CHAPTER 6

Pseudodifferential operators on manifolds

In this chapter the notion of a pseudodifferential on a manifold is discussed. Some preliminary material on manifolds is therefore necessary. However the discussion of the basic properties of differentiable manifolds is kept to a bare minimum. For a more leisurely treatement the reader might well consult XX or YY. Our main aims here are first, to be able to prove the Hodge theorem (given the deRham theorem). Then we describe some global object which are very useful in applications, namely a global quantization map, the structure of complex powers and the zeta function.

6.1. $\mathcal{C}^\infty$ structures

Let X be a paracompact Hausdorff topological space. A $\mathcal{C}^\infty$ structure on X is a subspace

$$\mathcal{F} \subset C^0(X) = \{u : X \longrightarrow \mathbb{R} \ \text{ continuous }\} \tag{6.1}$$

with the following property:

For each $\overline{x} \in X$ there exists elements $f_1, \dots, f_n \in \mathcal{F}$ such that for some open neighbourhood $\Omega \ni \overline{x}$

$$F : \Omega \ni x \longmapsto (f_1(x), \dots, f_n(x)) \in \mathbb{R}^n \tag{6.2}$$

is a homeomorphism onto an open subset of $\mathbb{R}^n$ and *every* $f \in \mathcal{F}$ satisfies

$$f \restriction \Omega = g \circ F \quad \text{ for some } g \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{6.3}$$

The map (6.2) is a coordinate system near $\overline{x}$. Two $\mathcal{C}^\infty$ structures $\mathcal{F}_1$ and $\mathcal{F}_2$ are 'compatible' if $\mathcal{F}_1 \cup \mathcal{F}_2$ is also a $\mathcal{C}^\infty$ structure. Compatibility in this sense is an equivalence relation on $\mathcal{C}^\infty$ structures. It therefore makes sense to say that:

Definition 6.1. *A $\mathcal{C}^\infty$ manifold is a (connected) paracompact Hausdorff topological space with a* maximal *$\mathcal{C}^\infty$ structure.*

The maximal $\mathcal{C}^\infty$ structure is conventionally denoted

$$\mathcal{C}^\infty(X) \subset \mathcal{C}^0(X). \tag{6.4}$$

It is necessarily an algebra. If we let $\mathcal{C}^\infty_c(W) \subset \mathcal{C}^\infty(X)$ denote the subspace of functions which vanish outside a compact subset of W then any local coordinates (6.2) have the property

$$F^* : \mathcal{C}^\infty_c(F(\Omega)) \longleftrightarrow \big\{u \in \mathcal{C}^\infty(X);\ u = 0 \ \text{ on } X\backslash K, K \subset\subset \Omega\big\}. \tag{6.5}$$

Futhermore $\mathcal{C}^\infty(X)$ is *local*:

$$\begin{gathered} u : X \longrightarrow \mathbb{R} \text{ and } \ \forall\, \overline{x} \in X\ \exists\ \Omega_{\overline{x}} \text{ open},\ \Omega_{\overline{x}} \ni \overline{x}, \\ \text{s.t. } u - f_{\overline{x}} = 0 \text{ on } \Omega_{\overline{x}} \text{ for some } f_{\overline{x}} \in \mathcal{C}^\infty(X) \Longrightarrow u \in \mathcal{C}^\infty(X). \end{gathered} \tag{6.6}$$

A map $G : X \longrightarrow Y$ between $\mathcal{C}^\infty$ manifolds X and Y is $\mathcal{C}^\infty$ if

$$G^* : \ \mathcal{C}^\infty(Y) \longrightarrow \mathcal{C}^\infty(X) \tag{6.7}$$

i.e. $G \circ u \in \mathcal{C}^\infty(X)$ for all $u \in \mathcal{C}^\infty(Y)$.

6.2. Form bundles

A vector bundle is a triple $\pi : V \longrightarrow X$ consisting of two manifolds, X and V, and a surjective $\mathcal{C}^\infty$ map π with each

$$V_x = \pi^{-1}(x) \tag{6.8}$$

having a *linear structure* such that

$$\mathcal{F} = \big\{u : V \longrightarrow \mathbb{R}, u \text{ is linear on each } V_x\big\} \tag{6.9}$$

is a $\mathcal{C}^\infty$ structure on V compatible with $\mathcal{C}^\infty(V)$ (i.e. contained in it, since it is maximal).

The basic example is the *cotangent bundle* which we defined before for open sets in $\mathbb{R}^n$. The same definition works here. Namely for each $\overline{x} \in X$ set

$$\begin{gathered} \mathcal{I}_{\overline{x}} = \big\{u \in \mathcal{C}^\infty(X); u(\overline{x}) = 0\big\} \\ \mathcal{I}^2_{\overline{x}} = \big\{u = \sum_{\text{finite}} u_i u_i';\ u_i, u_i' \in \mathcal{I}_{\overline{x}}\big\} \\ T^*_{\overline{x}}X = \mathcal{I}_{\overline{x}}/\mathcal{I}^2_{\overline{x}},\ T^*X = \bigcup_{\overline{x} \in X} T^*_{\overline{x}}X. \end{gathered} \tag{6.10}$$

So $\pi : T^*X \longrightarrow X$ just maps each $T^*_{\overline{x}}X$ to $\overline{x}$. We need to give T^*X a $\mathcal{C}^\infty$ structure so that "it" (meaning $\pi : T^*X \longrightarrow X$) becomes a vector bundle. To do so note that the *differential* of any $f \in \mathcal{C}^\infty(X)$

$$df : X \longrightarrow T^*X \quad df(\overline{x}) = [f - f(\overline{x})] \in T^*_{\overline{x}}X \tag{6.11}$$

is a section ($\pi \circ df = \mathrm{Id}$). Put

$$\mathcal{F} = \big\{u : T^*X \longrightarrow \mathbb{R}; u \circ df : X \longrightarrow \mathbb{R} \text{ is } \mathcal{C}^\infty\ \forall\ f \in \mathcal{C}^\infty(X)\big\}. \tag{6.12}$$

Then $\mathcal{F} = \mathcal{C}^\infty(T^*X)$ is a maximal $\mathcal{C}^\infty$ structure on T^*X and

$$\mathcal{F}_{\text{lin}} = \big\{u : T^*X \longrightarrow \mathbb{R},\ \text{linear on each } T^*_{\overline{x}}X; u \in \mathcal{F}\big\}$$

is therefore compatible with it. Clearly df is $\mathcal{C}^\infty$.

Any (functorial) operation on finite dimensional vector spaces can be easily seen to generate new vectors bundles from old. Thus *duality*, *tensor product*, *exterior powers* all lead to new vector bundles:

$$T_xX = (T^*_xX)^*,\ TX = \bigcup_{x \in X} T_xX \tag{6.13}$$

is the tangent bundle

$$\Lambda^k_x X = \big\{u : \overbrace{T_xX \times \cdots \times T_xX}^{k \text{ factors}} \longrightarrow \mathbb{R}; u \text{ is multilinear and antisymmetric }\big\}$$

leads to the k-form bundle

$$\Lambda^k X = \bigcup_{x \in X} \Lambda^k_x X,\ \Lambda^1 X \simeq T^*X$$

where equivalence means there exists (in this case a *natural*) $\mathcal{C}^\infty$ diffeomorphism mapping fibres to fibres linearly (and in this case projecting to the identity on X).

A similar construction leads to the *density bundles*

$$\Omega^\alpha_x X = \{u : \overbrace{T_x X \wedge \cdots \wedge T_x X}^{n=\dim X \text{ factors}} \longrightarrow \mathbb{R};\ \text{absolutely homogeneous of degree } \alpha\}$$

that is

$$u(tv_1 \wedge \dots v_n) = |t|^\alpha u(v_1 \wedge \cdots \wedge v_n).$$

These are important because of integration. In general if $\pi : V \longrightarrow X$ is a vector bundle then

$$\mathcal{C}^\infty(X;V) = \{u :\ X \longrightarrow V;\ \pi \circ u = \mathrm{Id}\}$$

is the space of sections. It has a natural linear structure. Suppose $W \subset X$ is a coordinate neighbourhood and $u \in \mathcal{C}^\infty(X;\Omega)$, $\Omega = \Omega^1 X$, has compact support in W. Then the coordinate map gives an identification

$$\Omega^*_x X \longleftrightarrow \Omega^*_{F(x)}\mathbb{R}^n \qquad \forall\ \alpha$$

and

$$\int u = \int_{\mathbb{R}^n} g_u(x), \quad u = g_u(x)|dx| \tag{6.14}$$

is defined independent of coordiantes. That is the integral

$$\int : \mathcal{C}^\infty_c(X;\Omega) \longrightarrow \mathbb{R} \tag{6.15}$$

is well-defined.

6.3. Pseudodifferential operators

We will start with a definition of pseudodifferential operators on a (not necessarily compact) manifold which has lots of properties but may be a bit hard to verify in practice.

Definition 6.2. *If X is a $\mathcal{C}^\infty$ manifold and $\mathcal{C}^\infty_c(X) \subset \mathcal{C}^\infty(X)$ is the space of $\mathcal{C}^\infty$ functions of compact support, then, for any $m \in \mathbb{R}$, $\Psi^m(X)$ is the space of linear operators*

$$A : \mathcal{C}^\infty_c(X) \longrightarrow \mathcal{C}^\infty(X) \tag{6.16}$$

with the following properties. First,

$$\begin{gathered}\textit{if } \phi,\ \psi \in \mathcal{C}^\infty(X) \textit{ have disjoint supports then } \exists\ K \in \mathcal{C}^\infty(X^2;\Omega_R) \\ \textit{such that } \forall\ u \in \mathcal{C}^\infty_c(X)\ \phi A \psi u = \int_X K(x,y)u(y),\end{gathered} \tag{6.17}$$

and secondly if $F : W \longrightarrow \mathbb{R}^n$ is a coordinate system in X and $\psi \in \mathcal{C}^\infty_c(X)$ has support in W then

$$\begin{gathered}\exists\ B \in \Psi^m_\infty(\mathbb{R}^n),\ \mathrm{supp}(B) \subset F(W) \times F(W)\ \textit{s.t.} \\ \psi A \psi u \restriction W = F^*(B((F^{-1})^*(\psi u)))\ \forall\ u \in \mathcal{C}^\infty_c(X).\end{gathered}$$

This seems a pretty horrible definition, since it requires us to check *every* coordinate system, at least in principle. In practice the coordinate-invariance we proved earlier (see Proposition 5.4) means that this is *not* necessary and also that there are plenty of examples as we proceed to see.

