r] & 0 \end{tikzcd} \tag{8.16}$$

with the horizontal sequence exact.

PROOF. The commutativity of the triangle follows directly from the definitions. The exactness of the horizontal sequence follows from the density of $\mathcal{C}^\infty_c(X^\circ;\Omega)$ in $\dot{\mathcal{C}}^\infty(X;\Omega)$. Indeed, this shows that $v \in \dot{\mathcal{C}}^{-\infty}_{\partial X}(X)$ maps to 0 in $\mathcal{C}^{-\infty}(X)$ since $v(\varphi) = 0 \ \forall \ \varphi \in \mathcal{C}^\infty_c(X^\circ;\Omega)$. Similarly, if $u \in \dot{\mathcal{C}}^{-\infty}(X)$ maps to zero in $\mathcal{C}^{-\infty}(X)$ then $u(\varphi) = 0$ for all $\varphi \in \mathcal{C}^\infty_c(X^\circ;\Omega)$, so $\operatorname{supp}(u) \cap X^\circ = \emptyset$. □

Note that both maps in (8.16) from $\mathcal{C}^\infty(X)$ into supported and extendible distributions are injective. We regard the map into $\mathcal{C}^{-\infty}(X)$ as an identification. In particular this is consistent with the action of differential operators. Thus $P \in \operatorname{Diff}^m(X)$ acts on $\mathcal{C}^\infty(X)$ and then the smoothness of the coefficients of P amount to the fact that it preserves $\mathcal{C}^\infty(X)$, as a subspace. The formal adjoint P^* with respect to the sesquilinear pairing for some smooth positive density, ν

$$\langle \varphi, \psi \rangle = \int_X \varphi \overline{\psi} \nu \quad \forall \ \varphi, \psi \in \mathcal{C}^\infty(X) \tag{8.17}$$

acts on $\dot{\mathcal{C}}^\infty(X)$:

$$\langle P^*\varphi, \psi \rangle = \langle \varphi P \psi \rangle \quad \forall \ \varphi \in \dot{\mathcal{C}}^\infty(X), \psi \in \mathcal{C}^\infty(X), \ P^* : \dot{\mathcal{C}}^\infty(X) \longrightarrow \dot{\mathcal{C}}^\infty(X). \tag{8.18}$$

However, $P^* \in \operatorname{Diff}^m(X)$ is fixed by its action over X°. Thus we do have

$$\langle P^*\varphi, \psi \rangle = \langle \varphi, P \psi \rangle \quad \forall \ \varphi \in \mathcal{C}^\infty(X), \psi \in \dot{\mathcal{C}}^\infty(X). \tag{8.19}$$

We define the action of P by duality. In view of the possibility of confusion, we denote P the action on $\mathcal{C}^{-\infty}(X)$ and by $\dot{P}$ the action on $\dot{\mathcal{C}}^\infty(X)$.

$$\begin{gathered} \langle Pu, \varphi \rangle = \langle u, P^*\varphi \rangle \quad \forall \ u \in \mathcal{C}^{-\infty}(X), \varphi \in \dot{\mathcal{C}}^\infty(X), \ P : \mathcal{C}^{-\infty}(X) \longrightarrow \mathcal{C}^{-\infty}(X) \\ \langle \dot{P}u, \varphi \rangle = \langle u, P^*\varphi \rangle \quad \forall \ u \in \dot{\mathcal{C}}^{-\infty}(X), \varphi \in \mathcal{C}^\infty(X), \ \dot{P} : \dot{\mathcal{C}}^{-\infty}(X) \longrightarrow \dot{\mathcal{C}}^{-\infty}(X). \end{gathered} \tag{8.20}$$

It is of fundamental importance that (8.19) does not hold for all $\varphi, \psi \in \mathcal{C}^\infty(X)$. This failure is reflected in Green's formula for the boundary terms, which appears below as the 'Jump formula'. This is a distributional formula for the difference

$$\dot{P}u_c - (Pu)_c \in \dot{\mathcal{C}}^{-\infty}_{\partial X}, u \in \mathcal{C}^\infty(X) P \in \operatorname{Diff}^m(X). \tag{8.21}$$

Recall that a product decomposition of $C \subset X$ near ∂X is fixed by an inward pointing vector field V. Let $x \in \mathcal{C}^\infty(X)$ be a corresponding boundary defining function, with $Vx = 0$ near ∂X, with $\chi_V : C \to \partial X$ the projection onto the boundary from the product neighborhood C. Then Taylor's formula for $u \in \mathcal{C}^\infty(X)$ becomes

$$u \sim \sum_k \frac{1}{k!} \chi_V^*(V^k u|_{\partial x}) x^k. \tag{8.22}$$

It has the property that a finite sum

$$u_N = \varphi u - \varphi \sum_{k=0}^N \frac{1}{k!} \chi_V^*(V^k u|_{\partial X}) x^k$$

where $\varphi \equiv 1$ near ∂X, $\operatorname{supp} \varphi \subset C$, satisfies

$$\dot{P}(u_N)_c = (Pu_N)_c, P \in \operatorname{Diff}^m(X), m < N. \tag{8.23}$$

Since $(1-\varphi)u \in \dot{\mathcal{C}}^\infty(X)$ also satisfies this identity, the difference in (8.21) can (of course) only depend on the $V^k u|_{\partial X}$ for $k \leq m$, in fact only for $k < m$.

Consider the Heaviside function $1_c \in \dot{\mathcal{C}}^{-\infty}(X)$, detained by cutting off the identity function of the boundary. We define distributions

$$\delta^{(j)}(x) = V^{j+1} 1_c \in \dot{\mathcal{C}}^{-\infty}_{\partial X}, j \geq 0. \tag{8.24}$$

Thus, $\delta^{(0)}(x) = \delta(x)$ is a 'Dirac delta function' at the boundary. Clearly $\operatorname{supp} \delta(x) \subset \partial X$, so the same is true of $\delta^{(j)}(x)$ for every j. If $\psi \in \mathcal{C}^\infty(\partial X)$ we define

$$\psi \cdot \delta^{(j)}(x) = \varphi(X_V^* \psi) \cdot \delta^{(j)}(x) \in \dot{\mathcal{C}}^{-\infty}_{\partial X}(X). \tag{8.25}$$

This, by the support property of $\delta^{(j)}$, is independent of the cut off φ used to define it.

Proposition 8.5. *For each $P \in \operatorname{Diff}^m(X)$ there are differential operators on the boundary $P_{ij} \in \operatorname{Diff}^{m-i-j-1}(\partial X)$, $i + j < m$, $i, j \geq 0$, such that*

$$\dot{P}u_c - (Pu)_c = \sum_{i,j} (P_{ij}(V_u^j|_{\partial X}) \cdot \delta^{(j)}(x), \ \forall \ u \in \mathcal{C}^\infty(X), \tag{8.26}$$

and $P_{0m-1} = i^{-m}\sigma(P, dx) \in \mathcal{C}^\infty(\partial X)$.

