CHAPTER 5

Microlocalization

5.1. Calculus of supports

Recall that we have already defined the support of a tempered distribution in the slightly round-about way:

$$\text{if } u \in \mathcal{S}'(\mathbb{R}^n),\ \operatorname{supp}(u) = \{x \in \mathbb{R}^n;\ \exists\ \phi \in \mathcal{S}(\mathbb{R}^n), \phi(x) \neq 0, \phi u = 0\}^{\complement}. \tag{5.1}$$

Now if $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is any continuous linear operator we can consider the support of the kernel:

$$\operatorname{supp}(A) = \operatorname{supp}(K_A) \subset \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}. \tag{5.2}$$

We write out the space as a product here to point to the fact that *any* subset of the product defines (is) a *relation* i.e. a map on subsets:

$$\begin{gathered} G \subset \mathbb{R}^n \times \mathbb{R}^n, \quad S \subset \mathbb{R}^n \Longrightarrow \\ G \circ S = \big\{x \in \mathbb{R}^n;\ \exists\ y \in S \text{ s.t. } (x, y) \in G\big\}. \end{gathered} \tag{5.3}$$

One can write this much more geometrically in terms of the two projection maps

$$\begin{tikzcd} & \mathbb{R}^{2n} \arrow[dl, "\pi_L"'] \arrow[dr, "\pi_R"] & \\ \mathbb{R}^n & & \mathbb{R}^n. \end{tikzcd} \tag{5.4}$$

Thus $\pi_R(x, y) = y$, $\pi_L(x, y) = x$. Then (5.3) can be written in terms of the action of maps on sets as

$$G \circ S = \pi_L\left(\pi_R^{-1}(S) \cap G\right). \tag{5.5}$$

From this it follows that if S is compact and G is closed, then $G \circ S$ is closed, since its intersection with any compact set is the image of a compact set under a continuous map, hence compact. Now, by the *calculus of supports* we mean the 'trivial' result.

Proposition 5.1. *If $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is a continuous linear map then*

$$\operatorname{supp}(A\phi) \subset \operatorname{supp}(A) \circ \operatorname{supp}(\phi)\ \forall\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n). \tag{5.6}$$

Proof. Since we want to *bound* $\operatorname{supp}(A\phi)$ we can use (5.1) directly, i.e. show that

$$x \notin \operatorname{supp}(A) \circ \operatorname{supp}(\phi) \Longrightarrow x \notin \operatorname{supp}(A\phi). \tag{5.7}$$

Since we know $\operatorname{supp}(A) \circ \operatorname{supp}(\phi)$ to be closed, the assumption that x is outside this set means that there exists $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with

$$\psi(x) \neq 0 \text{ and } \operatorname{supp}(\psi) \cap \operatorname{supp}(A) \circ \operatorname{supp}(\phi) = \emptyset.$$

From (5.3) or (5.5) this means

$$\operatorname{supp}(A) \cap (\operatorname{supp}(\psi) \times \operatorname{supp}(\phi)) = \emptyset \text{ in } \mathbb{R}^{2n}. \tag{5.8}$$

But this certainly implies that

$$\begin{gathered} K_A(x,y)\psi(x)\phi(y) = 0 \\ \Longrightarrow \psi A(\phi) = \int K_A(x,y)\psi(x)\phi(y)dy = 0. \end{gathered} \tag{5.9}$$

Thus we have proved (5.6) and the lemma. □

Diff ops.

5.2. Singular supports

As well as the support of a tempered distribution we can consider the singular support:

$$\operatorname{sing\,supp}(u) = \left\{x \in \mathbb{R}^n; \exists\ \phi \in \mathcal{S}(\mathbb{R}^n), \phi(x) \neq 0, \phi u \in \mathcal{S}(\mathbb{R}^n)\right\}^{\complement}. \tag{5.10}$$

Again this is a closed set since $x \notin \operatorname{sing\,supp}(u) \Longrightarrow \exists\ \phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi u \in \mathcal{S}(\mathbb{R}^n)$ and $\phi(x) \neq 0$ so $\phi(x') \neq 0$ for $|x - x'| < \epsilon$, some $\epsilon > 0$ and hence $x' \notin \operatorname{sing\,supp}(u)$ i.e. the complement of $\operatorname{sing\,supp}(u)$ is open.

Directly from the definition we have

$$\operatorname{sing\,supp}(u) \subset \operatorname{supp}(u)\ \forall\ u \in \mathcal{S}'(\mathbb{R}^n) \text{ and} \tag{5.11}$$

$$\operatorname{sing\,supp}(u) = \emptyset \Longleftrightarrow u \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{5.12}$$

Examples

5.3. Pseudolocality

We would like to have a result like (5.6) for singular support, and indeed we can get one for pseudodifferential operators. First let us work out the singular support of the kernels of pseudodifferential operators.

Proposition 5.2. *If $A \in \Psi^m_\infty(\mathbb{R}^n)$ then*

$$\operatorname{sing\,supp}(A) = \operatorname{sing\,supp}(K_A) \subset \left\{(x,y) \in \mathbb{R}^{2n}; x = y\right\}. \tag{5.13}$$

Proof. The kernel is defined by an oscillatory integral

$$I(a) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,y,\xi)d\xi. \tag{5.14}$$

If the order m is $< -n$ we can show by integration by parts that

$$(x-y)^\alpha I(a) = I\left((-D_\xi)^\alpha a\right), \tag{5.15}$$

and then this must hold by continuity for all orders. If a is of order m and $|\alpha| > m + n$ then $(-D_\xi)^\alpha a$ is of order less than $-n$, so

$$(x-y)^\alpha I(a) \in \mathcal{C}^0_\infty(\mathbb{R}^n), |\alpha| > m + n. \tag{5.16}$$

In fact we can also differentiate under the integral sign:

$$D^\beta_x D^\gamma_y (x-y)^\alpha I(a) = I\left(D^\beta_x D^\gamma_y (-D_\xi)^\alpha a\right) \tag{5.17}$$

so generalizing (5.16) to

$$(x-y)^\alpha I(a) \in \mathcal{C}^k_\infty(\mathbb{R}^n) \text{ if } |\alpha| > m + n + k. \tag{5.18}$$

This implies that $I(A)$ is $\mathcal{C}^\infty$ on the complement of the diagonal, $\{x = y\}$. This proves (5.13). □

An operator is said to be *pseudolocal* if it satisfies the condition

$$\operatorname{sing\,supp}(Au) \subset \operatorname{sing\,supp}(u)\ \forall\ u \in \mathcal{C}^{-\infty}(\mathbb{R}^n). \tag{5.19}$$