LEMMA 6.1. *The space* $\Psi^{-\infty}(X) = \bigcap_m \Psi^m(X)$ *contains all the smoothing operators on* X, *those with kernels* $K \in \mathcal{C}^\infty(X^2; \Omega_R)$.

In fact there is equality between $\Psi^{-\infty}(X)$ and the space of smoothing operators but it is easier to see this after a little more thought!

PROOF. Smoothing operators, having smooth kernels, satisfy the first part of the definition and also the second since smoothing operators with compactly supported kernels are pseudodifferential operators on $\mathbb{R}^n$. $\square$

LEMMA 6.2. *If* $G : U \longrightarrow \mathbb{R}^n$ *is a coordinate patch on* X *and* $B \in \Psi^m_\infty(\mathbb{R}^n)$ *has kernel with support* $\operatorname{supp}(B) \Subset G(U) \times G(U)$ *then*

$$Au = G^* B (G^{-1})^* (u\big|_U) \text{ defines } A \in \Psi^m(X). \tag{6.18}$$

PROOF. Since the kernel of a pseudodifferential operator is smooth outside the diagonal the first part of the definition holds for A – indeed if ϕ, $\psi \in \mathcal{C}^\infty(X)$ then

$$\phi A \psi = G^* B' (G^{-1})^* (u\big|_U),\ B' = ((G^{-1})^*\phi) B ((G^{-1})^*\psi) \in \Psi^{-\infty}_\infty(\mathbb{R}^n) \tag{6.19}$$

since $(G^{-1})^*\phi$, $(G^{-1})^*\psi \in \mathcal{C}^\infty(G(U))$ have disjoint supports. Similarly for the second part, the identity (6.19) still holds and if ϕ and ψ are both supported in some other coordinate patch $F : W \longrightarrow \mathbb{R}^n$ then the support of the kernel of B' is contained in $G(U \cap W) \times G(U \cap W)$ and $H = F \circ G^{-1}$ is a diffeomorphism from $G(U \cap W)$ to $F(U \cap W)$. The local coordinate invariance in Proposition 2.11 shows that $B'' = H^* B' (H^{-1})^* \in \Psi^m(\mathbb{R}^n)$ has kernel with support in $F(U \cap W) \times F(U \cap W)$ and (6.19) becomes

$$\phi A \psi = F^* B'' (F^{-1})^* (u\big|_W) \tag{6.20}$$

which implies the second condition. $\square$

Thus there are lots of examples – if $B \in \Psi^m_\infty(\mathbb{R}^n)$ and $\psi \in \mathcal{C}^\infty_c(X)$ has support in a coordinate patch with image ψ' in the local coordinates then applying (6.18) to $\psi' B \psi'$ gives an element of $\Psi^m(X)$. In fact each pseudodifferential operator is a sum of a smoothing operator and terms of this type. To see this, first note the following elementary result. Any open cover of a $\mathcal{C}^\infty$ manifold has a partition of unity subordinate to it, i.e. if $A_r \subset X$ are open sets for $r \in R$ and

$$X = \bigcup_{r \in R} A_r \tag{6.21}$$

there exists $\phi_i \in \mathcal{C}^\infty_c(X)$, all non-negative with locally finite supports:

$$\forall\ i\ \operatorname{supp}(\phi_i) \cap \operatorname{supp}(\phi_j) \neq \emptyset \text{ for a finite set of indices } j, \tag{6.22}$$

where each $\operatorname{supp}(\phi_i) \subset A_r$ for some $r = r(i)$ and

$$\sum_i \phi_i(x) = 1 \quad \forall\ x. \tag{6.23}$$

In fact one can do slightly better than this.

LEMMA 6.3. *Given an open cover* U_a *of* X *there exists a partition of unity* ϕ_i *(so with locally finite supports) and*

$$\forall\ i, j\ \exists\ a \text{ such that } \operatorname{supp}(\phi_j) \cap \operatorname{supp}_j \subset U_a. \tag{6.24}$$

Taking $i = j$ shows that the partition of unity is subordinate to the given open cover and the condition (6.24) is automatically satisfied if the intersection of supports is empty.

PROOF. Take any partition of unity ψ_a subordinate to the cover U_a and indexed so that $\mathrm{supp}(\psi_a) \subset U_a$. Thus, the support of each $\mathrm{supp}(\psi_a)$ is compact and only meets finitely many of the others. It follows that each point $p \in \mathrm{supp}(\phi_a)$ has a neighbourhood $V(p)$ which is contained in the intersection of all of the U_b such that $p \in \mathrm{supp}(\psi_b)$. For each a take a partition of unity of X subordinate to the cover by such $V(p)$'s and $X \setminus \mathrm{supp}(\psi_a)$. Then replace ψ_a by the finitely many non-zero products with this partition of unity (any term from a factor with support in $X \setminus \mathrm{supp}(\psi_a)$) gives zero. Taken together all the resulting (non-zero) functions give a partition of unity as desired since when two of the supports intersect they are contained in one of the $V(p)$'s. $\square$

PROPOSITION 6.1. *If ϕ_i is a partition of unity subordinate to a coordinate covering of X satisfying the condition of Lemma 6.3 and for each pair i, j such that $\mathrm{supp}(\phi_i) \cap \mathrm{supp}(\phi_j) \neq \emptyset$ $F_{ij} : \Omega_{ij} \longrightarrow \mathbb{R}^n$ is a coordinate system in a neighbourhood Ω_{ij} of this set, then an operator $A : \mathcal{C}^\infty_c(X) \longrightarrow \mathcal{C}^\infty(X)$ is a pseudodifferential operator on X if and only if*

$$\begin{aligned}(6.25)\quad \phi_i A \phi_j \textit{ has smooth kernel if } \mathrm{supp}(\phi_i) \cap \mathrm{supp}(\phi_j) = \emptyset \\ \textit{and otherwise is of the form } F^*_{ij} A_{ij} (F^{-1}_{ij})^* \textit{ with } A_{ij} \in \Psi^m_\infty(\mathbb{R}^n) \\ \textit{and kernel supported in } F(\Omega_{ij}) \times F(\Omega_{ij}).\end{aligned}$$

PROOF. The necessity of these conditions follows directly from the definition. Conversely if A satisifies all these conditions then for each ϕ, $\psi \in \mathcal{C}^\infty_c(X)$ $\phi A \psi$ is a finite sum (by local finiteness of the partition of unity) of terms to which either Lemma 6.1 or Lemma 6.2 applies. Thus it is an element of $\Psi^m(X)$. $\square$

So, this means that the original defintion can be replaced by the same one with respect to any given cover by coordinate patches – meaning that a pseudodifferential operator is just a (locally finite) sum of a smoothing operator plus pseudodifferential operators acting in a cover by coordinate patches $F_i : \Omega_i \longrightarrow \mathbb{R}^n$:

$$\begin{aligned}(6.26)\quad A \in \Psi^m(X) \Longrightarrow A = A' + \sum_i A_i,\ A' \in \mathcal{C}^\infty(X^2; \Omega_R),\ A_i = F^*_i B_i (F^{-1}_i)^*, \\ B_i \in \Psi^m_\infty(\mathbb{R}^n),\ \mathrm{supp}(B_i) \Subset_i (\Omega_i) \times F_i(\Omega_i).\end{aligned}$$

THEOREM 6.1. *Let X be a compact $\mathcal{C}^\infty$ manifold then the pseudodifferential operators $\Psi^*(X)$ form an order filtered ring.*

PROOF. The main point of course is that they form a ring, the order-filtering means that

$$(6.27)\qquad \Psi^m(X) \circ \Psi^{m'}(X) \subset \Psi^{m+m'}(X).$$

Since X is compact, $\mathcal{C}^\infty_c(X) = \mathcal{C}^\infty(X)$ and all the operators act on $\mathcal{C}^\infty(X)$, so the product is well-defined. From the remarks above, it suffices to consider the four cases of products $A \circ B$ where A and B are either smoothing operators or pseudodifferential operators with supports in a coordinate patch. In fact using a partition of unity as in Lemma 6.3 corresponding to a coordinate cover and then applying Proposition 6.1 if they are *both* pseudodifferential operators we can assume they have support in the same coordinate patch. Then the result follows from the local composition theorem of Chapter 2. So it is enough to suppose that at least one of the operators is a smoothing operator. If both are smoothing then this follows from the fact that the kernel of the composite is given in terms of the kernels of the factors by

$$(A \circ B)(p, p') = \int_X A(p, \cdot)B(\cdot, p') \in \mathcal{C}^\infty(X^2; \Omega_R). \tag{6.28}$$

When one factor is smoothing and the other is a local pseudodifferential the composte is smoothing since it is given by the action of the pseudodifferential operator (or its transpose) on the kernel of the smoothing operator, in one of the variables. □

Note that if X is not compact we cannot in general compose pseudodifferential operators, since the first one maps $\mathcal{C}_c^\infty(X)$ into $\mathcal{C}^\infty(X)$ and the second may not act on $\mathcal{C}^\infty(X)$. This is sorted out below.

Now, it is most important to show that the symbol maps still makes sense and has at leat most of the properties it had on $\mathbb{R}^n$. This is not quite obvious because of the non-uniqueness inherent in a presentation such as (6.25). First however we need to check that there is a place for the symbol to take values.