Proof. In the local product neighborhood C,

$$P = \sum_{0 \leq l \leq m} P_l V^l \tag{8.27}$$

where P_l is a differential operator of the order at most $m - l$, on X be depending on x as a parameter. Thus the basic cases we need to analyze arise from the application of V to powers of x :

$$\begin{aligned} &x^l \left(V^{j+1}(x^p)_c - (V^{j+1}x^p)_c \right) \\ &\qquad = \begin{cases} 0 & l + p > j \\ \frac{p!(j-p)!}{(j-p-l)!} & (-1)^{l\delta^{(j-p-l)}} \qquad l + p \leq j. \end{cases} \end{aligned} \tag{8.28}$$

Taking the Taylor sense of the P_l,

$$P_l \sim \sum_r x^r P_{l,r}$$

and applying P to (8.22) gives

$$Pu_c - (Pu)_c = \sum_{r+k<l} (-1)^r (P_{l,r}(V^k u|_{\partial x})) \quad \delta^{(l-1-r-k)}(x). \tag{8.29}$$

This is of the form (8.26). The only term with $l-1-r-k=m-1$ arises from $l-m$, $k=r=0$ so is the operator P_m at $x=0$. This is just $i^{-m}\sigma(P,dx)$. $\square$

8.5. Sobolev spaces

As with $\mathcal{C}^\infty$ functions we may define the standard (extendible) Sobolev spaces by restriction or intrinsically. Thus, if $\tilde{X}$ is an extension of a compact manifold with boundary, X, the we can define

$$H^m(X) = H^m_c(\tilde{X})|X,\ \forall\ m \in \mathbb{R}; H^m(X) \subset \mathcal{C}^{-\infty}(X). \tag{8.30}$$

That this is independent of the choice of $\tilde{X}$ follows from the standard properties of the Sobolev spaces, particularly their localizability and invariance under diffeomorphisms. The norm in $H^m(X)$ can be taken to be

$$\|u\|_m = \inf\left\{\|\tilde{u}\|_{H^m(\tilde{X})}; \tilde{u} \in H^m(\tilde{X}),\ u = \tilde{u}_X\right\}. \tag{8.31}$$

A more intrinsic defintion of these spaces is discussed in the problems.

There are also *supported* Sobolev spaces,

$$\dot{H}^m(X) = \left\{u \in H^m(\tilde{X}); \operatorname{supp}(u) \subset X\right\} \subset \dot{\mathcal{C}}^{-\infty}(X). \tag{8.32}$$

Sobolev space of sections of any vector bundle can be defined similarly.

Proposition 8.6. *For any $m \in \mathbb{R}$ and any compact manifold with boundary X, $H^m(X)$ is the dual of $\dot{H}^{-m}(X;\Omega)$ with respect to the continuous extension of the densely defined bilinear pairing*

$$(u,v) = \int_X uv,\ u \in \mathcal{C}^\infty(X),\ v \in \dot{\mathcal{C}}^\infty(X;\Omega).$$

Both $H^m(X)$ and $\dot{H}^m(X)$ are $\mathcal{C}^\infty(X)$-modules and for any vector bundle over X, $H^m(X;E) \equiv H^m(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X;E)$ and $\dot{H}^m(X;E) \equiv \dot{H}^m(X) \otimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X;E)$.

Essentially from the definition of the Sobolev spaces, any $P \in \operatorname{Diff}^k(X;E_1,E_2)$ defines a continuous linear map

$$P : H^m(X;E_1) \longrightarrow H^{m-k}(X;E_2). \tag{8.33}$$

We write the dual (to P^* of course) action

$$\dot{P} : \dot{H}^m(X;E_1) \longrightarrow \dot{H}^{m-k}(X;E_2). \tag{8.34}$$

These actions on Sobolev spaces are consistent with the corresponding actions on distributions. Thus

$$\mathcal{C}^{-\infty}(X;E) = \bigcup_m H^m(X),\ \mathcal{C}^\infty(X;E) = \bigcap_m H^m(X),$$
$$\dot{\mathcal{C}}^{-\infty}(X;E) = \bigcup_m \dot{H}^m(X),\ \dot{\mathcal{C}}^\infty(X;E) = \bigcap_m \dot{H}^m(X).$$

8.6. Dividing hypersurfaces

As already noted, the point of view we adopt for boundary problems is that they provide a parametrization of the space of solutions of a differential operator on a space with boundary. In order to clearly indicate the method pioneered by Calderòn, we shall initially consider the restrictive context of an operator of Dirac type on a compact manifold without boundary with an embedded separating hypersurface.

Thus, suppose initially that D is an elliptic first order differential operator acting between sections of two (complex) vector bundles V_1 and V_2 over a compact manifold without boundary, M. Suppose further that $H \subset M$ is a dividing hypersurface. That is, H is an embedded hypersuface with oriented (i.e. trivial) normal bundle and that $M = M_+ \cup M_-$ where $M_\pm$ are compact manifolds with boundary which intersect in their common boundary, H. The convention here is that M_+ is on the positive side of H with respect to the orientation.

In fact we shall make a further analytic assumption, that

$$D : \mathcal{C}^\infty(M; V_1) \longrightarrow \mathcal{C}^\infty(M; V_2) \text{ is an isomorphism.} \tag{8.35}$$

As we already know, D is always Fredholm, so this implies the topological condition that the index vanish. However we only assume (8.35) to simplify the initial discussion.

Our objective is to study the space of solutions on M_+. Thus consider the map

$$\left\{u \in \mathcal{C}^\infty(M_+; V_1); Du = 0 \text{ in } M_+^\circ\right\} \xrightarrow{b_H} \mathcal{C}^\infty(H; V_1),\ b_H u = u_{|\partial M_+}. \tag{8.36}$$

The idea is to use the boundary values to parameterize the solutions and we can see immediately that this is possible.

Lemma 8.3. *The assumption* (8.35) *imples that map b_H in* (8.36) *is injective.*

Proof. Consider the form of D in local coordinates near a point of H. Let the coordinates be $x, y_1, \dots, y_{n-1}$ where x is a local defining function for H and assume that the coordinate patch is so small that V_1 and V_2 are trivial over it. Then

$$D = A_0 D_x + \sum_{j=1}^{n-1} A_j D_{y_j} + A'$$

where the A_j and A' are local smooth bundle maps from V_1 to V_2. In fact the ellipticity of D implies that each of the A_j's is invertible. Thus the equation can be written locally

$$D_x u = Bu,\ B = -\sum_{j=1}^{n-1} A_0^{-1} D_{y_j} - A_0^{-1} A'.$$

The differential operator B is tangent to H. By assumption u vanishes when restricted to H so it follows that $D_x u$ also vanishes at H. Differentiating the equation with respect to x, it follows that all derivatives of u vanish at H. This in turn implies that the global section of V_1 over M

$$\tilde{u} = \begin{cases} u & \text{in } M_+ \\ 0 & \text{in } M_- \end{cases}$$

is smooth and satisfies $D\tilde{u} = 0$. Then assumption (8.35) implies that $\tilde{u} = 0$, so $u = 0$ in M_+ and b_H is injective. $\square$

In the proof of this Lemma we have used the strong assumption (8.35). As we show below, if it is assumed instead that D is of Dirac type then the Lemma remains true without assuming (8.35). Now we can state the basic result in this setting.

THEOREM 8.1. *If $M = M_+ \cup M_-$ is a compact manifold without boundary with separating hypersurface H as described above and $D \in \mathrm{Diff}^1(M; V_1, V_2)$ is a generalized Dirac operator then there is an element $\Pi_C \in \Psi^0(H; V)$, $V = V_1|H$, satisfying $\Pi_C^2 = \Pi_C$ and such that*

$$b_H : \{u \in \mathcal{C}^\infty(M_+; V_1); Du = 0\} \longrightarrow \Pi_C \mathcal{C}^\infty(H; V) \tag{8.37}$$

is an isomorphism. The projection Π_C can be chosen so that

$$b_H : \{u \in \mathcal{C}^\infty(M_-; V_1); Du = 0\} \longrightarrow (\mathrm{Id} - \Pi_C)\mathcal{C}^\infty(H; V) \tag{8.38}$$

then Π_C is uniquely determined and is called the Calderòn projection.