PROPOSITION 5.3. *Pseudodifferential operators are pseudolocal.*

PROOF. Suppose $u \in \mathcal{S}'(\mathbb{R}^n)$ has compact support and $\overline{x} \notin \operatorname{sing\,supp}(u)$. Then we can choose $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi \equiv 1$ near $\overline{x}$ and $\phi u \in \mathcal{S}(\mathbb{R}^n)$ (by definition). Thus

$$u = u_1 + u_2,\ u_1 = (1-\phi)u, \quad u_2 \in \mathcal{S}(\mathbb{R}^n). \tag{5.20}$$

Since $A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$, $Au_2 \in \mathcal{S}(\mathbb{R}^n)$ so

$$\operatorname{sing\,supp}(Au) = \operatorname{sing\,supp}(Au_1) \text{ and } \overline{x} \notin \operatorname{supp}(u_1). \tag{5.21}$$

Choose $\psi \in \mathcal{S}(\mathbb{R}^n)$ with compact support, $\psi(\overline{x}) = 1$ and

$$\operatorname{supp}(\psi) \cap \operatorname{supp}(1-\phi) = \emptyset. \tag{5.22}$$

Thus

$$\psi Au_1 = \psi A(1-\phi)u = \tilde{A}u \tag{5.23}$$

where

$$K_{\tilde{A}}(x,y) = \psi(x)K_A(x,y)(1-\phi(y)). \tag{5.24}$$

Combining (5.22) and (5.13) shows that $K_{\tilde{A}} \in \Psi_{\infty}^{-\infty}(\mathbb{R}^n)$ so, by Lemma 2.8, $\tilde{A}u \in \mathcal{C}^\infty(\mathbb{R}^n)$ and $\overline{x} \notin \operatorname{sing\,supp}(Au)$ by (5.13)(?). This proves the proposition. □

5.4. Coordinate invariance

If $\Omega \subset \mathbb{R}^n$ is an open set, put

$$\begin{aligned}\mathcal{C}_c^\infty(\Omega) &= \big\{u \in \mathcal{S}(\mathbb{R}^n); \operatorname{supp}(u) \Subset \Omega\big\}\\ \mathcal{C}_c^{-\infty}(\Omega) &= \big\{u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{supp}(u) \Subset \Omega\big\}\end{aligned} \tag{5.25}$$

respectively the space of $\mathcal{C}^\infty$ functions of compact support in Ω and of distributions of compact support in Ω. Here $K \Subset \Omega$ indicates that K is a compact subset of Ω. Notice that if $u \in \mathcal{C}_c^{-\infty}(\Omega)$ then u defines a continuous linear functional

$$\mathcal{C}^\infty(\Omega) \ni \phi \longmapsto u(\phi) = u(\psi\phi) \in \mathbb{C} \tag{5.26}$$

where if $\psi \in \mathcal{C}_c^\infty(\Omega)$ is chosen to be identically one near $\operatorname{supp}(u)$ then (5.26) is independent of ψ. [Think about what continuity means here!]

Now suppose

$$F : \Omega \longrightarrow \Omega' \tag{5.27}$$

is a diffeomorphism between open sets of $\mathbb{R}^n$. The pull-back operation is

$$F^* :\ \mathcal{C}_c^\infty(\Omega') \longleftrightarrow \mathcal{C}_c^\infty(\Omega),\ F^*\phi = \phi \circ F. \tag{5.28}$$

LEMMA 5.1. *If F is a diffeomorphism,* (5.27)*, between open sets of $\mathbb{R}^n$ then there is an extension by continuity of* (5.28) *to*

$$F^* : \mathcal{C}_c^{-\infty}(\Omega') \longleftrightarrow \mathcal{C}_c^{-\infty}(\Omega). \tag{5.29}$$

PROOF. The density of $\mathcal{C}_c^\infty(\Omega)$ in $\mathcal{C}_c^{-\infty}(\Omega)$, in the weak topology given by the seminorms from (5.26), can be proved in the same way as the density of $\mathcal{S}(\mathbb{R}^n)$ in $\mathcal{S}'(\mathbb{R}^n)$ (see Problem 5.5). Thus, we only need to show continuity of (5.29) in this sense. Suppose $u \in \mathcal{C}_c^\infty(\Omega)$ and $\phi \in \mathcal{C}_c^\infty(\Omega')$ then

$$\begin{aligned}(F^*u)(\phi) &= \int u(F(x))\phi(x)dx \\ &= \int u(y)\phi(G(y))|J_G(y)|dy\end{aligned} \tag{5.30}$$

where $J_G(y) = \left(\frac{\partial G(y)}{\partial y}\right)$ is the Jacobian of G, the inverse of F. Thus (5.28) can be written

$$F^*u(\phi) = \left(|J_G|u\right)(G^*\phi) \tag{5.31}$$

and since $G^* : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega')$ is continuous (!) we conclude that F^* is continuous as desired. $\square$

Now suppose that

$$A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$$

has

$$\operatorname{supp}(A) \Subset \Omega \times \Omega \subset \mathbb{R}^{2n}. \tag{5.32}$$

Then

$$A : \mathcal{C}_c^\infty(\Omega) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega) \tag{5.33}$$

by Proposition 5.1. Applying a diffeomorphism, F, as in (5.27) set

$$A_F : \mathcal{C}_c^\infty(\Omega') \longrightarrow \mathcal{C}_c^{-\infty}(\Omega'),\ A_F = G^* \circ A \circ F^*. \tag{5.34}$$

LEMMA 5.2. *If A satisfies* (5.32) *and F is a diffeomorphism* (5.27) *then*

$$K_{A_F}(x,y) = (G \times G)^* K \cdot |J_G(y)| \text{ on } \Omega' \times \Omega' \tag{5.35}$$

has compact support in $\Omega' \times \Omega'$.

PROOF. Essentially the same as that of (5.30). $\square$

PROPOSITION 5.4. *Suppose $A \in \Psi_\infty^m(\mathbb{R}^n)$ has kernel satisfying* (5.32) *and F is a diffeomorphism as in* (5.27) *then A_F, defined by* (5.34), *is an element of $\Psi_\infty^m(\mathbb{R}^n)$.*

PROOF. See Proposition 2.11. $\square$

5.5. Problems

PROBLEM 5.1. Show that *Weyl quantization*

$$S_\infty^\infty(\mathbb{R}^n;\mathbb{R}^n) \ni a \longmapsto q_W(a) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(\frac{x+y}{2},\xi)d\xi \tag{5.36}$$

is well-defined by continuity from $S_\infty^{-\infty}(\mathbb{R}^n;\mathbb{R}^n)$ and induces an isomorphism

$$S_\infty^m(\mathbb{R}^n;\mathbb{R}^n) \underset{q_W}{\overset{\sigma_W}{\rightleftarrows}} \Psi_\infty^m(\mathbb{R}^n)\ \forall\ m \in \mathbb{R}. \tag{5.37}$$

Find an asymptotic formula relating $q_W(A)$ to $q_L(A)$ for any $A \in \Psi_\infty^m(\mathbb{R}^n)$.