Recall that for an open set $\Omega \subset \mathbb{R}^n$ we defined the symbol spaces $S^m_\infty(\Omega; \mathbb{R}^p)$ as consisting of the smooth functions satisfying (2.1). Let $\pi : W \longrightarrow X$ be a real vector bundle over a manifold X. So X is covered by local coordinate patches Ω_i over which W is trivial, meaning there is a diffeomorphism

$$F_i : \pi^{-1}(\Omega_i) \longrightarrow \Omega'_i \times \mathbb{R}^p \tag{6.29}$$

which maps each fibre $\pi^{-1}(p)$ to the corresponding $\{p'\} \times \mathbb{R}^q$ and is a linear map. Then if we choose a partition of unity subordinate to the cover we can set

$$S^m(W) = \left\{ a : W \longrightarrow \mathbb{C}; a = \sum_i \phi_i F_i^* a_i \text{ for some } a_i \in S^m(\Omega'_i \times \mathbb{R}^p) \right\} \tag{6.30}$$

provided we show this is independent of choices.

Proposition 6.2. *If $W \longrightarrow X$ is a real vector bundle over a smooth manifold X then the space, $S^m(W)$, of symbols on W is well-defined for each $m \in \mathbb{R}$ by* (6.30).

Proof. We need to check to things here, what happens under changes of coordinate covering and changes of local trivializations. Notice that can move the ϕ into local coordinates to get $\phi'_i \in \mathcal{C}_c^\infty(\Omega'_i)$ and write (6.30) as

$$S^m(W) = \left\{ a : W \longrightarrow \mathbb{C}; a = \sum_i F_i^* \phi'_i a_i \text{ for some } a_i \in S^m(\Omega'_i \times \mathbb{R}^p) \right\}. \tag{6.31}$$

Then $\phi'_i a_i$ actually has compact support in the base variables, so is a global symbol on $\mathbb{R}^n \times \mathbb{R}^p$. If ψ_j is a partition of unity subordinate to another coordinate patch then we can lift these functions under π to W and write $a \in S^m(W)$ as

$$a = \sum_{i,j} \psi_j F_i^* \phi'_i a_i = \sum_{i,j} F_i^* \psi'_j \phi'_i a_i.$$

Thus each $\psi_j\phi_i$ is supported in the intersection of the two coordinate patches. Thus it suffices to show that if

$$F : \omega \times \mathbb{R}^p \longrightarrow \Omega' \times \mathbb{R}^p,\ F(x,\xi) = (f(x), A(x)\xi) \tag{6.32}$$

is a diffeomorphism, so f is a diffeomorphism and $A(x)$ is smooth and invertible, then $a \in S^m(\Omega';\mathbb{R}^p)$, $\operatorname{supp}(a) \subset K \times \mathbb{R}^p$ imples that $F^*a \in S^m(\Omega;\mathbb{R}^p)$. We can do this in two steps since $F = F' \circ (f, \mathrm{Id})$ where F' is of the same form with $f = \mathrm{Id}$. The second map amounts to a coordinate change and it is easy to see that the estimates in (2.1) are preserved by such a transformation. Thus it suffices to show that if $a \in S^m(\mathbb{R}^m;\mathbb{R}^p)$ has support in $K \times \mathbb{R}^p$ for some compact K and $A : \Omega \longleftrightarrow \mathrm{GL}(p,\mathbb{R})$ is a smooth map in an open neighbourhood of $\Omega \supset K$ then

$$a(x, A(x)\xi) \in S^m(\mathbb{R}^n;\mathbb{R}^p). \tag{6.33}$$

The basic symbol estimate

$$|a(x, A(x)\xi)| \le C \sup_K \langle A(x)\xi\rangle^m \le C'\langle\xi\rangle^m$$

therefore follows from the invertibility of $A(x)$ and the fact that a vanishes outside $K \times \mathbb{R}^p$. $V_{ij}\xi_i D_{\xi_j}$ and D_{x_k}. The symbol estimates on a function b just amount to requiring the estimate

$$|P(x, V, D_x)b(x,\xi)| \le C\langle\xi\rangle^m \tag{6.34}$$

for all polynomials P with smooth coefficient in x (since b vanishes outside $K \times \mathbb{R}^p$. The diffeomorphism $(x,\xi) \longmapsto (x, A(x)\xi)$ maps the space of these differential operators into itself, so the symbol estimates carry over. □

Suppose $A \in \Psi^m(X)$ and ρ_i is a square partition of unity subordinate to a coordinate cover $F_i : \Omega_i \longrightarrow \mathbb{R}^b$, so we can suppose

$$\operatorname{supp}(\rho_i) \subset \Omega_i,\ \sum_i \rho_i^2 = 1. \tag{6.35}$$

Then

$$A - \sum_i \rho_i A \rho_i \in \Psi^{m-1}(X) \tag{6.36}$$

since $[A, \rho_i] \in \Psi^{m-1}(X)$ as follows from (6.26) and the corresponding local property. This lead us to set

$$\sigma_m(\tau)(A) = \sum_{\{i, \pi(\tau) \in \operatorname{supp}(\rho_i)\}} b_i(x^{(i)}, \xi^{(i)}) \tag{6.37}$$

where

$$\tau = F_i^*(\sum_j \xi_j^{(i)} \cdot dx_j) \quad \xi^{(i)} \cdot dx \in T^*_{x^{(i)}}\mathbb{R}^n,\ x^{(i)} = F_i(\pi(\tau)) \tag{6.38}$$

and the b_i are representatives of the symbols of the $\rho_i A\rho_i$. This defines a function on $T^*X\backslash 0$, in fact the equivalence class

$$\sigma_m(A) \in S^{m-[1]}(T^*X) = S^m(T^*X)/S^{m-1}(T^*X) \tag{6.39}$$

is well-defined.

Proposition 6.3. *The principal symbol map in* (6.39)*, defined as in* (6.37)*, gives a short exact sequence:*

$$0 \hookrightarrow \Psi^{m-1}(X) \hookrightarrow \Psi^m(X) \xrightarrow{\sigma_m} S^{m-[1]}(T^*X) \longrightarrow 0. \tag{6.40}$$

Proof. First we need to check that $\sigma_m(A)$ is indeed well-defined. This involves checking what happens under a change of coordinate covering and a change of partition of unity subordinate to it. For a change of coordinate covering for a fixed square partition of unity it suffices to use the transformation law for the principal symbol under a diffeomorphism of $\mathbb{R}^n$.

Now, if ρ'_j is another square partition of unity, subordinate to the same covering note that

$$\sum_j \rho'_j \rho_i A \rho'_j \rho_i \equiv \rho_i A \rho_i$$

where equality is modulo Ψ^{m-1}, since $[\phi, \Psi^m] \subset \Psi^{m-1}$ for any $\mathcal{C}^\infty$ function ϕ. It follows from (6.40) that the principal symbols, defined by (6.37), for the two partitions are the same.

The principal symbol is therefore well defined. Moreover, it follows that if $\phi \in \mathcal{C}^\infty(X)$ then

$$\sigma_m(\phi A) = \phi \sigma_m(A) \text{ since } \rho_i(\phi A)\rho_i = \phi(\rho_i A \rho_i). \tag{6.41}$$

Certainly if $A \in \Psi^{m-1}(X)$ then $\sigma_m(N) \equiv 0$. Moreover if $A \in \Psi^m(X)$ and $\sigma_m(A) \equiv 0$ then it follows from (6.41) that $\sigma_m(\rho_i A \rho_i) = 0$ and hence, from the properties of operators on $\mathbb{R}^n$ that $\rho_i A \rho_i$ is actually of order $m-1$. This proves that the null space of σ_m is exactly $\Psi^{m-1}(X)$.

Thus it only remains to show that the map σ_m is surjective. If $a \in S^m(T^*X)$ choose $A_i \in \Psi^m_\infty(\mathbb{R}^n)$ by

$$\sigma_L(A_i) = \rho_i(x)(F^*)^{-1} a_i \rho_i(y) \in S^m_\infty(\mathbb{R}^n \times \mathbb{R}^n) \tag{6.42}$$

and set

$$A = \sum_i F_i^* A_i G_i^* \qquad G_i = F_i^{-1}. \tag{6.43}$$

Then, from (6.37) $\sigma_m(A) \equiv a$ by invariance of the principal symbol. □

6.4. The symbol calculus

The other basic properties of the calculus on a compact manifold are easily established. For example to check that

$$\sigma_{m+m'}(A \cdot B) = \sigma_m(A) \cdot \sigma_m(B) \tag{6.44}$$

if $A \in \Psi^m(X)$, $B \in \Psi^{m'}(X)$ note that

$$AB = \sum_{i,j} \rho_i^2 A \rho_j^2 B = \sum_{i,j} \rho_i A \rho_i \cdot \rho_j B \rho_i \qquad \text{mod } \Psi^{m+m'-1}. \tag{6.45}$$

In § 5.9 we used the symbol calculus to construct a left and right parametrix for an elliptic element of $\Psi^m(X)$, where X is compact, i.e. an element $B \in \Psi^{-m}(X)$, such that

$$AB - \mathrm{Id}, BA - \mathrm{Id} \in \Psi^{-\infty}(X). \tag{6.46}$$

As a consequence of this construction note that:

Proposition 6.4. *If $A \in \Psi^m(X)$ is elliptic, and X is compact, then*

$$A : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X) \tag{6.47}$$

is Fredholm*, i.e. has finite dimensional null space and closed range with finite dimensional complement. If ν is a non-vanishing $\mathcal{C}^\infty$ measure on X and a generalized inverse of A is defined by*

$$\begin{aligned} Gu &= f \ \text{if}\ u \in \operatorname{Ran}(A), Af = u, f \perp_\nu \operatorname{Nul}(A) \\ Gu &= 0 \ \text{if}\ u \perp_\nu \operatorname{Ran}(A) \end{aligned} \tag{6.48}$$

then $G \in \Psi^{-m}(X)$ satisfies

$$\begin{aligned} GA &= \operatorname{Id} - \pi_N \\ AG &= \operatorname{Id} - \pi_R \end{aligned} \tag{6.49}$$

where π_N and π_R are ν-orthogonal projections onto the null space of A and the ν-orthocomplement of the range of A respectively.