This result remains true for a general elliptic operator of first order if (8.35) is assumed, and even in a slightly weakened form without (8.35). Appropriate modifications to the proofs below are consigned to problems.

For first order operators the jump formula discussed above takes the following form.

LEMMA 8.4. *Let D be an elliptic differential operator of first order on M, acting between vector bundles V_1 and V_2. If $u \in \mathcal{C}^\infty(M_+; V_1)$ satisfies $Du = 0$ in M_+° then*

$$Du_c = \frac{1}{i}\sigma_1(D)(dx)(b_H u) \cdot \delta(x) \in \mathcal{C}^{-\infty}(M; V_2). \tag{8.39}$$

Since the same result is true for M_-, with an obvious change of sign, D defines a linear operator

$$\begin{aligned} D : \big\{u \in L^1(M; V_1); u_\pm = u|M_\pm \in \mathcal{C}^\infty(M_\pm; V_1),\ Du_\pm = 0 \text{ in } M_\pm^\circ\big\} \longrightarrow \\ \frac{1}{i}\sigma(D)(dx)(b_H u_+ - b_H u_-) \cdot \delta(x) \in \mathcal{C}^\infty(H; V_2) \cdot \delta(x). \end{aligned} \tag{8.40}$$

To define the Calderòn projection we shall use the 'inverse' of this result.

PROPOSITION 8.7. *If $D \in \mathrm{Diff}^1(M; V_1, V_2)$ is elliptic and satisfies (8.35) then (8.40) is an isomorphism, with inverse I_D, and*

$$\Pi_C v = b_H \left(I_D \frac{1}{i}\sigma(D)(dx) v \cdot \delta(x) \right)_+,\ v \in \mathcal{C}^\infty(H; V_1), \tag{8.41}$$

satisfies the conditions of Theorem 8.1.

PROOF. Observe that the map (8.40) is injective, since its null space consists of solutions of $Du = 0$ globally on M; such a solution must be smooth by elliptic regularity and hence must vanish by the assumed invertibility of D. Thus the main task is to show that D in (8.40) is surjective.

Since D is elliptic and, by assumption, an isomorphism on $\mathcal{C}^\infty$ sections over M it is also an isomorphism on distributional sections. Thus the inverse of (8.40) must be given by D^{-1}. To prove the surjectivity it is enough to show that

$$D^{-1}(w \cdot \delta(x))|M_\pm \in \mathcal{C}^\infty(M_\pm; V_1)\ \forall\ w \in \mathcal{C}^\infty(H; V_2). \tag{8.42}$$

There can be no singular terms supported on H since $w \cdot \delta(x) \in H^{-1}(M; V_2)$ implies that $u = D^{-1}(w \cdot \delta(x)) \in L^2(M; V_1)$.

Now, recalling that $D^{-1} \in \Psi^{-1}(M; V_2, V_1)$, certainly u is $\mathcal{C}^\infty$ away from H. At any point of H outside the support of w, u is also smooth. Since we may decompose w using a partition of unity, it suffices to suppose that w has support in a small coordinate patch, over which both V_1 and V_2 are trivial and to show that u is smooth 'up to H from both sides' in the local coordinate patch. Discarding smoothing terms from D^{-1} we may therefore replace D^{-1} by any local parametrix Q for D and work in local coordinates and with components:
(8.43)

$$Q_{ij}(w_j(y) \cdot \delta(x)) = (2\pi)^{-n} \int e^{i(x-x')\xi + i(y-y')\cdot\eta} q_{ij}(x, y, \xi, \eta) w(y') \delta(x') dx' dy' d\xi d\eta.$$

For a general pseudodifferential operator, even of order -1, the result we are seeking is not true. We must use special properties of the symbol of Q, that is D^{-1}.

8.7. Rational symbols

LEMMA 8.5. *The left-reduced symbol of any local parametrix for a generalized Dirac operator has an expansion of the form*
(8.44)

$$q_{ij}(z, \zeta) = \sum_{l=1}^{\infty} g(z, \zeta)^{-2l+1} p_{ij,l}(z, \zeta) \text{ with } p_{ij,l} \text{ a polynomial of degree } 3l - 2 \text{ in } \zeta;$$

here $g(z, \zeta)$ is the metric in local coordinates; each of the terms in (8.44) *is therefore a symbol of order $-l$.*

PROOF. This follows by an inductive arument, of a now familiar type. First, the assumption that D is a generalized Dirac operator means that its symbol $\sigma_1(D)(z, \zeta)$ has inverse $g(z, \zeta)^{-1} \sigma_1(D)^*(z, \zeta)$; this is the princiapl symbol of Q. Using Leibniz' formula one concludes that for any polynomial r_l of degree j

$$\partial_{\zeta_i} \left(g(z, \zeta)^{-2l+1} r_j(z, \zeta) \right) = g(z, \zeta)^{-2l} r'_{j+1}(z, \zeta)$$

where r_{j+1} has degree (at most) $j + 1$. Using this result repeatedly, and proceeding by induction, we may suppose that $q = q'_k + q''_{k+1}$ where q'_k has an expansion up to order k, and so may be taken to be such a sum, and q''_{k+1} is of order at most $-k - 1$. The composition formula for left-reduced symbols then shows that

$$\sigma_1(D) q''_{k+1} \equiv g^{-2k} q_{k+1} \mod S^{-k-1}$$

where q_{k+1} is a polynomial of degree at most $3k$. Inverting $\sigma_1(D)(\zeta)$ as at the initial step then shows that q''_{k+1} is of the desired form, $g^{-2k-1} r_{k+1}$ with r_{k+1} of degree $3k + 1 = 3(k + 1) - 2$, modulo terms of lower order. This completes the proof of the lemma. □

With this form for the symbol of Q we proceed to the proof of Proposition 8.7. That is, we consider (8.43). Since we only need to consider each term, we shall drop the indicies. A term of low order in the amplitude q_N gives an operator with kernel in $\mathcal{C}^{N-d}$. Such a kernel gives an operator

$$\mathcal{C}^\infty(H; V_2) \longrightarrow \mathcal{C}^{N-d}(M; V_1)$$

with kernel in $\mathcal{C}^{N-d}$. The result we want will therefore follow if we show that each term in the expansion of the symbol q gives an operator as in (8.42).

To be more precise, we can assume that the amplitude q is of the form

$$q = (1 - \phi) g^{-2l} q'$$

where q' is a polynomial of degree $3l-2$ and $\phi = \phi(\xi,\eta)$ is a function of compact support which is identically one near the origin. The cutoff function is to remove the singularity at $\zeta = (\xi,\eta) = 0$. Using continuity in the symbol topology the integrals in x' and y' can be carried out. By assumption $w \in \mathcal{C}_c^\infty(\mathbb{R}^{n-1})$, so the resulting integral is absolutely convergent in η. If $l > 1$ it is absolutely convergent in ξ as well, so becomes

$$Q(w(y)\cdot\delta(x)) = (2\pi)^{-n}\int e^{ix\xi+iy\cdot\eta}q(x,y,\xi,\eta)\hat{w}(\eta)d\xi d\eta.$$

In $|\xi| > 1$ the amplitude is a rational function of ξ, decaying quadratically as $\xi \to \infty$. If we assume that $x > 0$ then the exponential factor is bounded in the half plane $\Im\xi \geq 0$. This means that the limit as $R \to \infty$ over the integral in $\Im\xi \geq 0$ over the semicircle $|\xi| = R$ tends to zero, and does so with uniform rapid decrease in η. Cauchy's theorem shows that, for $R > 1$ the real integral in ξ can be replaced by the contour integral over $\gamma(R)$, which is, for$R >> |\eta|$ given by the real interval $[-R, R]$ together with the semicircle of radius R in the upper half plane. If $|\eta| > 1$ the integrand is meromorphic in the upper half plane with a possible pole at the singular point $g(x,y,\xi,\eta) = 0$; this is at the point $\xi = ir^{\frac{1}{2}}(x,y,\eta)$ where $r(x,y,\eta)$ is a positive-definite quadratic form in η. Again applying Cauchy's theorem

$$Q(w(y)\delta(x) = (2\pi)^{-n+1}i\int e^{xr^{\frac{1}{2}}(x,y,\eta)+iy\cdot\eta}q'(x,y,\eta)\hat{w}(\eta)d\eta$$

where q' is a symbol of order $-k+1$ in η.