PROBLEM 5.2. Show that if $A \in \Psi_\infty^m(\mathbb{R}^n)$ then $A^* = A$ if and only if $\sigma_W(A)$ is real-valued.

PROBLEM 5.3. Is it true that every $E \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ defines a map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$?

PROBLEM 5.4. Show that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ by proving that if $\phi \in \mathcal{C}^\infty(\mathbb{R}^n)$ has compact support and is identically equal to 1 near the origin then

$$u_n(x) = (2\pi)^{-n}\phi(\frac{x}{n})\int e^{ix\cdot\xi}\phi(\xi/n)\hat{u}(\xi)d\xi \in \mathcal{S}(\mathbb{R}^n) \text{ if } u \in L^2(\mathbb{R}^n) \tag{5.38}$$

and $u_n \to u$ in $L^2(\mathbb{R}^n)$. Can you see any relation to pseudodifferential operators here?

PROBLEM 5.5. Check carefully that with the definition

$$H^k(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n); u = \sum_{|\alpha|\le -k} D^\alpha u_\alpha,\ u_\alpha \in L^2(\mathbb{R}^n) \right\} \tag{5.39}$$

for $-k \in \mathbb{N}$ one does have

$$u \in H^k(\mathbb{R}^n) \Longleftrightarrow \langle D\rangle^k u \in L^2(\mathbb{R}^n) \tag{5.40}$$

as claimed in the text.

PROBLEM 5.6. Suppose that $a(x) \in \mathcal{C}_\infty^\infty(\mathbb{R}^n)$ and that $a(x) \ge 0$. Show that the operator

$$A = \sum_{j=1}^n D_{x_j}^2 + a(x) \tag{5.41}$$

can have no solution which is in $L^2(\mathbb{R}^n)$.

PROBLEM 5.7. Show that for any open set $\Omega \subset \mathbb{R}^n$, $\mathcal{C}_c^\infty(\Omega)$ is dense in $\mathcal{C}_c^{-\infty}(\Omega)$ in the weak topology.

PROBLEM 5.8. Use formula (2.204) to find the principal symbol of A_F; more precisely show that if $F^* : T^*\Omega' \longrightarrow T^*\omega$ is the (co)-differential of F then

$$\sigma_m(A_F) = \sigma_m(A) \circ F^*.$$

We have now studied special distributions, the Schwartz kernels of pseudodifferential operators. We shall now apply this knowledge to the study of general distributions. In particular we shall examine the wavefront set, a refinement of singular support, of general distributions. This notion is fundamental to the general idea of 'microlocalization.'

5.6. Characteristic variety

If $A \in \Psi_\infty^m(\mathbb{R}^n)$, the left-reduced symbol is elliptic at $(\overline{x}, \overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$ if there exists $\epsilon > 0$ such that

$$\begin{gathered}\left|\sigma_L(A)(x,\xi)\right| \ge \epsilon|\xi|^m \quad \text{in} \\ \left\{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})\,;|x-\overline{x}| \le \epsilon,\ \left|\frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\overline{\xi}|}\right| \le \epsilon,\ |\xi| \ge \frac{1}{\epsilon}\right\}.\end{gathered} \tag{5.42}$$

Directly from the definition, ellipticity at $(\overline{x}, \overline{\xi})$ is actually a property of the principal symbol, $\sigma_m(A)$ and if A is elliptic at $(\overline{x}, \overline{\xi})$ then it is elliptic at $(\overline{x}, t\overline{\xi})$ for any $t > 0$. Clearly

$$\left\{(\overline{x}, \overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\});\ A \text{ is elliptic (of order } m) \text{ at } (\overline{x}, \overline{\xi})\right\}$$

is an *open* cone in $\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$. The complement

$$\Sigma_m(A) = \{(\bar{x},\bar{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})\,;\ A \text{ is } \textit{not} \text{ elliptic of order } m \text{ at } (\bar{x},\bar{\xi})\} \tag{5.43}$$

is therefore a closed conic subset of $\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$; it is the characteristic set (or variety) of A. Since the product of two symbols is only elliptic at $(\bar{x},\bar{\xi})$ if they are both elliptic there, if follows from the composition properties of pseudodifferential operators that

$$\Sigma_{m+m'}(A \circ B) = \Sigma_m(A) \cup \Sigma_{m'}(B). \tag{5.44}$$

5.7. Wavefront set

We adopt the following bald definition:

$$\begin{gathered}\text{If } u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n);\ \operatorname{supp}(u) \Subset \mathbb{R}^n\} \text{ then}\\ \operatorname{WF}(u) = \bigcap\{\Sigma_0(A);\ A \in \Psi^0_\infty(\mathbb{R}^n) \text{ and } Au \in \mathcal{C}^\infty(\mathbb{R}^n)\}.\end{gathered} \tag{5.45}$$

Thus $\operatorname{WF}(u) \subset \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$ is always a closed conic set, being the intersection of such sets. The first thing we wish to show is that $\operatorname{WF}(u)$ is a refinement of $\operatorname{sing\,supp}(u)$. Let

$$\pi:\ \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\}) \ni (x,\xi) \longmapsto x \in \mathbb{R}^n \tag{5.46}$$

be projection onto the first factor.

Proposition 5.5. *If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ then*

$$\pi(\operatorname{WF}(u)) = \operatorname{sing\,supp}(u). \tag{5.47}$$

Proof. The inclusion $\pi(\operatorname{WF}(u)) \subset \operatorname{sing\,supp}(w)$ is straightforward. Indeed, if $\bar{x} \notin \operatorname{sing\,supp}(u)$ then there exists $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\phi(\bar{x}) \neq 0$ such that $\phi u \in \mathcal{C}^\infty(\mathbb{R}^n)$. Of course as a multiplication operator, $\phi \in \Psi^0_\infty(\mathbb{R}^n)$ and $\Sigma_0(\phi) \not\ni (\bar{x},\bar{\xi})$ for any $\bar{\xi} \neq 0$. Thus the definition (5.45) shows that $(\bar{x},\bar{\xi}) \notin \operatorname{WF}(u)$ for all $\bar{\xi} \in \mathbb{R}^n\diagdown 0$ proving the inclusion.