Proof. The main point to note is that $E \in \Psi^{-\infty}(X)$ is smoothing,

$$E : \mathcal{C}^{-\infty}(X) \longrightarrow \mathcal{C}^\infty(X) \quad \forall\ E \in \Psi^{-\infty}(X). \tag{6.50}$$

Such a map is *compact* on $L^2(X)$, i.e. maps bounded sets into precompact sets by the theorem of Ascoli and Arzela. The second thing to recall is that a Hilbert space with a compact unit ball is finite dimensional. Then

$$\operatorname{Nul}(A) = \{u \in \mathcal{C}^\infty(X); Au = 0\} = \{u \in L^2(X); Au = 0\} \tag{6.51}$$

since, from (6.51) $Au = 0 \Longrightarrow (BA - \operatorname{Id})u = -Eu$, $E \in \Psi^\infty(X)$, so $Au = 0$, $u \in \mathcal{C}^{-\infty}(X) \Longrightarrow u \in \mathcal{C}^\infty(X)$. Then

$$\operatorname{Nul}(A) = \{u \in L^2(X); Au = 0 \int |u|^2\nu = 1\} \subset L^2(X) \tag{6.52}$$

is compact since it is closed (A is continuous) and so $\operatorname{Nul}(A) = E(\operatorname{Nul}(A))$ is *precompact*. Thus $\operatorname{Nul}(A)$ is finite dimensional.

Next let us show that $\operatorname{Ran}(A)$ is closed. Suppose $f_j = Au_j \longrightarrow f$ in $\mathcal{C}^\infty(X)$, $u_j \in \mathcal{C}^\infty(X)$. By what we have just shown we can assume that $u_j \perp_\nu \operatorname{Nul}(A)$. Now if B is the parametrix

$$u_j = Bf_j + Eu_j,\ E \in \Psi^{-\infty}(X). \tag{6.53}$$

Suppose, along some subsequence, $\|u_j\|_\nu \longrightarrow \infty$. Then

$$\frac{u_j}{\|u_j\|_\nu} = B\left(\frac{f_j}{\|u_j\|_\nu}\right) + E\left(\frac{u_j}{\|u_j\|_\nu}\right) \tag{6.54}$$

shows that $\frac{u_j}{\|u_j\|_\nu}$ lies in a precompact subset of L^2, $\frac{u_j}{\|u_j\|_\nu} \longrightarrow u$. This is a contradiction, since $Au = 0$ but $\|u\| = 1$ and $u \perp_\nu \operatorname{Nul}(A)$. Thus the norm sequence $\|u_j\|$ is bounded and therefore the sequence has a weakly convergent subsequence, which we can relabel as u_j. The parametrix shows that $u = Bf_j + Eu_j$ is strongly convergent with limit u, which satisfies $Au = f$.

Finally we have to show that $\operatorname{Ran}(A)$ has a finite dimensional complement. If π_R is orthogonal projection off $\operatorname{Ran}(A)$ then from the second part of (6.46) $f = \pi_R E' f$ for some smoothing operator E. This shows that the orthocomplement has compact unit ball, hence is finite dimensional. □

Notice that it follows that the two projections in (6.49) are both smoothing operators of finite rank.

6.5. Pseudodifferential operators on vector bundles

Perhaps unwisely I have carried through the discussion above for pseudodifferential operators acting on functions. The extension to operators between sections of vector bundles is mainly notational.

THEOREM 6.2. *If $W \longrightarrow Y$ is a $\mathcal{C}^\infty$ vector bundle and $F : X \longrightarrow Y$ is a $\mathcal{C}^\infty$ map then $F^*W \longrightarrow X$ is a well-defined $\mathcal{C}^\infty$ vector bundle over X with total space*

$$F^*W = \bigcup_{x\in X} W_{F(x)}; \tag{6.55}$$

if $\phi \in \mathcal{C}^\infty(Y;W)$ then $F^*\phi$ is a section of F^*W and $\mathcal{C}^\infty(X;F^*W)$ is spanned by $\mathcal{C}^\infty(X)\cdot F^*\mathcal{C}^\infty(Y;W)$.

Distributional sections of any $\mathcal{C}^\infty$ vector bundle can be defined in two equivalent ways:

$$\text{“Algebraically”}\ \mathcal{C}^{-\infty}(X;W) = \mathcal{C}^{-\infty}(X) \bigotimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X;W) \tag{6.56}$$

or as the dual space

$$\text{“Analytically”}\ \mathcal{C}^{-\infty}(X;W) = \left[\mathcal{C}^\infty_c(X;\Omega\otimes W')\right]' \tag{6.57}$$

where W' is the dual bundle and Ω the density bundle over X. In order to use (6.57) we need to define a topology on $\mathcal{C}^\infty_c(X;U)$ for any vector bundle U over X. One can do this by reference to local coordinates.

We have just shown that any elliptic pseudodifferential operator, $A \in \Psi^m(X)$ on a compact manifold X has a generalized inverse $B \in \Psi^{-m}(X)$, meaning

$$\begin{aligned} BA &= \mathrm{Id} - \pi_N \\ AB &= \mathrm{Id} - \pi_R \end{aligned} \tag{6.58}$$

where π_N and π_R are the orthogonal projections onto the null space of A and the orthocomplement of the range of A with respect to a prescribed $\mathcal{C}^\infty$ positive density ν, both are elements of $\Psi^{-\infty}(X)$ and have finite rank. To use this theorem in geometric situations we need first to make the “trivial” extension to operators on sections of vector bundles.

As usual there are two ways (at least) to approach this extension; the high road and the low road. The “low” road is to go back to the definition of $\Psi^m(X)$ and to generalize to $\Psi^m(X;V,W)$. This just requires to take the definition, following (6.16), but using a covering with respect to which the bundles V, W are *both* locally trivial. The local coordinate representatives of the pseudodifferential operator are then matrices of pseudodifferential opertors. The symbol mapping becomes

$$\Psi^m(X;V,W) \longrightarrow S^{m-[1]}\left(T^*X;\mathrm{Hom}(V,W)\right) \tag{6.59}$$

where $\mathrm{Hom}(V,W) \simeq V \otimes W'$ is the bundle of homomorphisms from V to W and the symbol space consists of symbolic sections of the lift of this bundle to T^*X. We leave the detailed description and proof of these results to the enthusiasts.

So what is the “high” road. This involves only a little sheaf-theoretic thought. Namely we want to define the space $\Psi^m(X;V,W)$ using $\Psi^m(X)$ by:

$$\Psi^m(X;V,W) = \Psi^m(X) \bigotimes_{\mathcal{C}^\infty(X^2)} \mathcal{C}^\infty(X^2; V\boxtimes W'). \tag{6.60}$$

To make sense of this we first note that $\Psi^m(X)$ is a $\mathcal{C}^\infty(X^2)$-module as is the space $\mathcal{C}^\infty(X^2; V \boxtimes W')$ where $V \boxtimes W'$ is the "exterior" product:

$$(V \boxtimes W')_{(x,y)} = V_x \otimes W'_y. \tag{6.61}$$

The tensor product in (6.60) means that

$$A \in \Psi^m(X; V, W) \text{ is of the form } A = \sum_i A_i \cdot G_i \tag{6.62}$$

where $A_i \in \Psi^m(X)$, $G_i \in \mathcal{C}^\infty(X^2; V \boxtimes W')$ and equality is fixed by the relation

$$\phi A \cdot G - A \cdot \phi G \equiv 0. \tag{6.63}$$

Now what we really need to note is:

Proposition 6.5. *For any compact $\mathcal{C}^\infty$ manifold Y and any vector bundle U over Y*

$$\mathcal{C}^{-\infty}(Y; U) \equiv \mathcal{C}^{-\infty}(Y) \bigotimes_{\mathcal{C}^\infty(Y)} \mathcal{C}^\infty(Y; U). \tag{6.64}$$

Proof. $\mathcal{C}^{-\infty}(Y; U) = (\mathcal{C}^\infty(Y; \Omega \otimes U'))'$ is the definition. Clearly we have a mapping

$$\mathcal{C}^{-\infty}(Y) \bigotimes_{\mathcal{C}^\infty(Y)} \mathcal{C}^\infty(Y; U) \ni \sum_i A_i \cdot g_i \longrightarrow \mathcal{C}^{-\infty}(Y; U) \tag{6.65}$$

given by

$$\sum_i u_i \cdot g_i(\psi) = \sum_i u_i(g_i \cdot \psi) \tag{6.66}$$

since $g_i \psi \in \mathcal{C}^\infty(Y; \Omega)$ and linearity shows that the map descends to the tensor product. To prove that the map is an isomorphism we construct an inverse. Since Y is compact we can find a finite number of sections $g_i \in \mathcal{C}^\infty(Y; U)$ such that *any* $u \in \mathcal{C}^\infty(Y; U)$ can be written

$$u = \sum_i h_i g_i \quad h_i \in \mathcal{C}^\infty(Y). \tag{6.67}$$

By reference to local coordinates the same is true of distributional sections with

$$h_i = u \cdot q_i \quad q_i \in \mathcal{C}^\infty(Y; U'). \tag{6.68}$$

This gives a left and right inverse. □

Theorem 6.3. *The calculus extends to operators on sections of vector bundles over any compact $\mathcal{C}^\infty$ manifold.*

6.6. Hodge theorem

The identification of the deRham cohomology of a compact manifold with the finite dimensional vector space of harmonic forms goes back to Hodge in the algebraic setting and to Hermann Weyl in the general case. It is a rather direct consequence of the Fredholm properties on smooth sections of the Laplacian. In fact this has nothing much to do with the explicit form of the deRham complex, so let's do it in the natural context of an elliptic complex over a compact manifold M.