The product $e^{xr^{\frac{1}{2}}(x,y,\eta)}q'(x,y,\eta)$ is uniformly a symbol of order $-k+1$ in $x > 1$, with x derivatives of order p being uniformly symbols of order $-k+1+p$. It follows from the properties of pseudodifferential operators that $Q(w\cdot\delta(x))$ is a smooth function in $x > 0$ with all derivatives locally uniformly bounded as $x \downarrow 0$.

8.8. Proofs of Proposition 8.7 and Theorem 8.1

This completes the proof of (8.42), since a similar argument applies in $x < 0$, with contour deformation into the lower half plane. Thus we have shown that (8.40) is an isomorphism which is the first half of the statement of Proposition (8.7). Furthermore we see that the limiting value from above is a pseudodifferential operator on H :

$$Q_0 w = \lim_{x\downarrow 0} D^{-1}(w\cdot\delta(x)),\ Q_0 \in \Psi^0(H; V_2, V_1). \tag{8.45}$$

This in turn implies that Π_C, defined by (8.41) is an element of $\Psi^0(H; V_1)$, since it is $Q_0 \circ \frac{1}{i}\sigma(D)(dx)$.

Next we check that Π_C is a projection, i.e.
that $\Pi_C^2 = \Pi_C$. If $w = \Pi_C v$, $v \in \mathcal{C}^\infty(H; V_1)$, then $w = b_H u$, $u = I_D\frac{1}{i}\sigma(D)(dx)v|_{M_+}$, so $u \in \mathcal{C}^\infty(M_+; V_1)$ satisfies $Du = 0$ in M_+°. In particular, by (8.39), $Pu_c = \frac{1}{i}\sigma_1(D)(dx)w\cdot\delta(x)$, which means that $w = \Pi_C w$ so $\Pi_C^2 = \Pi_C$. This also shows that the range of Π_C is precisely the range of b_H as stated in (8.37). The same argument shows that this choice of the projection gives (8.38). $\square$

8.9. Inverses

Still for the case of a generalized Dirac operator on a compact manifold with dividing hypersurface, consider what we have shown. The operator D defines a map in (8.39) with inverse

$$I_D : \{v \in \mathcal{C}^\infty(H; V_1); \Pi_C v = v\} \longrightarrow \{u \in \mathcal{C}^\infty(M_+; V_1); Du = 0 \text{ in } M_+\}. \tag{8.46}$$

This operator is the 'Poisson' operator for the canonical boundary condition given by the Calderòn operator, that is $u = I_D v$ is the unique solution of

$$Du = 0 \text{ in } M_+,\ u \in \mathcal{C}^\infty(M_+; V_1),\ \Pi_C b_H u = v. \tag{8.47}$$

We could discuss the regularity properties of I_D but we shall postpone this until after we have treated the 'one-sided' case of a genuine boundary problem.

As well as I_D we have a natural right inverse for the operator D as a map from $\mathcal{C}^\infty(M_+; V_1)$ to $\mathcal{C}^\infty(M_-; V_2)$. Namely

Lemma 8.6. *If $f \in \mathcal{C}^\infty(M_+; V_2)$ then $u = D^{-1}(f_c)|_{M_+} \in \mathcal{C}^\infty(M_+; V_1)$ and the map $R_D : f \longmapsto u$ is a right inverse for D, i.e.*
$D \circ R_D = \mathrm{Id}$.

Proof. Certainly $D(D^{-1}(f_c) = f_c$, so $u = D^{-1}(f_c)|_{M_+} \in \mathcal{C}^{-\infty}(M_+; V_1)$ satisfies $Du = f$ in the sense of extendible distributions. Since $f \in \mathcal{C}^\infty(M_+; V_2)$ we can solve the problem $Du \equiv f$ in the sense of Taylor series at H, with the constant term freely prescibable. Using Borel's lemma, let $u' \in \mathcal{C}^\infty(M_+; V_1)$ have the appropriate Taylor series, with $b_H u' = 0$.. Then $D(u'_c) - f_c = g \in \dot{\mathcal{C}}^\infty(M +_; V_2)$. Thus $u'' = D^{-1} g \in \mathcal{C}^\infty(M; V_1)$. Since $D(u' - u'') = f_c$, the uniqueness of solutions implies that $u = (u' - u'')|_{M_+} \in \mathcal{C}^\infty(M_+; V_1)$. □

Of course R_D cannot be a two-sided inverse to D since it has a large null space, described by I_D.

Problem 8.2. Show that, for D as in Theorem 8.1 if $f \in \mathcal{C}^\infty(M_+; V_2)$ and $v \in \mathcal{C}^\infty(H; V_1)$ there exists a unique $u \in \mathcal{C}^\infty(M_+; V_2)$ such that $Du = f$ in $\mathcal{C}^\infty(M_+; V_2)$ and $b_H u = \Pi_C v$.

8.10. Smoothing operators

The properties of smoothing operators on a compact manifold with boundary are essentially the same as in the boundaryless case. Rather than simply point to the earlier discussion we briefly repeat it here, but in an abstract setting.

Let $\mathcal{H}$ be a separable Hilbert space. In the present case this would be $L^2(X)$ or $L^2(X; E)$ for some vector bundle over X, or some space $H^m(X; E)$ of Sobolev sections. Let $\mathcal{B} = \mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$ and $\mathcal{K} = \mathcal{K}(\mathcal{H})$ the ideal of compact operators. Where necessary the norm on $\mathcal{B}$ will be written $\| \ \|_{\mathcal{B}}$; $\mathcal{K}$ is a closed subspace of $\mathcal{B}$ which is the closure of the ideal $\mathcal{F} = \mathcal{F}(\mathcal{H})$ of finite rank bounded operators.

We will consider a subspace $\mathcal{J} = \mathcal{J}(\mathcal{H}) \subset \mathcal{B}$ with a stronger topology. Thus we suppose that $\mathcal{J}$ is a Fréchet algebra. That is, it is a Fréchet space with countably many norms $\| \ \|_k$ such that for each k there exists k' and C_k with

$$\|AB\|_k \le C_k \|A\|_{k'} \|B\|_{k'} \quad \forall\ A, B \in \mathcal{J}. \tag{8.48}$$

In particular of course we are supposing that $\mathcal{J}$ is a subalgebra (but *not* an ideal) in $\mathcal{B}$. To make it a topological *-subalgebra we suppose that

$$\|A\|_{\mathcal{B}} \le C \|A\|_k \quad \forall\ A \in \mathcal{J},\ * : \mathcal{J} \longrightarrow \mathcal{J}. \tag{8.49}$$

In fact we may suppose that $k = 0$ by renumbering the norms. The third condition we impose on $\mathcal{J}$ implies that it is a subalgebra of $\mathcal{K}$, namely we insist that

$$\mathcal{F} \cap \mathcal{J} \text{ is dense in } \mathcal{J}, \tag{8.50}$$

in the Fréchet topology. Finally, we demand, in place of the ideal property, that $\mathcal{J}$ be a bi-ideal in $\mathcal{B}$ (also called a 'corner') that is,

$$\begin{gathered} A_1, A_2 \in \mathcal{J}, B \in \mathcal{B} \Longrightarrow A_1 B A_2 \in \mathcal{J}, \\ \forall\ k\ \exists\ k' \text{ such that } \|A_1 B A_2\|_k \le C\|A_1\|_{k'}\|B\|_{\mathcal{B}}\|A_2\|_{k'}. \end{gathered} \tag{8.51}$$

Proposition 8.8. *The space of operators with smooth kernels acting on sections of a vector bundle over a compact manifold satisfies* (8.48)–(8.52) *with* $\mathcal{H} = H^m(X; E)$ *for any vector bundle* E.