Using the calculus of pseudodifferential operators, the opposite inclusion,

$$\pi(\operatorname{WF}(u)) \supset \operatorname{sing\,supp}(u) \tag{5.48}$$

is only a little more complicated. Thus we have to show that if $(\bar{x},\bar{\xi}) \notin \operatorname{WF}(u)$ for all $\bar{\xi} \in \mathbb{R}^n\diagdown 0$ then $\bar{x} \notin \operatorname{sing\,supp}(u)$. The hypothesis is that for each $(\bar{x},\bar{\xi}), \bar{\xi} \in \mathbb{R}^n\diagdown 0$, there exists $A \in \Psi^0_\infty(\mathbb{R}^n)$ such that A is elliptic at $(\bar{x},\bar{\xi})$ and $Au \in \mathcal{C}^\infty(\mathbb{R}^n)$. The set of elliptic points is open so there exists $\epsilon = \epsilon(\bar{\xi}) > 0$ such that A is elliptic on

$$\{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n\diagdown 0); |x-\bar{x}| < \epsilon, \Big|\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\bar{\xi}|}\Big| < \epsilon\}. \tag{5.49}$$

Let B_j, $j = 1,\dots,N$ be a finite set of such operators associated to $\bar{\xi}_j$ and such that the corresponding sets in (5.49) cover $\{\bar{x}\} \times (\mathbb{R}^n\diagdown 0)$; the finiteness follows from the compactness of the sphere. Then consider

$$B = \sum_{j=1}^N B_j^* B_j \Longrightarrow Bu \in \mathcal{C}^\infty(\mathbb{R}^n).$$

This operator B is elliptic at $(\overline{x},\xi)$, for all $\xi \neq 0$. Thus if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $0 \le \phi(x) \le 1$, has support sufficiently close to $\overline{x}$, $\phi(x) = 1$ in $|x-\overline{x}| < \epsilon/2$ then, since B has non-negative principal symbol

$$B + (1-\phi) \in \Psi^0_\infty(\mathbb{R}^n) \tag{5.50}$$

is globally elliptic. Thus, by Lemma 2.7, there exists $G \in \Psi^0_\infty(\mathbb{R}^n)$ which is a parametrix for $B + (1-\phi)$:

$$\mathrm{Id} \equiv G \circ B + G(1-\phi) \mod \Psi^{-\infty}_\infty(\mathbb{R}^n). \tag{5.51}$$

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}(\psi) \subset \{\phi = 1\}$ and $\psi(\overline{x}) \neq 0$. Then, from the reduction formula

$$\psi \circ G \circ (1-\phi) \in \Psi^{-\infty}_\infty(\mathbb{R}^n).$$

Thus from (5.51) we find

$$\psi u = \psi G \circ Bu + \psi G(1-\phi)u \in \mathcal{C}^\infty(\mathbb{R}^n).$$

Thus $\overline{x} \notin \operatorname{sing\,supp}(u)$ and the proposition is proved. □

We extend the definition to general tempered distributions by setting

$$\mathrm{WF}(u) = \bigcup_{\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)} \mathrm{WF}(\phi u),\ u \in \mathcal{S}'(\mathbb{R}^n). \tag{5.52}$$

Then (5.47) holds for every $u \in \mathcal{S}'(\mathbb{R}^n)$.

5.8. Essential support

Next we shall consider the notion of the essential support of a pseudodifferential operator. If $a \in S^m_\infty(\mathbb{R}^N;\mathbb{R}^n)$ we define the cone support of a by

$$\begin{aligned} \operatorname{cone\,supp}(a) = &\big\{(\overline{x},\overline{\xi}) \in \mathbb{R}^N \times (\mathbb{R}^n \setminus 0); \exists\ \epsilon > 0 \text{ and } \forall\ M \in \mathbb{R}, \exists\ C_M \text{ s.t.} \\ &|a(x,\xi)| \le C_M \langle\xi\rangle^{-M} \text{ if } |x-\overline{x}| \le \epsilon, \Big|\frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\overline{\xi}|}\Big| \le \epsilon\big\}^{\complement}. \end{aligned} \tag{5.53}$$

This is clearly a closed conic set in $\mathbb{R}^N \times (\mathbb{R}^n \setminus 0)$. By definition the symbol decays rapidly outside this cone, in fact even more is true.

Lemma 5.3. *If $a \in S^\infty_\infty(\mathbb{R}^N;\mathbb{R}^n)$ then*

$$\begin{aligned} &(\overline{x},\overline{\eta}) \notin \operatorname{cone\,supp}(a) \Longrightarrow \\ &\exists\ \epsilon > 0 \textit{ s.t. } \forall\ M,\alpha,\beta\ \exists\ C_M \textit{ with} \\ &\big|D_x^\alpha D_\xi^\beta a(x,\eta)\big| \le C_M\langle\eta\rangle^{-M} \textit{ if } |x-\overline{x}| < \epsilon, \Big|\frac{\eta}{|\eta|} - \frac{\overline{\eta}}{|\overline{\eta}|}\Big| < \epsilon. \end{aligned} \tag{5.54}$$

Proof. To prove (5.54) it suffices to show it to be valid for $D_{x_j}a$, $D_{\xi_k}a$ and then use an inductive argument, i.e. to show that

$$\operatorname{cone\,supp}(D_{x_j}a), \operatorname{cone\,supp}(D_{\xi_k}a) \subset \operatorname{cone\,supp}(a). \tag{5.55}$$

Arguing by contradiction suppose that $D_{x_\ell}a$ does not decay to order M in any cone around $(\overline{x},\overline{\xi}) \notin \operatorname{cone\,supp}$. Then there exists a sequence (x_j,ξ_j) with

$$\begin{cases} x_j \longrightarrow \overline{x},\ \big|\frac{\xi_j}{|\xi_j|} - \frac{\overline{\xi}}{|\overline{\xi}|}\big| \longrightarrow 0,\ |\xi_j| \longrightarrow \infty \\ \text{and } \big|D_{x_\ell}a(x_j,\xi_j)\big| > j\langle\xi_j\rangle^M. \end{cases} \tag{5.56}$$

We can assume that $M < m$, since $a \in S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^N\right)$. Applying Taylor's formula with remainder, and using the symbol bounds on $D^2_{x_j}a$, gives

$$a(x_j+te_\ell,\xi_j)=a(x_j,\xi_j)+it(D_{x_j}a)(x_j,\xi_j)+O\left(t^2\langle\xi_j\rangle^m\right),\ (e_\ell)_j=\delta_{\ell j} \tag{5.57}$$

providing $|t|<1$. Taking $t=\langle\xi_j\rangle^{M-m}\longrightarrow 0$ as $j\longrightarrow\infty$, the first and third terms on the right in (5.57) are small compared to the second, so

$$\left|a\left(x_j+\langle\xi_j\rangle^{\frac{M-m}{2}},\xi_j\right)\right|>\langle\xi_j\rangle^{2M-m}, \tag{5.58}$$

contradicting the assumption that $(\overline{x},\overline{\xi})\notin\operatorname{cone\,supp}(a)$. A similar argument applies to $D_{\xi_\ell}a$ so (5.54), and hence the lemma, is proved. □

For a pseudodifferential operator we define the *essential support* by

$$\mathrm{WF}'(A)=\operatorname{cone\,supp}\left(\sigma_L(A)\right)\subset\mathbb{R}^n\times(\mathbb{R}^n\setminus 0). \tag{5.59}$$