Thus let E_i, $i = 0, \dots, N$ be complex vector bundles and suppose $d_i \in \operatorname{Diff}^1(M; E_i, E_{i+1})$, $i < N$, form a complex of differential operators, meaning that for each $i < N$ d_{i+1} annihilates the range of d_i which means just that

$$d_{i+1} d_i = 0 \in \operatorname{Diff}^1(M; E_i; E_{i+2}),\ i < N. \tag{6.69}$$

Such a complex is said to be exact (on $\mathcal{C}^\infty$ sections) if

$$\mathcal{C}^\infty(M; E_0) \xrightarrow{d_0} \mathcal{C}^\infty(M; E_1) \xrightarrow{d_1} \dots \qquad \mathcal{C}^\infty(M; E_{N-1}) \xrightarrow{d_{N-1}} \mathcal{C}^\infty(M; E_N) \tag{6.70}$$

is exact, meaning that conversely

$$\operatorname{null}(d_{i+1}) = d_i \mathcal{C}^\infty(M : E_i)\ \forall\ i < N. \tag{6.71}$$

The principal symbol $\sigma_i(d_i) \in \mathcal{C}^\infty(T^*M; \pi^* \hom(E_i, E_{i+1})$ is a homogeneous polynomial of degree 1 and from (6.69) these bundle maps for a complex over T^*M. Of course the all vanish at the zero section so, excluding that, we say the original complex is *elliptic* if the symbol complex

$$\mathcal{C}^\infty(T^*M \setminus 0; \pi^* E_0) \xrightarrow{\sigma_1(d_0)} \mathcal{C}^\infty(T^*M \setminus 0; \pi^* E_1) \xrightarrow{\sigma_1(d_1)} \dots$$
$$\mathcal{C}^\infty(T^*M \setminus 0; \pi^* E_{N-1}) \xrightarrow{\sigma_1(d_{N-1})} \mathcal{C}^\infty(T^*M \setminus 0; \pi^* E_N) \tag{6.72}$$

is exact.

Now, choose an Hermitian inner product on each of the E_i and a smooth density on M so that we can define the adjoints δ_i of the d_{i-1} (so that the subscript corresponds to the subscript of the vector space on which the operator acts)

$$\delta_i = (d_{i-1})^* \in \operatorname{Diff}^1(M; E_i, E_{i-1}),\ i = 1, \dots, N. \tag{6.73}$$

Then form the Hodge operator and the Laplacian

$$(d+\delta)_i \in \operatorname{Diff}^1(M; E_i, E_{i-1} \oplus E_{i+1}),\ \varDelta_i = \delta_{i+1} d_i + d_{i-1} \delta_i^2 \in \operatorname{Diff}^2(M; E_i). \tag{6.74}$$

We can also take the direct sum of all the terms in the complex and set

$$d = \oplus_i d_i \in \operatorname{Diff}^1(M; E_*),\ \delta = \oplus_i \delta_i \in \operatorname{Diff}^1(M; E_*). \tag{6.75}$$

Then (6.69) and the induced identity $\delta_{i-1}\delta_i = 0$ together show that

$$(d+\delta)^2 = \oplus_i \varDelta_i \in \operatorname{Diff}^2(M; E_*) \tag{6.76}$$

since applied to $\mathcal{C}^\infty(M; E_i)$

$$(d+\delta)^2\big|_{\mathcal{C}^\infty(M;E_i)} = (d+\delta)d_i + (d+\delta)\delta_i = (d_{i+1} + \delta_{i+1})d_i + (d_{i-1} + \delta_{i-1})\delta_i = \varDelta_i. \tag{6.77}$$

THEOREM 6.4. *For an elliptic complex the operators $d + \delta$ and all the $\varDelta_i$ are elliptic,*

$$\operatorname{null}(\varDelta_i) = \{u \in \mathcal{C}^\infty(M; E_i); d_i u = 0,\ \delta_i u = 0\} \tag{6.78}$$

and the inclusion of this space into the null space of d_i induces an isomorphism of vector spaces

$$\operatorname{null}(\varDelta_i) \simeq \{u \in \mathcal{C}^\infty(M; E_i); d_i u = 0\} / d_{i-1}\mathcal{C}^\infty(M; E_i). \tag{6.79}$$

In particular the vector spaces on the right in (6.79) are finite dimensional; these are the (hyper-)cohomology spaces of the original complex.

PROOF. The symbol of Δ_i is exactly

$$\sigma_2(\Delta_i) = \sigma_1(\delta_{i+1})\sigma_1(d_i) + \sigma_1(d_{i-1})\sigma_1(\delta_i). \tag{6.80}$$

Over points of $T^*M \setminus 0$ we can use the (pointwise) inner product on the E_i's and the fact that $\sigma_1(\delta_i) = (\sigma_1(d^*_{i-1})$ to see that

$$\langle f, \sigma_1(\Delta_i)f\rangle = \langle f, \sigma_1(\delta_{i+1})\sigma_1(d_i)f\rangle + \langle f, \sigma_1(d_{i-1})\sigma_1(\delta_i)f\rangle = |\sigma_1(d_i)f|^2 + |\sigma_1(\delta_i)f|^2.$$

Thus an element of the null space of $\sigma_2(\Delta_i)$ is in the intersection of the null spaces of $\sigma_1(d_i)$ and $\sigma_1(\delta_i)$. The null space of the latter is precisely the orthocomplement to the range of the former, so (by the assumed ellipticity) $\sigma_2(\Delta_i)$ is injective and hence an isomorphism. As an elliptic operator the null space of Δ_i, even acting on distributions, consists of elements of $\mathcal{C}^\infty(M; E_i)$. Moreover integration by parts then gives

$$\Delta_i u = 0 \Longrightarrow \int_M \langle u, \Delta_i u\rangle \nu = \|d_i u\|^2_{L^2} + \|\delta_i u\|^2_{L^2} \Longrightarrow d_i u = 0,\ \delta_i u = 0. \tag{6.81}$$

The converse is obvious, so this proves (6.78).

We know that any elliptic operator on a compact manifold is Fredholm. Moreover Δ_i is self-adjoint, directly from the definition in (6.74). Thus the range of Δ_i is precisely the orthocomplement (with respect to the L^2 inner product) of its own null space:

$$\mathcal{C}^\infty(M; E_i) = \operatorname{null}(\Delta_i) \oplus \Delta\mathcal{C}^\infty(M; E_i). \tag{6.82}$$

Now expanding out Δ_i we can decompose each element of the second term as

$$\Delta u = d_{i-1}\delta_i u + \delta_{i+1} d_i u = d_{i-1} v_{i-1} + \delta_{i+1} w_{i+1}. \tag{6.83}$$

The two terms here are orthogonal in $L^2(M; E_i)$ and this allows us to rewrite (6.82) as *The Hodge Decomposition*

$$\mathcal{C}^\infty(M; E_i) = \operatorname{null}(\Delta_i) \oplus d_{i-1}\mathcal{C}^\infty(M; E_{i-1}) \oplus \delta_{i+1}\mathcal{C}^\infty(M; E_{i+1}). \tag{6.84}$$

Indeed, all three terms here are orthogonal as follows by integration by parts and the fact that $d^2 = 0$ and hence there must be equality in (6.84) since each element has such a decomposition, as follows from (6.82) and (6.83).

Now if $u \in \mathcal{C}^\infty(M; E_i)$ satisfies $d_i u = 0$, consider its Hodge decomposition

$$u = u_0 + du_1 + \delta v. \tag{6.85}$$

The last term must vanish since applying d to (6.85), $d\delta v = 0$ and then integrating by parts

$$\int_M \langle v, d\delta v\rangle \nu = \|\delta v\|^2_{L^2} = 0. \tag{6.86}$$

The map $u\rangle u_0$ therefore takes the left side of (6.79) to the right. It is injective, since $u_0 = 0$ means that u is 'exact' and it is surjective since u_0 is itself closed and the decomposition (6.85) is unique, so it is mapped to u_0. This gives the *Hodge isomorphism* (6.79). □

In fact the same argument works with distributional sections of the various bundles. We know that, as an elliptic operator

$$\Delta_i : \mathcal{C}^{-\infty}(M; E_i) \longrightarrow \mathcal{C}^{-\infty}(M; E_i) \tag{6.87}$$

also has range exactly the annihilator of the null space of its adjoint, also Δ_i, on $\mathcal{C}^\infty$ sections. Thus we get a distributional decomposition

$$\mathcal{C}^{-\infty}(M;E_i) = \operatorname{null}(\Delta_i) \oplus \Delta\mathcal{C}^{-\infty}(M;E_i) \tag{6.88}$$

which we can still think of as 'orthogonal' since the pairing exists between the smooth harmonic forms and the general distibutional sections. A distributional form of the Hodge decomposition follows as before which we can write as

$$\mathcal{C}^{-\infty}(M;E_i) = \operatorname{null}(\Delta_i) \oplus \left(d_{i-1}\mathcal{C}^{-\infty}(M;E_{i-1}) \dot{+} \delta_{i+1}\mathcal{C}^{-\infty}(M;E_{i+1})\right). \tag{6.89}$$

Here the second two terms do not formally 'pair' under extension of the L^2 inner product so we just claim that the intersection is empty. This follows from the fact that an element of the intersection is harmonic and hence smooth and thus, from (6.88), vanishes. This lead immediately to a distributional Hodge isomorphism

$$\operatorname{null}(\Delta_i) = \left\{u \in \mathcal{C}^{-\infty}(M;E_i); d_i u = 0\right\} / d_{i-1}\mathcal{C}^{-\infty}(M;E_i) \tag{6.90}$$

completely analogous to (6.78). The proof is almost the same. A closed distributional form has a decomposition as in (6.89), $u = u_0 + du' + \delta v$ where u_1 and v are now distributional sections. However applying d we see that $d\delta v = 0$ and $\delta\delta v = 0$ so δv is harmonic, hence smooth, and the integration by parts argument as before shows that $\delta v = 0$ (not of course that $v = 0$). This gives a map from right to left in (6.90) which is an isomorphism just as before.