Proof. The smoothing operators on sections of a bundle E can be written as integral operators

$$Au(x) = \int_E A(x, y)u(y), A(x, y) \in \mathcal{C}^\infty(X^2; \operatorname{Hom}(E) \otimes \Omega_R). \tag{8.52}$$

Thus $\mathcal{J} = \mathcal{C}^\infty(X^2; \operatorname{Hom}(E) \otimes \Omega_R)$ and we make this identification topological. The norms are the C^k norms. If $P_1, \dots, p_{N^{(m)}}$ is a basis, on $\mathcal{C}^\infty(X^2)$, for the differential operators of order m on $\operatorname{Hom}(E) \otimes \Omega_L$ then we may take

$$\|A\|_m = \sup_j \|P_j A\|_{L^\infty} \tag{8.53}$$

for some inner products on the bundles. In fact $\operatorname{Hom}(E) = \pi_L^* E \otimes \pi_R^* E^*$ from it which follows easily that this is a basis $P_j = P_{j,k} \otimes P_{j,R}$ decomposing as products. From this (8.48) follows easily since

$$\|AB\|_m = \sup_j \|(P_{jL}A) \cdot (P_{j,R}B)\|_\infty \|AB\|_{L^\infty} \le C\|A\|_{L^\infty}\|B\|_{L^\infty} \tag{8.54}$$

by the compactnes of X. From this (8.53) follows with $k = 0$.

The density (8.50) is the density of the finite tensor product $\mathcal{C}^\infty(X; E) \otimes \mathcal{C}^\infty(X; E^* \otimes \Omega_L)$ in $\mathcal{C}^\infty(X^2; \operatorname{Hom}(E) \otimes \Omega_L)$. This follows from the boundaryless case by doubling (or directly). Similarly the bi-ideal condition (8.52) can be seen from the regularity of the kernel. A more satisfying argument using distribution theory follows from the next result.

□

Proposition 8.9. *An operator* $A : \dot{\mathcal{C}}^\infty(X; E) \to \mathcal{C}^{-\infty}(X; F)$ *is a smoothing operator if and only if it extends by continuity to* $\dot{\mathcal{C}}^{-\infty}(X; E)$ *and then has range in* $\mathcal{C}^\infty(X; F) \hookrightarrow \mathcal{C}^{-\infty}(X; F)$.

Proof. If A has the stated mapping property then compose with a Seeley extension operator, then $EA = \tilde{A}$ is a continuous linear map

$$\tilde{A} : \dot{\mathcal{C}}^{-\infty}(X; E) \to \mathcal{C}^\infty(\tilde{X}; \tilde{F}),$$

for an extension of F to $\tilde{F}$ over the double $\tilde{X}$. Localizing in the domain to trivialize E and testing with a moving delta function we recover the kernel of $\tilde{A}$ as

$$\tilde{A}(x, y) = \tilde{A} \cdot \delta_y \in \mathcal{C}^\infty(\tilde{X}; \tilde{F}).$$

Thus it follows that $\tilde{A} \in \mathcal{C}^\infty(\tilde{X} \times X; \operatorname{Hom}(E, \tilde{F}) \otimes \Omega_R)$. The converse is more obvious.

□

Returning to the general case of a bi-ideal as in (8.48)–(8.52) we may consider the invertibility of $\mathrm{Id}+A$, $A \in \mathcal{J}$.

PROPOSITION 8.10. *If* $A \in \mathcal{J}$*, satisfying* (8.48)–(8.52)*, then* $\mathrm{Id}+A$ *has a generalized inverse of the form* $\mathrm{Id}+B$, $B \in \mathcal{J}$, *with*

$$AB = \mathrm{Id} - \pi_R, BA = \mathrm{Id} - \pi_L \in \mathcal{J} \cap \mathcal{F}$$

both finite rank self-adjoint projections.

PROOF. Suppose first that $A \in \mathcal{J}$ and $\|A\|_{\mathcal{B}} < 1$. Then $\mathrm{Id}+A$ is invertible in $\mathcal{B}$ with inverse $Id + B \in \mathcal{B}$,

$$B = \sum_{j \geq 1} (-1)^j A^j. \tag{8.55}$$

Not only does this Neumann series converge in $\mathcal{B}$ but also in $\mathcal{J}$ since for each k

$$\|A^j\|_k \leq C_k \|A\|_{k'} \|A^{j-2}\|_{\mathcal{B}} \|A\|_{k'} \leq C'_k \|A\|_{\mathcal{B}}^{j-2}, j \geq 2. \tag{8.56}$$

Thus $B \in \mathcal{J}$, since by assumption $\mathcal{J}$ is complete (being a Fréchet space). In this case $\mathrm{Id}+B \in \mathcal{B}$ is the unique two-sided inverse.

For general $A \in \mathcal{J}$ we use the assumed approximability in (8.50). Then $A = A' + A''$ when $A' \in \mathcal{F} \cap \mathcal{J}$ and $\|A''\|_{\mathcal{B}} \leq C\|A''\|_k < 1$ by appropriate choice. It follows that $\mathrm{Id}+B'' = (\mathrm{Id}+A'')^{-1}$ is the inverse for $\mathrm{Id}+A''$ and hence a parameterix for $\mathrm{Id}+A$:

$$\begin{aligned} (\mathrm{Id}+B'')(\mathrm{Id}+A) &= \mathrm{Id}+A' + B''A' \\ (\mathrm{Id}+A)(\mathrm{Id}+B'') &= \mathrm{Id}+A' + A'B'' \end{aligned} \tag{8.57}$$

Unfinished. Lemma on subprojections.