Lemma 5.4. *For every $A\in\Psi^m_\infty(\mathbb{R}^n)$*

$$\mathrm{WF}'(A)=\operatorname{cone\,supp}(\sigma_R(A)). \tag{5.60}$$

Proof. Using (5.54) and the formula relating $\sigma_R(A)$ to $\sigma_L(A)$ we conclude that

$$\operatorname{cone\,supp}(\sigma_L(A))=\operatorname{cone\,supp}(\sigma_R(A)), \tag{5.61}$$

from which (5.60) follows. □

A similar argument shows that

$$\mathrm{WF}'(A\circ B)\subset\mathrm{WF}'(A)\cap\mathrm{WF}'(B). \tag{5.62}$$

Indeed the asymptotic formula for $\sigma_L(A\circ B)$ in terms of $\sigma_L(A)$ and $\sigma_L(B)$ shows that

$$\operatorname{cone\,supp}(\sigma_L(A\circ B))\subset\operatorname{cone\,supp}\left(\sigma_L(A)\right)\cap\operatorname{cone\,supp}\left(\sigma_L(B)\right) \tag{5.63}$$

which is the same thing.

5.9. Microlocal parametrices

The concept of essential support allows us to refine the notion of a parametrix for an elliptic operator to that of a *microlocal parametrix.*

Lemma 5.5. *If $A\in\Psi^m_\infty(\mathbb{R}^n)$ and $z\notin\Sigma_m(A)$ then there exists a microlocal parametrix at z, $B\in\Psi^{-m}_\infty(\mathbb{R}^n)$ such that*

$$z\notin\mathrm{WF}'(\mathrm{Id}-AB)\ \textit{and}\ z\notin\mathrm{WF}'(\mathrm{Id}-BA). \tag{5.64}$$

Proof. If $z=(\overline{x},\overline{\xi})$, $\overline{\xi}\neq 0$, consider the symbol

$$\gamma_\epsilon(x,\xi)=\phi\left(\frac{x-\overline{x}}{\epsilon}\right)(1-\phi)(\epsilon\xi)\phi\left(\left(\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|}\right)/\epsilon\right) \tag{5.65}$$

where as usual $\phi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$, $\phi(\zeta)=1$ in $|\zeta|\leq\frac{1}{2}$, $\phi(\zeta)=0$ in $|\zeta|\geq 1$. Thus $\gamma_\epsilon\in S^0_\infty(\mathbb{R}^n;\mathbb{R}^n)$ has support in

$$|x-\overline{x}|\leq\epsilon,\ |\xi|\geq\frac{1}{2\epsilon},\ \left|\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|}\right|\leq\epsilon \tag{5.66}$$

and is identically equal to one, and hence elliptic, on a similar smaller set

$$|x-\overline{x}|<\frac{\epsilon}{2}, |\xi|\geq\frac{1}{\epsilon},\ \left|\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|}\right|\leq\frac{\epsilon}{2}. \tag{5.67}$$

Define $L_\epsilon\in\Psi^0_\infty(\mathbb{R}^n)$ by $\sigma_L(L_\epsilon)=\gamma_\epsilon$. Thus, for any $\epsilon>0$,

$$z\notin \mathrm{WF}'(\mathrm{Id}-L_\epsilon),\ \mathrm{WF}'(L_\epsilon)\subset\left\{(x,\xi);|x-\overline{x}|\leq\epsilon\text{ and }\left|\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|}\right|\leq\epsilon\right\}. \tag{5.68}$$

Let $G_{2m}\in\Psi^{2m}_\infty(\mathbb{R}^n)$ be a globally elliptic operator with positive principal symbol. For example take $\sigma_L(G_{2m})=(1+|\xi|^2)^m$, so $G_s\circ G_t=G_{s+t}$ for any s, $t\in\mathbb{R}$. Now consider the operator

$$J=(\mathrm{Id}-L_\epsilon)\circ G_{2m}+A^*A\in\Psi^{2m}_\infty(\mathbb{R}^n). \tag{5.69}$$

The principal symbol of J is $(1-\gamma_\epsilon)(1+|\xi|^2)^m+|\sigma_m(A)|^2$ which is globally elliptic if $\epsilon>0$ is small enough (so that $\sigma_m(A)$ is elliptic on the set (5.66)). According to Lemma 2.75, J has a global parametrix $H\in\Psi^{-2m}_\infty(\mathbb{R}^n)$. Then

$$B=H\circ A^*\in\Psi^{-m}_\infty(\mathbb{R}^n) \tag{5.70}$$

is a microlocal right parametrix for A in the sense that $B\circ A-\mathrm{Id}=R_R$ with $z\notin\mathrm{WF}'(R_R)$ since

$$\begin{aligned}R_R=B\circ A-\mathrm{Id}&=H\circ A^*\circ A-\mathrm{Id}\\&=(H\circ J-\mathrm{Id})+H\circ(\mathrm{Id}-L_\epsilon)G_{2m}\circ A\end{aligned} \tag{5.71}$$

and the first term on the right is in $\Psi^{-\infty}_\infty(\mathbb{R}^n)$ whilst z is not in the operator wavefront set of $(\mathrm{Id}-L_\epsilon)$ and hence not in the operator wavefront set of the second term.

By a completely analogous construction we can find a left microlocal parametrix. Namely $(\mathrm{Id}-L_\epsilon)\circ G_{2m}+A\circ A^*$ is also globally elliptic with parametrix H' and then $B'=A^*\circ H'$ satisfies

$$B'\circ A-\mathrm{Id}=R_L,\ z\notin\mathrm{WF}'(R_L). \tag{5.72}$$

Then, as usual,

$$B=(B'\circ A-R_L)\,B=B'\,(A\circ B)-R_LB=B'+B'R_R-R_LB \tag{5.73}$$

so $z\notin\mathrm{WF}'(B-B')$, which implies that B is both a left and right microlocal parametrix. □

In fact this argument shows that such a left parametrix is essentially unique. See Problem 5.29.

5.10. Microlocality

Now we can consider the relationship between these two notions of wavefront set.

PROPOSITION 5.6. *Pseudodifferential operators are* microlocal *in the sense that*

$$\mathrm{WF}(Au)\subset\mathrm{WF}'(A)\cap\mathrm{WF}(u)\quad\forall\ A\in\Psi^\infty_\infty(\mathbb{R}^n),\ u\in\mathcal{C}^{-\infty}_c(\mathbb{R}^n). \tag{5.74}$$

PROOF. We need to show that

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \text{ and } \mathrm{WF}(Au) \subset \mathrm{WF}(u). \tag{5.75}$$

the second being the usual definition of microlocality. The first inclusion is easy. Suppose $(\overline{x},\overline{\xi}) \notin \operatorname{cone\,supp} \sigma_L(A)$. If we choose $B \in \Psi^0_\infty(\mathbb{R}^n)$ with $\sigma_L(B)$ supported in a small cone around $(\overline{x},\overline{\xi})$ then we can arrange

$$(\overline{x},\overline{\xi}) \notin \Sigma_0(B),\ \mathrm{WF}'(B) \cap \mathrm{WF}'(A) = \emptyset. \tag{5.76}$$

Then from (5.62), $\mathrm{WF}'(BA) = \emptyset$ so $BA \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$ and $BAu \in \mathcal{C}^\infty(\mathbb{R}^n)$. Thus $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(Au)$.