In particular this shows that the 'distributional deRham' and 'smooth deRham' cohomologies are isomorphic. In fact the isomorphism is natural, even though both isomorphisms (6.78) and (6.90) depend on the choice of inner product and smooth density (since of course the harmonic forms depend on these choices). Namely the isomorphism is induced by the natural 'inclusion map'

$$\{u \in \mathcal{C}^\infty(M;E_i); d_i u = 0\} / d_{i-1}\mathcal{C}^\infty(M;E_i) \longrightarrow \left\{u \in \mathcal{C}^{-\infty}(M;E_i); d_i u = 0\right\} / d_{i-1}\mathcal{C}^{-\infty}(M;E_i). \tag{6.91}$$

In many applications in differential geometry it is important to go a little further than this. The Hodge theorem above identifies the null space of the Laplacian with the intersections of the null spaces of d and δ. More generally consider the spectral decomposition associated to the Δ_i.

PROPOSITION 6.6. *If* $(\mathcal{C}^\infty(M;E_i))^+$ *is the orthocomplement to* $\operatorname{null}(\Delta_i)$ *for each* i *then the* d_i *induce and exact complex*

$$(\mathcal{C}^\infty(M;E_0))^+ \xrightarrow{d_0} (\mathcal{C}^\infty(M;E_1))^+ \xrightarrow{d_1} \dots \qquad (\mathcal{C}^\infty(M;E_{N-1}))^+ \xrightarrow{d_{N-1}} (\mathcal{C}^\infty(M;E_N))^+ \tag{6.92}$$

which restricts to an exact finite-dimensional complex on the subspaces which are eigenspaces of Δ_i *for a fixed* $\lambda > 0$.

PROOF. All the null spaces vanish and exactness follows from the Hodge decomposition. □

Of course the adjoint complex is the one for δ and the same result holds for distributional sections. Note that this means that the eigenspaces of Δ_i, corresponding to non-zero eigenvalues, can be decomposed into exact and coexact parts. Thus even though the Hodge operator $d + \delta$ mixes form degrees, all its eigenvectors are can be decomposed into eigenvectors of Δ which have 'pure degree'.

6.7. Sobolev spaces and boundedness

[Following discussions with Sheel Ganatra]

In the discussion above, I have shown that elliptic pseudodifferential operators are Fredholm on the spaces of $\mathcal{C}^\infty$ sections directly from the existence of parameterices, rather than using the more standard argument on Sobolev spaces. However, let me now recall this starting with operators of order 0. In fact it is convenient to define the Sobolev spaces for other orders so that boundedness is 'obvious' and then check that the definition is sensible.

Lemma 6.4. *On any compact manifold M each $A \in \Psi^0(M; V, W)$ for vector bundles V and W extends by continuity from $\mathcal{C}^\infty(M; V)$ to a bounded operator*

$$A : L^2(M; V) \longrightarrow L^2(M; W). \tag{6.93}$$

Proof. There are two obvious alternatives here. The first is to use the same construction of approximate square roots as before. That is, using the symbol calculus onie can see that if A is as above and we choose inner products on V and W and a smooth volume form on M so that A^* is defined then for a large positive constant C there exists $B \in \Psi^0(M; V)$ so that

$$C - B^*B = A^*A + G,\ G \in \Psi^{-\infty}(M; V). \tag{6.94}$$

This starts by solving the equation at a symbolic level, so showing that $\sigma_0(A)$ exists such that

$$C - \sigma_0(B)^*\sigma_0(B) = \sigma_0(A)^2,\ \sigma_0^*(A) = \sigma_0(A). \tag{6.95}$$

Thus $\sigma_0(A)$ is the square root of the positive definite matrix $C - \sigma_0^*(B)\sigma_0(B)$. Then one can proceed inductively using the symbol calculus, as before, to solve the problem modulo smoothing.

Alternatively we can simply use the known boundedness of smoothing operators on M and of pseudodifferential operators on $\mathbb{R}^n$. Thus the local (matrices of) operators, or order 0, as in (6.18) are bounded on $L^2(\mathbb{R}^n)$ and since $u \in L^2(M; V)$ is equivalent to $(F_i)^*\psi_i u_j \in L^2(\mathbb{R}^n)$ for a partition of unity ψ_i subordinate to a coordinate cover over each element of which the bundle is trivial, the boundedness (6.93) follows. Of course we are also using the density of $\mathcal{C}^\infty(M; W)$ in $L^2(M; W)$ which follows from the same argument. □

defines

Definition 6.3. *On a compact manifold and for a vector bundle W we set*

$$H^s(M; W) = \left\{u \in \mathcal{C}^{-\infty}(M; W); Au \in L^2(M; V)\ \forall\ A \in \Psi^{-s}(M; W, V)\right\},\ s \in \mathbb{R}. \tag{6.96}$$

Here we are demanding this for all pseudodifferential operators and all vector bundles V. This of course is gross overkill.

Proposition 6.7. *For each $s \in \mathbb{R}$, $\mathcal{C}^\infty(M; V)$ is dense in $H^s(M; V)$, every element $A \in \Psi^m(M; V, W)$ extends by continuity to a bounded linear operator*

$$A : H^s(M; V) \longrightarrow H^{s-m}(M; W)\ \forall\ s \in \mathbb{R},\ \forall\ s \in \mathbb{R} \tag{6.97}$$

and if $A \in \Psi^m(M; V, W)$ is elliptic then

$$Au \in H^s(M; V) \Longrightarrow u \in H^{s+m}(M; W). \tag{6.98}$$

PROOF. Since I have not quite fixed the topology on $H^s(M;V)$, the density statement is to be interpreted as meaning that if $u \in H^s(M;V)$ then there is a sequence $u_n \in \mathcal{C}^\infty(M;V)$ such that $Pu_n \to Pu$ in $L^2(M;W)$ for every $P \in \Psi^s(M;V,W)$. In fact the simplest thing to prove is that, with the ugly definition (6.96) of $H^s(M;V)$ that

$$P \in \Psi^s(M;V,W),\ u \in H^s(M;V) \Longrightarrow Pu \in L^2(M;W) \tag{6.99}$$

since this is precisely what the definition requires. Conversely we can see that

$$P \in \Psi^s(M;V,W),\ u \in L^2(M;V) \Longrightarrow Pu \in H^{-s}(M;W) \tag{6.100}$$

Here we are using the action of pseudodifferential operators on distributions. Indeed, if $A \in \Psi^{-s}(M;W,U)$ for some other vector bundle U then we just need to show that $APu \in L^2(M;U)$. However, by the composition theorem, $AP \in \Psi^0(M;V,U)$ so this follows from Lemma 6.4.

Combining these two special cases of (6.97) we can get the general case. Note that there is always an elliptic element $P_s \in \psi^s(M;V)$ for any $s \in \mathbb{R}$ and any vector bundle V. There is certainly an elliptic symbol, say $(1+|\xi|^2)^{\frac{s}{2}}\,\mathrm{Id}_V$ where $|\cdot|$ is some Riemannian metric. The surjectivity of the symbol maps shows that there is in fact a pseudodifferential operator P_s with this symbol, which is therefore elliptic. By the elliptic construction above this operator has a parameterix $Q_s \in \Psi^{-s}(M;V)$ which is also elliptic and satisfies

$$Q_sP_s - \mathrm{Id},\ P_sQ_s - \mathrm{Id} \in \Psi^{-\infty}(M;V). \tag{6.101}$$

Now, given a general $A \in \Psi^m(M;W,V)$ composing with this identity shows that

$$A = (AQ_s)P_s + G = BP_s + G,\ B = AQ_s \in \Psi^{m-s}(M;V,W),\ G \in \Psi^{-\infty}(M;V,W). \tag{6.102}$$

A smoothing operator certainly satisfies (6.97) (since $\mathcal{C}^\infty(M;V) \subset H^s(M;V)$ for all s) so it suffices to consider BP_s in place of A. Applying (6.99) to P_s and (6.100) to B, with s replaced by $m-s$ shows that

$$H^s(M;V) \xrightarrow{P_s} L^2(M;V) \xrightarrow{B} H^{s-m}(M;W) \tag{6.103}$$

which gives (6.97).

If A is elliptic then (6.98) follows since if $Q \in \Psi^{-m}(M;W,V)$ is a parametrix for A then

$$QA = \mathrm{Id} - G,\ G \in \Psi^{-\infty}(M;V),\ Au \in H^s(M;W) \Longrightarrow u = QAu + Gu \in H^{s+m}(M;V). \tag{6.104}$$

This means that the original definition can be written in the much simpler form

$$H^s(M;W) = \left\{u \in \mathcal{C}^{-\infty}(M;W); P_{-s}u \in L^2(M;W) \text{ for some elliptic } P_{-s}s \in \Psi^{-s}(M;W)\right\}. \tag{6.105}$$

Here of course 'some' means for any one elliptic element.

Finally then the density also follows. Namely, if $u \in H^s(M;V)$ then

$$u = Q_s(P_su) + Gu,\ P_s \in \Psi^s(M;V),\ Q_s \in \Psi^{-s}(M;V),\ G \in \Psi^{-\infty}(M;V). \tag{6.106}$$

Thus $P_su \in L^2(M;V)$. Let $v_n \to P_su$ in $L^2(M;V)$ then $Gu \in \mathcal{C}^\infty(M;V)$ and

$$Q_su_n + Gu \to u \in H^s(M;V) \text{ since } PQ_su_n + PGu \to Gu \in L^2(M;W)\ \forall\, P \in \Psi^s(M;V,W). \tag{6.107}$$

$\square$

Using the Fredholm properties of elliptic operators we see that if $P_{s/2} \in \Psi^{s/2}(M;V)$ is elliptic then, if $s > 0$,

$$B_s = P_{s/2}^* P_{s/2} + 1 \in \Psi^s(M;V) \tag{6.108}$$

is an isomorphism. Indeed, it is elliptic so we know that any element u of its null space is in $\mathcal{C}^\infty(M;V)$. However integration by parts is then justified and shows that

$$B_s u = 0 \Longrightarrow \langle P_{s/2}u, P_{s/2}u\rangle + \|u\|_{L^2}^2 = 0 \Longrightarrow u \equiv 0. \tag{6.109}$$

Thus its null space consists of $\{0\}$ and since it is (formally) self-adjoint, the same is true of the null space of its adjoint. Thus, being Fredholm, it is an isomorphism. In fact its inverse

$$B_s^{-1} \in \Psi^{-s}(M;V) \tag{6.110}$$

is also invertible. We have already shown (6.110) since B_s^{-1} is the generalized inverse.