with both 'error' terms in $\mathcal{F} \cap \mathcal{J}$. □

8.11. Left and right parametrices

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces and $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a bounded linear operator between them. Let $\mathcal{J}_1 \subset \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{J}_2 \subset \mathcal{B}(\mathcal{H}_2)$ be bi-ideals as in the previous section. A left parametrix for A, modulo $\mathcal{J}_1$, is a bounded linear map $B_L : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ such that

$$B_L \circ A = \mathrm{Id} + J_L,\ J_L \in \mathcal{J}_1. \tag{8.58}$$

Similarly a right parametrix for A, modulo $\mathcal{J}_2$ is a bounded linear map $B_R : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ such that

$$A \circ B_R = \mathrm{Id} + J_R,\ J_R \in \mathcal{J}_2. \tag{8.59}$$

PROPOSITION 8.11. *If a bounded linear operator* $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ *has a left parametrix* B_L *modulo a bi-ideal* $\mathcal{J}_1$*, satisfying* (8.48)–(8.52)*, then* A *has closed range, null space of finite dimension and there is a generalized left inverse, differing from the original left parametrix by a term in* $\mathcal{J}_1$*, such that*

$$B_L \circ A = \mathrm{Id} - \pi_L,\ \pi_L \in \mathcal{J}_1 \cap \mathcal{F}, \tag{8.60}$$

with π_L *the self-adjoint projection onto the null space of* A*.*

Proof. Applying Proposition 8.10, $\mathrm{Id}+J_L$ has a generalized inverse $\mathrm{Id}+J$, $J \in \mathcal{J}_1$, such that $(\mathrm{Id}+J)(\mathrm{Id}+J_L) = (\mathrm{Id}-\pi'_L)$, $\pi'_L \in \mathcal{J}_1 \cap \mathcal{F}$. Replacing B_L by $\tilde{B}_L = (\mathrm{Id}+J)B_L$ gives a new left parametrix with error term $\pi'_L \in \mathcal{J}_1 \cap \mathcal{F}$. The null space of A is contained in the null space of $B'_L \circ A$ and hence in the range of F_L; thus it is finite dimensional. Furthermore the self-dajoint projection π_L onto the null space is a subprojection of π'_L, so is also an element of $\mathcal{J}_1 \cap \mathcal{F}$. The range of A is closed since it has finite codimension in $\mathrm{Ran}(A(\mathrm{Id}-\pi_L))$ and if $f_n \in \mathrm{Ran}(A(\mathrm{Id}-\pi_L)) = Au_n$, $u_n = (\mathrm{Id}-\pi_L)u_n$, converges to $f \in \mathcal{H}_2$, then $u_n = B_L f_n$ converges to $u \in \mathcal{H}_1$ with $A(\mathrm{Id}-\pi_L)u = f$. □

Proposition 8.12. *If a bounded linear operator $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ has a right parametrix B_R modulo a bi-ideal $\mathcal{J}_2$, satisfying* (8.48)–(8.52)*, then it has closed range of finite codimension and there is a generalized right inverse, differing from the original right parametrix by a term in $\mathcal{J}_2$, such that*

$$A \circ B_R = \mathrm{Id}-\pi_R,\ \pi_R \in \mathcal{J}_2 \cap \mathcal{F}, \tag{8.61}$$

with $\mathrm{Id}-\pi_R$ *the self-adjoint projection onto the range space of A.*

Proof. The operator $\mathrm{Id}+J_R$ has, by Proposition 8.10, a generalized inverse $\mathrm{Id}+J$ with $J \in \mathcal{J}_1$. Thus $B'_R = B_R \circ (\mathrm{Id}+J)$ is a right parametrix with error term $\mathrm{Id}-\pi'_R$, $\pi'_R \in \mathcal{J}_1 \cap \mathcal{F}$ being a self-adjoint projection. Thus the range of A contains the range of $\mathrm{Id}-\pi'_R$ and is therefore closed with a finite-dimensional complement. Furthemore the self-adjoint projection onto the range of A is of the form $\mathrm{Id}-\pi_R$ where π_R is a subprojection of π'_R, so also in $\mathcal{J}_1 \cap \mathcal{F}$. □

The two cases, of an operator with a right or a left parametrix are sometimes combined in the term 'semi-Fredholm.' Thus an operator $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is semi-Fredholm if it has closed range and either the null space or the orthocomplement to the range is finite dimensional. The existence of a right or left parametrix, modulo the ideal of compact operators, is a necessary and sufficient condition for an operator to be semi-Fredholm.

8.12. Right inverse

In treating the 'general' case of an elliptic operator on compact manifold with boundary we shall start by constructing an analogue of the right inverse in Lemma 8.6. So now we assume that $D \in \mathrm{Diff}^1(X; V_1, V_2)$ is an operator of Dirac type on a compact manifold with boundary.

To construct a right inverse for D we follow the procedure in the boundaryless case. That is we use the construction of a pseudodifferential parametrix. In order to make this possible we need to extend M and D 'across the boundary.' This is certainly possible for X, since we may double it to a compact manifold without boundary, $2X$. Then there is not obstruction to extending D 'a little way' across the boundary. We shall denote by M an open extension of X (of the same dimension) so $X \subset M$ is a compact subset and by $\tilde{D}$ an extension of Dirac type to M.

The extension of D to $\tilde{D}$, being elliptic, has a parametrix $\tilde{Q}$. Consider the map

$$\tilde{Q}' : L^2(X; V_2) \longrightarrow H^1(X; V_1),\ \tilde{Q}' f = \tilde{Q} f_c | X \tag{8.62}$$

where f_c is the extension of f to be zero outside X. Then $\tilde{Q}'$ is a right parametrix, $D\tilde{Q}' = \mathrm{Id}+E$ where E is an operator on $L^2(X; V_2)$ with smooth kernel on X^2.

Following Proposition 8.12, D has a generalized right inverse $\tilde{Q}'' = \tilde{Q}'(\mathrm{Id} + E')$ up to finite rank smoothing and

$$D : H^1(X; V_1) \longleftrightarrow L^2(X; V_2) \tag{8.63}$$

has closed range with a finite dimensional complement in $\mathcal{C}^\infty(X; V_2)$.

PROPOSITION 8.13. *The map* (8.63) *maps* $\mathcal{C}^\infty(X; V_2)$ *to* $\mathcal{C}^\infty(X; V_1)$, *it is surjective if and only if the only solution of* $D^*u = 0$, $u \in \dot{\mathcal{C}}^\infty(X; V_2)$ *is the trivial solution.*

PROOF. The regularity statement, that $Q'\mathcal{C}^\infty(X; V) \subset \mathcal{C}^\infty(X; V_1)$ follows as in the proof of Lemma 8.6. Thus Q' maps $\mathcal{C}^\infty(X; V_1)$ to $\mathcal{C}^\infty(X; V_2)$ if and only if any paramatrix $\tilde{Q}'$ does so. Given $f \in \mathcal{C}^\infty(X; V_2)$ we may solve $Du' \equiv f$ in Taylor series at the boundary, with $u' \in \mathcal{C}^\infty(X; V_1)$ satisfying $b_H u' = 0$. Then $D(u')_c - f \in \dot{\mathcal{C}}^\infty(X; V_2)$ so it follows that $Q'(f_c)|_X \in \mathcal{C}^\infty(X; V_1)$.

Certainly any solution of $D^*u = 0$ with $u \in \dot{\mathcal{C}}^\infty(X; V_2)$ is orthogonal to the range of (8.63) so the condition is necessary. So, suppose that (8.63) is not surjective. Let $f \in L^2(X; V_2)$ be in the orthocomplement to the range. Then Green's formula gives the pairing with any smooth section

$$(Dv, f)_X = (D\tilde{v}, f_c)_{\tilde{X}} = (\tilde{v}, D^* f_c)_{\tilde{X}} = 0.$$

This means that $D^* f_c = 0$ in $\tilde{X}$, that is as a supported distribution. Thus, $f \in \dot{\mathcal{C}}^\infty(X; V_2)$ satisfies $D^* f = 0$. □

As noted above we will proceed under the assumption that $D^* f$ has no such non-trivial solutions in $\dot{\mathcal{C}}^\infty(X; V_2)$. This condition is discussed in the next section.

THEOREM 8.2. *If unique continuation holds for* D^* *then* D *has a right inverse*

$$Q : \mathcal{C}^\infty(X : V_2) \longrightarrow \mathcal{C}^\infty(X; V_1), \ DQ = \mathrm{Id} \tag{8.64}$$

where $Q = \tilde{Q}' + E$, $\tilde{Q}' f = \tilde{Q} f|X$ *where* $\tilde{Q}$ *is a parametrix for an extension of* D *across the boundary and* E *is a smoothing operator on* X.