Similarly suppose $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(u)$. Then there exists $G \in \Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at $(\overline{x},\overline{\xi})$ with $Gu \in \mathcal{C}^\infty(\mathbb{R}^n)$. Let B be a microlocal parametrix for G at $(\overline{x},\overline{\xi})$ as in Lemma 5.5. Thus

$$u = BGu + Su,\ (\overline{x},\overline{\xi}) \notin \mathrm{WF}'(S). \tag{5.77}$$

Now apply A to this identity. Since, by assumption, $Gu \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ the first term on the right in

$$Au = ABGu + ASu \tag{5.78}$$

is smooth. Since, by (5.62), $(\overline{x},\overline{\xi}) \notin \mathrm{WF}'(AS)$ it follows from the first part of the argument above that $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(ASu)$ and hence $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(Au)$. □

We can deduce from the existence of microlocal parametrices at elliptic points a partial converse of (8.24).

PROPOSITION 5.7. *For any $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ and any $A \in \Psi^m_\infty(\mathbb{R}^n)$*

$$\mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \Sigma_m(A). \tag{5.79}$$

PROOF. If $(\overline{x},\overline{\xi}) \notin \Sigma_m(A)$ then, by definition, A is elliptic at $(\overline{x},\overline{\xi})$. Thus, by Lemma 5.5, A has a microlocal parametrix B, so

$$u = BAu + Su,\ (\overline{x},\overline{\xi}) \notin \mathrm{WF}'(S). \tag{5.80}$$

It follows that $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(Au)$ implies that $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(u)$ proving the Proposition. □

5.11. Explicit formulations

From this discussion of $\mathrm{WF}'(A)$ we can easily find a 'local coordinate' formulations of $\mathrm{WF}(u)$ in general.

LEMMA 5.6. *If $(\overline{x},\overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ then $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(u)$ if and only if there exists $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\phi(\overline{x}) \neq 0$ such that for some $\epsilon > 0$, and for all M there exists C_M with*

$$\big|\widehat{\phi u}(\xi)\big| \le C_M \langle\xi\rangle^M \text{ in } \Big|\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\xi|}\Big| < \epsilon. \tag{5.81}$$

PROOF. If $\zeta \in \mathcal{C}^\infty(\mathbb{R})$, $\zeta(\xi) \equiv 1$ in $|\xi| < \frac{\epsilon}{2}$ and $\operatorname{supp}(\zeta) \subset \big[\frac{-3\epsilon}{4}, \frac{3\epsilon}{4}\big]$ then

$$\gamma(\xi) = (1-\zeta)(\xi) \cdot \zeta\big(\frac{\xi}{|\xi|} - \frac{\overline{x}}{|\overline{x}|}\big) \in S^0_\infty(\mathbb{R}^n) \tag{5.82}$$

is elliptic at $\bar{\xi}$ and from (5.81)

$$\gamma(\xi) \cdot \widehat{\phi u}(\xi) \in \mathcal{S}(\mathbb{R}^n). \tag{5.83}$$

Thus if $\sigma_R(A) = \phi_1(x)\gamma(\xi)$ then $A(\phi_2 u) \in \mathcal{C}^\infty$ where $\phi_1\phi_2 = \phi$, $\phi_1(\overline{x}), \phi_2(\overline{x}) \neq 0$, $\phi_1, \phi_2 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Thus $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$. Conversely if $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$ and A is chosen as above then $A(\phi_1 u) \in \mathcal{S}(\mathbb{R}^n)$ and Lemma 5.6 holds. □

5.12. Wavefront set of K_A

At this stage, a natural thing to look at is the wavefront set of the kernel of a pseudodifferential operator, since these kernels are certainly an interesting class of distributions.

Proposition 5.8. *If $A \in \Psi_\infty^m(\mathbb{R}^n)$ then*

$$\begin{aligned} \mathrm{WF}(K_A) = \Big\{ &(x, y, \xi, \eta) \in \mathbb{R}^{2n} \times \big(\mathbb{R}^{2n} \setminus 0\big)\,; \\ &x = y,\ \xi + \eta = 0 \text{ and } (x, \xi) \in \mathrm{WF}'(A)\Big\}. \end{aligned} \tag{5.84}$$

In particular this shows that $\mathrm{WF}'(A)$ determines $\mathrm{WF}(K_A)$ and conversely.

Proof. Using Proposition 5.5 we know that $\pi\left(\mathrm{WF}(K_A)\right) \subset \big\{(x,x)\big\}$ so

$$\mathrm{WF}(K_A) \subset \big\{(x, x; \xi, \eta)\big\}.$$

To find the wave front set more precisely consider the kernel

$$K_A(x, y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} b(x, \xi) d\xi$$

where we can assume $|x - y| < 1$ on $\mathrm{supp}(K_A)$. Thus is $\phi \in \mathcal{C}_c^\infty(X)$ then

$$g(x, y) = K_A(x, y) \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$$

and

$$\begin{aligned} \hat{g}(\zeta, \eta) &= (2\pi)^{-n} \int e^{-i\zeta x - i\eta y} e^{i(x-y)\cdot\zeta} (\phi b)(x, \xi) d\zeta dx dy \\ &= \int e^{-i(\zeta+\eta)\cdot x} (\phi b)(x, -\eta) dx \\ &= \widehat{\phi b}(\zeta + \eta, -\eta). \end{aligned}$$

The fact that ϕb is a symbol of compact support in x means that for every M

$$\left|\widehat{\phi b}(\zeta + \eta, -\eta)\right| \leq C_M \left(\langle \zeta + \eta \rangle\right)^{-M} \langle \eta \rangle^m.$$

This is rapidly decreasing if $\zeta \neq -\eta$, so

$$\mathrm{WF}(K_A) \subset \big\{(x, x, \eta, -\eta)\big\} \text{ as claimed.}$$

Moreover if $(\overline{x}, \overline{\eta}) \notin \mathrm{WF}'(A)$ then choosing ϕ to have small support near $\overline{x}$ makes $\widehat{\phi b}$ rapidly decreasing near $-\overline{\eta}$ for all ζ. This proves Proposition 5.8. □