Thus we have shown the main part of

Proposition 6.8. *For any compact manifold M and any vector bundle V over M there is an invertible element $B_s \in \Psi^s(M;V)$ for each s and then*

$$H^s(M;V) = \{u \in \mathcal{C}^{-\infty}(M;V); B_s u \in L^2(M;V)\},\ \|u\|_s = \|B_s u\|_{L^2} \tag{6.111}$$

shows that $H^s(M;V)$ is a Hilbert space. Moreover $\psi_i u$ has entries in $H^s(\mathbb{R}^n)$ for any covering of M by coordinate patches over each of which the bundle is trivial and for any partition of unity subordinate to it.

Proof. The last part just follows by looking at the local coordinate representative of B_s. Namely $\psi_i u$ is a (vector of) compactly supported distributions in the coordinate patch and $(1 + |D|^2)^{-s/2}\psi_i u \in L^2(\mathbb{R}^n)$ since it is smooth outside the image of the support of ψ_i by pseudolocality and inside the coordinate patch by the boundedness of pseudodifferential operators discussed above. □

Proposition 6.9. *The L^2 pairing with respect to an inner product and smooth volume form extends by continuity to a non-degenerate pairing*

$$H^s(M;V) \times H^{-s}(M;V) \longrightarrow \mathbb{C} \tag{6.112}$$

which allows $H^{-s}(M;V)$ to be identified with the dual of $H^s(M;V)$ for any s.

Proof. Exercise! □

6.8. Pseudodifferential projections

We are interested in constructing projections in the pseudodifferential algebra corresponding to arbitrary symbolic projections.

Theorem 6.5. *If M is compact, E is a complex vector bundle over M and $p \in \mathcal{C}^\infty(S^*M; \hom(E))$ is valued in the projections in the sense that $p^2 = p$ then there exists an element $P \in \Psi^0(M;E)$ with symbol p which is itself a projection.*

First we work modulo smoothing operators, for later applications we shall do this without assuming the compactness of M.

LEMMA 6.5. *If $E \longrightarrow M$ is a complex vector bundle and $p \in \mathcal{C}^\infty(S^*M; E)$ satisfies $p^2 = p$ then there exists $Q \in \Psi^0(M; E)$ which is properly supported and such that*

$$Q^2 - Q \in \Psi^{-\infty}(M; E). \tag{6.113}$$

PROOF. Of course the first step is simply to choose $Q_0 \in \Psi^0(M; E)$ which is properly supported and has $\sigma_0(Q) = p$. This gives a version of (6.113) but only modulo terms of order -1 :

$$Q_0^2 - Q_0 = E_1 \in \Psi^{-1}(M; E). \tag{6.114}$$

However note, by composing with Q_0 first on the left and then on the right, that $Q_0 E_1 = E_1 Q_0$. It follows that

$$(\operatorname{Id} - P)E_i P,\ PE_i(\operatorname{Id} - P) \in \Psi^{-i-1}(M; E) \tag{6.115}$$

for $i = 1$. Then set $Q_1 = -Q_0 E_1 Q_0 + (\operatorname{Id} - Q)E_0(\operatorname{Id} - Q_0)$ and $Q_{(1)} = Q_0 + Q_1$. It follows from (6.114) and (6.115) that

$$Q_{(i)}^2 - Q_{(i)} = E_{i+1} = E_i + Q_{(i)}Q_i + Q_i Q_{(i)} - Q_i \in \Psi^{-i-1}(M; E). \tag{6.116}$$

Thus we can proceed by induction and successively find $Q_j \in \Psi^{-j}(M; E)$, always properly supported, such that

$$Q_{(i)} = \sum_{j=1}^{i} Q_j \text{ satisfies (6.116) for all } i. \tag{6.117}$$

Then taking Q to be a properly supported asympotic sum of this series gives an operator as claimed. □

PROPOSITION 6.10. *If M is compact, E is a complex vector bundle over M and $Q \in \Psi^0(M; E)$ is such that $Q^2 - Q \in \Psi^{-\infty}(M; E)$ then there exists $P \in \Psi^0(M; E)$ such that $P^2 = P$ and $P - Q \in \Psi^{-\infty}(M; E)$.*

PROOF. As a bounded operator on $L^2(M; E)$, Q has discrete spectrum outside $\{0, 1\}$. Indeed, if $\tau \notin \{0, 1\}$ then

$$(Q - \tau \operatorname{Id})((1-\tau)^{-1}Q - \tau^{-1}(\operatorname{Id} - Q)) = \operatorname{Id} + (1-\tau)^{-1}\tau^{-1}(Q^2 - Q) \tag{6.118}$$

gives a parametrix for $Q - \tau \operatorname{Id}$. The right side is invertible for $|\tau|$ large and hence for all τ outside a discrete subset of $\mathbb{C} \setminus \{0, 1\}$ with inverse $\operatorname{Id} + S(\tau)$ where $S(\tau)$ is meromorphic with values in $\Psi^{-\infty}(M; E)$. Letting Γ be the circle of radius $\frac{1}{2} - \epsilon$ around the origin for $\epsilon > 0$ sufficiently small it follows that $Q - \tau \operatorname{Id}$ is invertible on Γ with inverse $((1-\tau)^{-1}Q - \tau^{-1}(\operatorname{Id} - Q))(\operatorname{Id} + S(\tau)$. Thus, by Cauchy's theorem,

$$\operatorname{Id} - P = \frac{1}{2\pi i} \oint_\Gamma (\tau - Q)^{-1} d\tau = \operatorname{Id} - Q + S,\ S \in \Psi^{-\infty}(M; E) \tag{6.119}$$

and moreover P is a projection since choosing Γ' to be a circle with slightly larger radius than Γ,

$$\begin{aligned}(\operatorname{Id} - P)^2 &= \frac{1}{2\pi i} \frac{2\pi i}{\oint}{}_{\Gamma'} \oint_\Gamma (\tau' - Q)^{-1}(\tau - Q)^{-1} d\tau' d\tau \\ &= \frac{1}{2\pi i} \frac{1}{2\pi i} \oint_{\Gamma'} \oint_\Gamma \left((\tau' - \tau)^{-1}(\tau' - Q)^{-1} + (\tau' - \tau)^{-1}(\tau - Q)^{-1}\right) d\tau' d\tau \\ &= \operatorname{Id} - P\end{aligned} \tag{6.120}$$

since in the first integral the integrand is holomorphic in τ inside Γ and in the second the τ' integral has a single pole at $\tau' = \tau$ inside Γ. $\square$

The following more qualitative version is used in the discussion of the Calderón projection .

PROPOSITION 6.11. *If M is compact, E is a complex vector bundle over M and $Q \in \Psi^0(M;E)$ is such that $Q^2 - Q \in \Psi^{-\infty}(M;E)$ and $F \subset H^s(M;E)$ is a closed subspace corresponding to which there are smoothing operators $A, B \in \Psi^{-\infty}(M;E)$ with $\operatorname{Id} - Q = A$ on F and $(Q+B)L^2(M;E) \subset F$ then there is a smoothing operator $B' \in \Psi^{-\infty}(M:E)$ such that $F = \operatorname{Ran}(Q+B')$ and $(Q+B')^2 = Q+B'$.*

PROOF. Assume first that $s = 0$, so F is a closed subspace of $L^2(X;E)$. Applying Proposition 6.10 to Q we may assume that it is a projection P, without affecting the other conditions. Consider the intersection $E = F \cap \operatorname{Ran}(\operatorname{Id} - P)$. This is a closed subspace of $L^2(M;E)$. With A as in the statement of the proposition, $E \subset \operatorname{Nul}(\operatorname{Id} - A)$. Indeed P vanishes on $\operatorname{Ran}(\operatorname{Id} - P)$ and hence on E and by hypothesis $\operatorname{Id} - P - A$ vanishes on F and hence on E. From the properties of smoothing operators, E is contained in a finite dimensional subspace of $\mathcal{C}^\infty(M;E)$, so is itself such a space. We may modify P by adding a smoothing projection onto E to it, and so assume that $F \cap \operatorname{Ran}(\operatorname{Id} - P) = \{0\}$.

Consider the sum $G = F + \operatorname{Ran}(\operatorname{Id} - P)$ and the operator $\operatorname{Id} + B = (P+B) + (\operatorname{Id} - P)$, with B as in the statement of the Proposition. The range of $\operatorname{Id} + B$ is contained in G. Thus G must be a closed subspace of $L^2(M;E)$ with a finite dimensional complement in $\mathcal{C}^\infty(M;E)$. Adding a smoothing projection onto such a complement we can, again by altering P by smoothing term, arrange that

$$L^2(M;E) = F \oplus \operatorname{Ran}(\operatorname{Id} - P) \tag{6.121}$$

is a (possibly non-orthogonal) direct sum. Since P has only been altered by a smoothing operator the hypotheses of the Proposition continue to hold. Let Π be the projection with range F and null space equal to the range of $\operatorname{Id} - P$. It follows that $P' = P + (\operatorname{Id} - P)RP$ for some bounded operator R (namely $R = (\operatorname{Id} - P)(P' - P)P$.) Then restricted to F, $P' = \operatorname{Id}$ and $P = \operatorname{Id} + A$ so $R = -A$ on F. In fact $R = AP \in \Psi^{-\infty}(M;E)$, since they are equal on F and both vanish on $\operatorname{Ran}(\operatorname{Id} - P)$. Thus P' differs from P by a smoothing operator.