PROOF. As just noted, unique continuation for D^* implies that D in (8.63) is surjective. Since the parametrix maps $\mathcal{C}^\infty(X; V_2)$ to $\mathcal{C}^\infty(X; V_1)$, D must be surjective as a map from $\mathcal{C}^\infty(X; V_1)$ to $\mathcal{C}^\infty(X; V_2)$. The parametrix modulo finite rank operators can therefore be corrected to a right inverse for D by the addition of a smoothing operator of finite rank. □

8.13. Boundary map

The map b from $\mathcal{C}^\infty(X; E)$ to $\mathcal{C}^\infty(\partial X; E)$ is well defined, and hence is well defined on the space of smooth solutions of D. We wish to show that it has closed range. To do so we shall extend the defintion to the space of square-integrable solutions. For any $s \in \mathbb{R}$ set

$$\mathcal{N}^s(D) = \{u \in H^s(X; E); Du = 0\}. \tag{8.65}$$

Of course the equation $Du = 0$ is to hold in the sense of extendible distributions, which just means in the interior of X. Thus $\mathcal{N}^\infty(D)$ is the space of solutions of D smooth up to the boundary.

LEMMA 8.7. *If* $u \in \mathcal{N}^0(D)$ *then*

$$\dot{D}u_c = v \cdot \delta(x),\ v \in H^{-\frac{1}{2}}(\partial X; E) \tag{8.66}$$

defines an injective bounded map $\tilde{b} : \mathcal{N}^0(D) \longrightarrow H^{-\frac{1}{2}}(\partial X; E)$ *by* $\tilde{b}(u) = i\sigma(D)(dx)v$ *which is an extension of* $b : \mathcal{N}^\infty(D) \longrightarrow \mathcal{C}^\infty(\partial X; E)$ *defined by restriction to the boundary.*

PROOF. Certainly $\dot{D}u_c \subset \dot{\mathcal{C}}^\infty_{\partial X}(X; E)$ has support in the boundary, so is a sum of products in any product decomposition of X near ∂X,

$$D(u_c) = \sum_j v_j \cdot \delta^{(j)}(x).$$

Since D is a first order operator and $u_c \in L^2(\tilde{X}; E)$, for any local extension, $\dot{D}u_c \in \dot{H}^{-1}(X; E)$. Localizing so that E is trivial and the localized v_j have compact supports this means that

$$(1 + |\eta|^2 + |\xi|^2)^{-\frac{1}{2}} \widehat{v_j}(\eta)\xi^j \in L^2(\mathbb{R}^n). \tag{8.67}$$

If $v_j \neq 0$ for some $j > 0$ this is not true even in some region $|\eta| < C$. Thus $v_j \equiv 0$ for $j > 0$ and (8.66) must hold. Furthermore integration in ξ gives

$$\begin{gathered}\int_{\mathbb{R}} (1 + |\eta|^2 + |\xi|^2)^{-1} d\xi = c(1 + |\eta|^2)^{-\frac{1}{2}},\ c > 0, \quad \text{so} \\ \int_{\mathbb{R}^{n-1}} (1 + |\eta|^2)^{-\frac{1}{2}} |\hat{v}(\eta)|^2 d\eta < 0.\end{gathered} \tag{8.68}$$

Thus $v \in H^{-\frac{1}{2}}(\partial X; E)$ and $\tilde{b}$ is well defined. The jumps formula shows it to be an extension of b. The injectivity of $\tilde{b}$ follows from the assumed uniqueness of solutions to $\dot{D}u = 0$ in X. □

Notice that (8.68) is actually reversible. That is if $v \in H^{-\frac{1}{2}}(\partial X; E)$ then $v \cdot \delta(x) \in H^{-1}(X; E)$. This is the basis of the construction of a left parametrix for $\tilde{b}$, which then shows its range to be closed.

LEMMA 8.8. *The boundary map* $\tilde{b}$ *in Lemma 8.7 has a continuous left parametrix* $\widetilde{I_D} : H^{-\frac{1}{2}}(\partial X; E) \longrightarrow \mathcal{N}^0(D)$, $I_D \circ \tilde{b} = \mathrm{Id} + G$, *where* G *has smooth kernel on* $X \times \partial X$, *and the range of* $\tilde{b}$ *is therefore a closed subspace of* $H^{-\frac{1}{2}}(\partial X; E)$.

PROOF. The parametrix $\widetilde{I_D}$ is given directly by the parametrix $\tilde{Q}$ for $\tilde{D}$, and extension to $\tilde{X}$. Applying $\tilde{Q}$ to (8.66) gives

$$u = \widetilde{I_D}v + Ru,\ \widetilde{I_D} = R_X \circ \tilde{Q} \circ \frac{1}{i}\sigma(D)(dx) \tag{8.69}$$

with R having smooth kernel. Since $\widetilde{I_D}$ is bounded from $H^{-\frac{1}{2}}(\partial X; E)$ to $L^2(X; E)$ and R is smoothing it follows from Proposition 8.11 that the range of $\tilde{b}$ is closed. □

8.14. Calderòn projector

Having shown that the range of $\tilde{b}$ in Lemma 8.7 is closed in $H^{-\frac{1}{2}}(\partial X; E)$ we now deduce that there is a pseudodifferential projection onto it. The discussion above of the boundary values of the $\tilde{Q}(w \cdot \delta(x))$ is local, and so applies just as well in the present more general case. Since this is just the definition of the map $\widetilde{I_D}$ in Lemma 8.8, we conclude directly that

$$Pv = \lim_{X^\circ} \widetilde{I_D} v,\ v \in \mathcal{C}^\infty(\partial X; E) \tag{8.70}$$

defines $P \in \Psi^0(\partial X; E)$.

Lemma 8.9. *If P is defined by* (8.70) *then $P^2 - P \in \Psi^{-\infty}(\partial X; E)$ and there exist A, $B \in \Psi^{-\infty}(\partial X; E)$ such that $P - \mathrm{Id} = A$ on $\mathrm{Ran}(\tilde{b})$ and $\mathrm{Ran}(P + B) \subset \mathrm{Ran}(\tilde{b})$.*

Proof needs clarification.

Proof. That $P^2 - P \in \Psi^{-\infty}(\partial X; E)$ follows, as above, from the fact that $\tilde{Q}$ is a two-sided parametrix on distributions supported in X. Similarly we may use the right inverse of D to construct B. If $v \in H^{-\frac{1}{2}}(\partial X; E)$ then by construction,

$$D\widetilde{I_D} v = R' v$$

where R' has a smooth kernel on $X \times \partial X$. Applying the right inverse Q it follows that $u' = \widetilde{I_D} v - (Q \circ R') v \in \mathcal{N}^0(D)$, where $Q \circ R'$ also has smooth kernel on $X \times \partial X$. Thus $\tilde{b}(u') = (P + B)v \in \mathrm{Ran}(\tilde{b})$ where B has kernel arising from the restriction of the kernel of $A \circ R'$ to $\partial X \times \partial X$, so $B \in \Psi^{-\infty}(\partial X; E)$. □

Now we may apply Proposition 6.11 with $F = \mathrm{Ran}(\tilde{b})$ and $s = -\frac{1}{2}$ to show the existence of a Calderòn projector.