5.13. Hypersurfaces and Hamilton vector fields

In the Hamiltonian formulation of classical mechanics the dynamical behaviour of a 'particle' is fixed by the choice of an energy function ('the Hamiltonian') $h(x, \xi)$ depending on the position and momentum vectors (both in $\mathbb{R}^3$ you might think, but maybe in $\mathbb{R}^{3N}$ because there are really N particles). In fact one can think of a system confined to a surface in which case the variables are in the cotangent bundle of a manifold. However, in the local coordinate description the motion of the particle is given by Hamilton's equations:-

$$\frac{dx_i}{dt} = \frac{\partial h}{\partial \xi_i}(x,\xi),\ \frac{d\xi_i}{dt} = -\frac{\partial h}{\partial x_i}(x,\xi). \tag{5.85}$$

This means that the trajectory $(x(t),\xi(t))$ of a particle is an integral curve of the vector field

$$H_h(x,\xi) = \sum_i \left(\frac{\partial h}{\partial \xi_i}(x,\xi)\frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i}(x,\xi)\frac{\partial}{\partial \xi_i}(x,\xi)\right). \tag{5.86}$$

This, of course, is called the Hamilton vector field of h. The most important basic fact is that h itself is constant along integral curves of H_h, namely

$$H_h h = \sum_i \left(\frac{\partial}{\partial h \xi_i}(x,\xi)\frac{\partial h}{\partial x_i}(x,\xi) - \frac{\partial h}{\partial x_i}(x,\xi)\frac{\partial h}{\partial \xi_i}(x,\xi)h(x,\xi)\right) = 0. \tag{5.87}$$

More generally the action of H_h on any other function defines the Poisson bracket between h and g and

$$H_h g = \{h,g\} = -\{g,h\} = -H_g h \tag{5.88}$$

from which (5.87) again follows. See Problem 5.18.

More invariantly the Hamilton vector can be constructed using the symplectic form

$$\omega = \sum_i d\xi_i \wedge dx_i = d\alpha,\ \alpha = \sum_i \xi_i dx_i. \tag{5.89}$$

Here α is the 'tautological' 1-form. If we think of $\mathbb{R}^n_x \times \mathbb{R}^n_\xi = (x,\xi)'$ as the pull back under $\pi : (x,\xi) \longmapsto x$ of β as a 1-covector on $\mathbb{R}^n$. In this sense the tautological form α is well defined on the cotangent bundle of any manifold and has the property that if one introduces local coordinates in the manifold x and the canonically dual coordinates in the cotangent bundle (by identifying a 1-covector as $\xi \cdot dx$) then it takes the form of α in (5.89). Thus the symplectic form, as $d\alpha$, is well-defined on T^*X for any manifold X.

Returning to the local discussion it follows directly from (5.86) that

$$\omega(\cdot, H_h) = dh(\cdot) \tag{5.90}$$

and conversely this determines H_h. See Problem 5.19.

Now, we wish to apply this discussion of 'Hamiltonian mechanics' to the case that $h = p(x,\xi)$ is the principal symbol of some pseudodifferential operator. We shall in fact take p to be homogeneous of degree m (later normalized to 1) in $|\xi| > 1$. That is,

$$p(x,s\xi) = s^m p(x,\xi)\ \forall\ x \in \mathbb{R}^n,\ |\xi| \geq 1, s|\xi| \geq 1,\ s > 0. \tag{5.91}$$

The effect of this is to ensure that

$$H_p \text{ is homogeneous of degree } m-1 \text{ under } (x,\xi) \longmapsto (x,s\xi) \tag{5.92}$$

in the same region. One consequence of this is that

$$H_p : S^M_{\mathrm{c}}(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow S^{M-1}_{\mathrm{c}}(\mathbb{R}^n;\mathbb{R}^n). \tag{5.93}$$

(where the subscript 'c' just means supports are compact in the first variable). To see this it is convenient to again rewrite the definition of symbol spaces. Since supports are compact in x we are just requiring uniform smoothness in those variables. Thus, we are first requiring that symbols be smooth. Now, consider any point $\bar{\xi} \neq 0$. Thus $\bar{\xi}_j \neq 0$ for some j and we can consider a conic region around $\bar{\xi}$ of the form

$$\xi_j/\bar{\xi}_j \in (0,\infty),\ |\xi_k/\xi_j - \bar{\xi}_j/\bar{\xi}_j| < \epsilon \tag{5.94}$$

where $\epsilon > 0$ is small. Then the symbolic conditions on $a \in S^M_{\mathrm{c}}(\mathbb{R}^n;\mathbb{R}^n)$ imply

$$\begin{gathered} b(x,t,r) = a(x, rt_1, \dots, rt_{j-1}, r\operatorname{sgn}\bar{\xi}_j, rt_j, \dots, rt_{n-1}) \\ \text{satisfies} |D^\alpha_x D^\gamma_t D^k_r b(x,t,r)| \leq C_{\alpha,\gamma,k} r^{M-k} \text{ in } r \geq 1. \end{gathered} \tag{5.95}$$

See Problem 5.20.

For the case of a homogeneous function (away from $\xi = 0$) such as p the surface $\Sigma_m(P) = \{p = 0\}$ has already been called the 'characteristic variety' above. Correspondingly the integral curves of H_p on $\Sigma_m(p)$ (so the ones on which p vanishes) are called null bicharacteristics, or sometimes just bicharacteristics. Note that $\Sigma_m(P)$ may well have singularities, since dp may vanish somewhere. However this is not a problem with the general discussion, since H_p vanishes at such points – and it is only singular in this sense of vanishing. The integral curves through such a point are necessarily constant.

Now we are in a position to state at least a local form of the propagation theorem for operators of 'real principal type'. This means $dp \neq 0$, and in fact even more, that dp and α are linearly independent. The theorems below in fact apply in general when p is real even if there are points where dp is a multiple of α – they just give no information in those cases.

THEOREM 5.1 (Hörmander's propagation theorem, local version). *Suppose $P \in \Psi^m_\infty(M)$ has real principal symbol homogeneous of degree m, that $c : (a,b) \longrightarrow \Sigma_m(P)$ is an interval of a null bicharacteristic curve (meaning $c_*(\frac{d}{dt}) = H_p$) and that $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfies*

$$c(a,b) \cap \mathrm{WF}(Pu) = \emptyset \tag{5.96}$$

then

$$\begin{cases} \textit{either} & c(a,b) \cap \mathrm{WF}(u) = \emptyset \\ \textit{or} & c(a,b) \subset \mathrm{WF}(u). \end{cases} \tag{5.97}$$

5.14. Relative wavefront set

Although we could proceed directly by induction over the (Sobolev) order of regularity to prove a result such as Theorem 5.1 it is probably better to divide up the proof a little. To do this we can introduce a refinement of the notion of wavefront set, which is actually the wavefront set relative to a Sobolev space. So, fixing $s \in \mathbb{R}$ we can simply define by direct analogy with (5.45)

$$\mathrm{WF}_s(u) = \bigcap \left\{ \Sigma_0(A); A \in \Psi^0_\infty(\mathbb{R}^n); Au \in H^s(\mathbb{R}^n) \right\},\ u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n). \tag{5.98}$$