The case of general s follows by conjugating with a pseudodifferential isomorphism of $H^s(M;E)$ to $L^2(M;E)$ since this preserves both the assumptions and the conclusions. $\square$

6.9. The Toeplitz algebra

6.10. Semiclassical algebra

Recall the notion of a semiclassical 1-parameter family of pseudodifferential operators (which we will nevertheless call a semiclassical operator) on Euclidean space in Section 2.19. Following the model in Section 6.3 above we can easily 'transfer' this definition to a manifold M, compact or not. The main thing to decide is what to require of the part of the kernel away from the diagonal. This however is clear from (2.210). Namely in any compact set of M^2 which does not meet the diagonal, the kernel should be smooth uniformly down to $\epsilon = 0$, including in ϵ itself, and it should vanish there to infinite order. This motivates the following definition modelled closely on Definition 6.2 and the discussion of operators between sections of vector bundles in Section 6.5. This time I have chosen to define the classical operators, of course the spaces $\Psi^m_{\text{sl-}\infty}(X,\mathbb{E})$ have a similar definition.

Definition 6.4. *If X is a $\mathcal{C}^\infty$ manifold and $\mathbb{E}=(E_+,E_-)$ is a pair of complex vector bundles over X then, for any $m\in\mathbb{R}$, $\Psi^m_{\text{sl}}(X;\mathbb{E})$ is the space of linear operators*

$$A_\epsilon:\mathcal{C}^\infty_c([0,1]\times X;E_+)\longrightarrow\mathcal{C}^\infty([0,1]\times X;E_-) \tag{6.122}$$

with the following properties. First,

$$\begin{gathered}\text{if } \phi,\ \psi\in\mathcal{C}^\infty(X) \text{ have disjoint supports then } \exists\ K_\epsilon\in\mathcal{C}^\infty([0,1]_\epsilon\times X^2;\Omega_R\otimes\operatorname{Hom}(\mathbb{E})),\\ K_\epsilon\equiv0 \text{ at } \{\epsilon=0\} \text{such that } \forall\ u\in\mathcal{C}^\infty_c([0,1]\times X;E_+)\ \phi A\psi u=\int\limits_X K_\epsilon(x,y)u(y),\end{gathered} \tag{6.123}$$

and secondly if $F:W\longrightarrow\mathbb{R}^n$ is a coordinate system in X over which $\mathbb{E}$ is trivial, with trivializations $h_\pm E_\pm\big|_W\longleftrightarrow W\times\mathbb{C}^{N_\pm}$, and $\psi\in\mathcal{C}^\infty_c(X)$ has support in W then

$$\begin{gathered}\exists\ B_\epsilon\in\Psi^m_{\text{sl}}(\mathbb{R}^n;\mathbb{C}^{N_+},\mathbb{C}^{N_-}),\ \operatorname{supp}(B_\epsilon)\subset[0,1]\times F(W)\times F(W)\ s.t.\\ \psi A_\epsilon\psi u\restriction W=h_-F^*(B_\epsilon((F^{-1})^*(h_+^{-1}\psi u)))\ \forall\ u\in\mathcal{C}^\infty_c([0,1]\times X;E_+).\end{gathered}$$

A semiclassical operator (always a family of course) is said to be properly supported if its kernel has proper support in $[0,1]\times X\times X$, that is proved the two maps

$$\begin{tikzcd} & \operatorname{supp}(B_\epsilon) \arrow[dl, "\pi_L"'] \arrow[dr, "\pi_R"] & \\ X & & X \end{tikzcd} \tag{6.124}$$

are both proper, meaning the inverse image of a compact set is compact. Since

$$\pi_X\operatorname{supp}(B_\epsilon u)\subset\pi_L(\operatorname{supp}(B_\epsilon)\cap\pi_R^{-1}(\pi_X\operatorname{supp}(u)) \tag{6.125}$$

(where $\operatorname{supp}(u)\subset[0,1]\times X$ and π_X is projection onto the second factor) it follows that a properly supported operator satisifes

$$B_\epsilon:\mathcal{C}^\infty_c([0,1]\times X;E_+)\longrightarrow\mathcal{C}^\infty_c([0,1]\times X;E_-). \tag{6.126}$$

The same is true of the adjoint, so in fact by duality

$$B_\epsilon:\mathcal{C}^\infty([0,1]\times X;E_+)\longrightarrow\mathcal{C}^\infty([0,1]\times X;E_-). \tag{6.127}$$

The discussion above now carries over to give similar results for semiclassical families.

Proposition 6.12. *The subspaces of properly supported semiclassial operators for any manifold have short exact symbol sequences*

$$\begin{gathered}0\hookrightarrow\Psi^{m-1}_{\text{sl}}(X)\hookrightarrow\Psi^m(X)\xrightarrow{\sigma_m}S^{m-[1]}(T^*X)\longrightarrow0,\\ 0\hookrightarrow\epsilon\Psi^m_{\text{sl}}(X)\hookrightarrow\Psi^m_{\text{sl}}(X)\xrightarrow{\sigma_m}\longrightarrow0,\end{gathered} \tag{6.128}$$

compose as operators (6.126) *and* (6.127) *and their symbols, standard and semiclassical, compose as well:*

$$\begin{gathered}\sigma_{m+m'}(A_\epsilon B_\epsilon)=\sigma_m(A_\epsilon)\circ\sigma_{m'}(B_\epsilon),\\ \sigma_{\text{sl}}(A_\epsilon B_\epsilon)=\sigma_{\text{sl}}(A_\epsilon)\circ\sigma_{\text{sl}}(B_\epsilon).\end{gathered} \tag{6.129}$$

The L^2 boundedness in Proposition 2.14 carries over easily to the manifold case.

Proposition 6.13. *If M is compact and E is a complex vector bundle over M then $A_\epsilon \in \Psi^0_{\mathrm{sl}}(M;E)$ then*

$$\sup_{0<\epsilon\le 1} \|A_\epsilon\|_{L^2(M;E)} < \infty. \tag{6.130}$$

We are particularly interested in semiclassical operators below because they make it possible to easily 'quantize' projections.

Proposition 6.14. *Suppose $p \in \mathcal{C}^\infty(\overline{{}^{sl}T^*X}; \hom(E))$ is a smooth family of projections for a compact manifold X then there exists a semiclassical family of projections $P_\epsilon \in \Psi^0_{\mathrm{sl}}(X;E)$ such that $\sigma_{\mathrm{sl}}(P_\epsilon) = p$.*

Proof. By the surjectivity of the semiclassical symbol map we can choose $A_\epsilon \in \Psi^0_{\mathrm{sl}}(X;E)$ with $\sigma_{\mathrm{sl}}(A_\epsilon) = p$ and we can arrange that $\sigma_0(A_\epsilon)$ is the constant family of projections defined by p on the sphere bundle at infinity. Then

$$A_\epsilon^2 - A_\epsilon = E_\epsilon \in \epsilon\Psi^{-1}_{\mathrm{sl}}(X;E). \tag{6.131}$$

Composing on the left in (6.131) gives the same result as composing on the right, so

$$A_\epsilon E_\epsilon = E_\epsilon A_\epsilon \Longrightarrow \sigma(e) = p\sigma(e)p + (\mathrm{Id} - p)\sigma(e)(\mathrm{Id} - p) \tag{6.132}$$

where the symbolic identity is true in both sense, for $\sigma(e) = \sigma_{\mathrm{sl}}(e)$ and $\sigma(e) = \sigma_{-1}(e)$.

Now, we wish to 'correct' A_ϵ so this error term is smoothing and vanishes to infinite order at $\epsilon = 0$. First we add the term

$$A^{(1)}_\epsilon = A_\epsilon A^{(1)}_\epsilon A_\epsilon - (\mathrm{Id} - A_\epsilon)A^{(1)}_\epsilon(\mathrm{Id} - A_\epsilon) \in \Psi^{-1}_{\mathrm{sl}}(X;E)$$

to A_ϵ. This modifies (6.131) to

$$(A_\epsilon + A^{(1)}_\epsilon)^2 - A_\epsilon - A^{(1)}_\epsilon = E_\epsilon + A_\epsilon A^{(1)}_\epsilon + A^{(1)}_\epsilon A_\epsilon - A^{(1)}_\epsilon \in \epsilon^2\Psi^{-2}_{\mathrm{sl}}(X;E). \tag{6.133}$$

Repeating this step generates an asymptotic solution and summing the asymptotic series gives a solution modulo rapidly decreasing smoothing error terms.

□

6.11. Heat kernel

6.12. Resolvent

6.13. Complex powers

6.14. Problems

Problem 6.1. Show that compatibility in the sense defined before Definition 6.1 is an equivalence relation on $\mathcal{C}^\infty$ structures. Conclude that there is a unique maximal $\mathcal{C}^\infty$ structure containing any give $\mathcal{C}^\infty$ structure.

Problem 6.2. Let $\mathcal{F}$ be a $\mathcal{C}^\infty$ structure on X and let O_a, $a \in A$, be a covering of X by coordinate neighbourhoods, in the sense of (6.2) and (6.3). Show that the maximal $\mathcal{C}^\infty$ structure containing $\mathcal{F}$ consists of ALL functions on X which are of the form (6.3) on each of these coordinate patches. Conclude that the maximal $\mathcal{C}^\infty$ structure is an algebra.

PROBLEM 6.3 (Partitions of unity). Show that any $\mathcal{C}^\infty$ manifold admits partitions of unity. That is, if O_a, $a \in A$, is an open cover of X then there exist elements $\rho_{a,i} \in \mathcal{C}^\infty(X)$, $a \in A$, $i \in \mathbb{N}$, with $0 \le \rho_{a,i} \le 1$, with each $\rho_{a,i}$ vanishing outside a compact subset $K_{a,i} \subset O_a$ such that only finite collections of the $\{K_{a,i}\}$ have non-trivial intersection and for which

$$\sum_{a \in A, i \in \mathbb{N}} \rho_{a,i} = 1.$$