Proposition 8.14. *If D is a generalized Dirac operator on X then there is an element $\Pi_C \in \Psi^0(\partial X; E)$ such that $\Pi_C^2 = \Pi_C$, $\mathrm{Ran}(\Pi_C) = \mathrm{Ran}(\tilde{b})$ on $H^{-\frac{1}{2}}(\partial X; E)$, $\Pi_C - P \in \Psi^{-\infty}(\partial X; E)$ where P is defined by* (8.70) *and $\mathrm{Ran}(\Pi_C) = \mathrm{Ran}(b)$ on $\mathcal{C}^\infty(\partial X; E)$.*

Proof. The existence of psuedodifferential projection, Π_C, differing from P by a smoothing operator and with range $\mathrm{Ran}(\tilde{b})$ is a direct consequence of the application of Proposition 6.11. It follows that $\mathrm{Ran}(\tilde{b}) \cap \mathcal{C}^\infty(\partial X; E)$ is dense in $\mathrm{Ran}(\tilde{b})$ in the topology of $H^{-\frac{1}{2}}(\partial X; E)$. Furthermore, if follows that if $v \in \mathrm{Ran}(\tilde{b}) \cap \mathcal{C}^\infty(\partial X; E)$ then $u \in \mathcal{N}^0(D)$ such that $\tilde{b}u = v$ is actually in $\mathcal{C}^\infty(X; E)$, i.e. it is in $\mathcal{N}^\infty(D)$. Thus the range of b is just $\mathrm{Ran}(\tilde{b}) \cap \mathcal{C}^\infty(\partial X; E)$ so $\mathrm{Ran}(b)$ is the range of Π_C acting on $\mathcal{C}^\infty(\partial X; E)$. □

In particular $\tilde{b}$ is just the continuous extension of b from $\mathcal{N}^\infty(D)$ to $\mathcal{N}^0(D)$, of which it is a dense subset. Thus we no longer distinguish between these two maps and set $\tilde{b} = b$.

8.15. Poisson operator

8.16. Unique continuation

8.17. Boundary regularity

8.18. Pseudodifferential boundary conditions

The discussion above shows that for any operator of Dirac type the 'Calderòn realization' of D,

$$D_{\mathcal{C}} : \{u \in H^s(X; E_1); \Pi_{\mathcal{C}} bu = 0\} \longrightarrow H^{s-1}(X; E_2),\ s > \frac{1}{2} \tag{8.71}$$

is an isomorphism.

We may replace the Calderòn projector in (8.71) by a more general projection Π, acting on $\mathcal{C}^\infty(\partial X, V_1)$, and consider the map

$$D_\Pi : \{u \in \mathcal{C}^\infty(X; V_1); \Pi b u = 0\} \longrightarrow \mathcal{C}^\infty(X; V_2). \tag{8.72}$$

In general this map will not be particularly well-behaved. We will be interested in the case that $\Pi \in \Psi^0(\partial X; V_1)$ is a pseudodifferential projection. Then a condition for the map D_Π to be Fredholm can be given purely in terms of the relationship between Π and the (any) Calderòn projector $\Pi_{\mathcal{C}}$.

Theorem 8.3. *If $D \in \operatorname{Diff}^1(X; E_1, E_2)$ is of Dirac type and $Pi \in \Psi^0(\partial X; E_1)$ is a projection then the map*

$$D_\Pi : \{u \in \mathcal{C}^\infty(X; E_1); \Pi(u_{\partial X}) = 0\} \xrightarrow{D} \mathcal{C}^\infty(X; E_2) \tag{8.73}$$

is Fredholm if and only if

$$\Pi \circ \Pi_C : \operatorname{Ran}(\Pi_C) \cap \mathcal{C}^\infty(\partial V_1) \longrightarrow \operatorname{Ran}(\Pi) \cap \mathcal{C}^\infty(\partial E_1) \text{ is Fredholm} \tag{8.74}$$

and then the index of D_Π is equal to the relative index of $\Pi_{\mathcal{C}}$ and Π, that is the index of (8.74).

Below we give a symbolic condition equivalent which implies the Fredholm condition. If enough regularity conditions are imposed on the generalized inverse to (8.71) then this symbolic is also necessary.

Proof. The null space of D_Π is easily analysed. Indeed $Du = 0$ implies that $u \in \mathcal{N}$, so the null space is isomorphic to its image under the boundary map:

$$\{u \in \mathcal{N}; \Pi b u = 0\} \xrightarrow{b} \{v \in \mathcal{C}; \Pi v = 0\}.$$

Since $\mathcal{C}$ is the range of $\Pi_{\mathcal{C}}$ this gives the isomorphism

$$\operatorname{Nul}(D_\Pi) \simeq \operatorname{Nul}(\Pi \circ \Pi_{\mathcal{C}} : \mathcal{C} \longrightarrow \operatorname{Ran}(\Pi)). \tag{8.75}$$

In particular, the null space is finite dimensional if and only if the null space of $\Pi \circ \Pi_{\mathcal{C}}$ is finite dimensional.

Similarly, consider the range of D_Π. We construct a map

$$\tau : \mathcal{C}^\infty(\partial X; V_1) \longrightarrow \mathcal{C}^\infty(X; V_2)/\operatorname{Ran}(D_\Pi). \tag{8.76}$$

Indeed each $v \in \mathcal{C}^\infty(\partial X; V_1)$ is the boundary value of some $u \in \mathcal{C}^\infty(X : V_1)$, let $\tau(v)$ be he class of DU. This is well-defined since any other extension u' is such that $b(u - u') = 0$, so $D(u - u') \in \operatorname{Ran}(D_\Pi)$. Furthermore, τ is surjective, since $D_{\mathcal{C}}$ is surjective. Consider the null space of τ. This certainly contains the null space of Π. Thus consider the quotient map

$$\tilde{\tau} : \operatorname{Ran}(\Pi) \longrightarrow \mathcal{C}^\infty(X : V_2)/\operatorname{Ran}(D_\Pi),$$

which is still surjective. Then $\tilde{\tau}(v) = 0$ if and only if there exists $v' \in \mathcal{C}$ such that $\Pi(v - v') = 0$. That is, $\tilde{\tau}(v) = 0$ if and only if $\Pi(v) = \Pi \circ \Pi_{\mathcal{C}}$. This shows that the finer quotient map

$$\tau' : \operatorname{Ran}(\Pi)/\operatorname{Ran}(\Pi \circ \Pi_{\mathcal{C}}) \longleftrightarrow \mathcal{C}^\infty(X; V_2)/\operatorname{Ran}(D_\Pi) \tag{8.77}$$

is an isomorphism. This shows that the range is closed and of finite codimension if $\Pi \circ \Pi_{\mathcal{C}}$ is Fredholm.

The converse follows by reversing these arguments. □

8.19. Gluing

Returning to the case of a compact manifold without boundary, M, with a dividing hypersurface H we can now give a gluing result for the index.

THEOREM 8.4. *If* $D \in \operatorname{Diff}^1(M; E_1, E_2)$ *is of Dirac type and* $M = M_1 \cap M_2$ *is the union of two manifolds with boundary intersecting in their common boundary* $\partial M_1 \cap \partial M_2 = H$ *then*

$$\operatorname{Ind}(D) = \operatorname{Ind}(\Pi_{1,\mathcal{C}}, \operatorname{Id} - \Pi_{2,\mathcal{C}}) = \operatorname{Ind}(\Pi_{2,\mathcal{C}}, \operatorname{Id} - \Pi_{1,\mathcal{C}}) \tag{8.78}$$

where $\Pi_{i,\mathcal{C}}$, $i = 1, 2$, *are the Calderòn projections for* D *acting over* M_i.

8.20. Local boundary conditions

8.21. Absolute and relative Hodge cohomology

8.22. Transmission condition