Notice that this would not be a very good definition if extended directly to $u \in \mathcal{S}'(\mathbb{R}^n)$ if we want to think of it as only involving local regularity (because growth of u might stop Au from being in $H^s(\mathbb{R}^n)$ even if it is smooth). So we will just localize the definition in general

$$\mathrm{WF}_s(u) = \bigcap \left\{\Sigma_0(A); A \in \Psi^0_\infty(\mathbb{R}^n); A(\psi u) \in H^s(\mathbb{R}^n) \ \forall \ \psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)\right\}, \quad u \in \mathcal{C}^{-\infty}(\mathbb{R}^n). \tag{5.99}$$

In this sense the regularity is with respect to $H^s_{\text{loc}}(\mathbb{R}^n)$ – is purely local.

LEMMA 5.7. *If $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ then $\mathrm{WF}_s(u) = \emptyset$ if and only if $u \in H^s_{\text{loc}}(\mathbb{R}^n)$.*

PROOF. The same proof as in the case of the original wavefront set works, only now we need to use Sobolev boundedness as well. Certainly if $u \in H^s_{\text{loc}}(\mathbb{R}^n)$ then $\psi u \in H^s(\mathbb{R}^n)$ for each $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ and hence $A(\psi u) \in H^s(\mathbb{R}^n)$ for every $A \in \Psi^0_\infty(\mathbb{R}^n)$. Thus $\mathrm{WF}_s(u) = \emptyset$.

Conversely if $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and $\mathrm{WF}_s(u) = \emptyset$ then for each point (x, ξ) with $x \in \operatorname{supp}(u)$ and $|\xi| = 1$ there exists $A_{x,\xi} \in \Psi^0_\infty(\mathbb{R}^n)$ such that $Au \in H^s(\mathbb{R}^n)$ with $(x, \xi) \notin \Sigma_0(A_{x,\xi})$. That is $A_{(x,\xi)}$ is elliptic at (x, ξ). By compactness (given the conic property of the elliptic set) a finite collection $A_i = A_{(x_i,\xi_i)}$ have the property that the union of their elliptic sets cover some set $K \times (\mathbb{R}^n \setminus 0)$ where K is compact and $\operatorname{supp}(u)$ is contained in the interior of K. We can then choose $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $0 \le \phi \le 1$, $\operatorname{supp}(\phi) \subset K$ and $\phi = 1$ on $\operatorname{supp}(u)$ and

$$B = (1 - \phi) + \sum_i A_i^* A_u \in \Psi^0_\infty(\mathbb{R}^n)$$

is globally elliptic in $\Psi^0_\infty(\mathbb{R}^n)$ and $Bu \in H^s(\mathbb{R}^n)$ by construction (since $(1 - \phi)u = 0$). Thus $u \in H^s(\mathbb{R}^n)$. Applying this argument to ψu for each $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ for $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ we see that $\mathrm{WF}_s(u) = \emptyset$ implies $\psi u \in H^s(\mathbb{R}^n)$ and hence $u \in H^s_{\text{loc}}(\mathbb{R}^n)$. □

Of course if $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ then $\mathrm{WF}_s(u) = \emptyset$ is equivalent to $u \in H^s(\mathbb{R}^n)$.

It also follows directly from this definition that pseudodifferential operators are ‘appropriately’ microlocal given their order.

LEMMA 5.8. *If $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ then*

$$\mathrm{WF}(u) \supset \bigcup_s \mathrm{WF}_s(u). \tag{5.100}$$

and coversely if $\gamma \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ is an open cone then

$$\gamma \cap \mathrm{WF}_s(u) = \emptyset \ \forall \ s \Longrightarrow \gamma \cap \mathrm{WF}(u) = \emptyset. \tag{5.101}$$

The combination of these two statements is that

$$\mathrm{WF}(u) = \overline{\bigcup_s \mathrm{WF}_s(u)}. \tag{5.102}$$

Note that there is not in general equality in (5.100).

PROOF. If $(\bar{x}, \bar{\xi}) \in \mathrm{WF}_s(u)$ for some s then by definition there exists $\psi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\psi(\bar{x}) \neq 0$ and $A \in \Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at $(\bar{x}, \bar{\xi})$ and is such that $A(\psi u) \notin H^s(\mathbb{R}^n)$. This certainly implies that $(\bar{x}, \bar{\xi}) \in \mathrm{WF}(u)$ proving (5.100).

To prove the partial converse if suffices to assume that $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and to fix a point $(\bar{x}, \bar{\xi}) \in \gamma$ and deduce from (5.101) that $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u)$. Since γ is an open cone we may choose $\epsilon > 0$ such that $G = \{(x, \xi); |x - \bar{x}| \le \epsilon, |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{\bar{\xi}}| \le \epsilon\} \subset \gamma$. Now for each s the covering argument in the proof of Lemma 5.7 shows that we may find $A_s \in \Psi^0_\infty(\mathbb{R}^n)$ such that $A_s(u) \in H^s(\mathbb{R}^n)$ and $G \cap \Sigma_0(A_s) = \emptyset$. Now choose one $A \in \Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at $(\bar{x}, \bar{\xi})$ and has $\mathrm{WF}'(A) \subset \{(x,\xi); |x - \bar{x}| < \epsilon, |\frac{\xi}{|\xi|} - \frac{\bar{\xi}}{\bar{\xi}}| < \epsilon\}$, which is the interior of G. Since A_s has a microlocal parametrix in a neighbourhood of G, $B_s A_s = \mathrm{Id} + E_s$, $\mathrm{WF}'(E_s) \cap G = \emptyset$ it follows that

$$Au = A(B_sA_s - E_s)u = (AB_s)A_su - AE_su \in H^s(\mathbb{R}^n)\ \forall\ s, \tag{5.103}$$

since $AE_s \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. Thus $Au \in \mathbb{S}(\mathbb{R}^n)$ (since u is assumed to have compact support) so $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(u)$, proving (5.101). □

Lemma 5.9. *If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $A \in \Psi^m_\infty(\mathbb{R}^n)$ then*

$$\mathrm{WF}_{s-m}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}_s(u)\ \forall\ s \in \mathbb{R}. \tag{5.104}$$

Proof. See the proof of the absolute version, Proposition 5.6. This shows that if $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}'(A)$ then $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}(Au)$, so certainly $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}_{s-m}(Au)$. Similarly, if $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}_s(u)$ then there exists $B \in \Psi^0_\infty(\mathbb{R}^n)$ which is elliptic at $(\bar{x}, \bar{\xi})$ and such that $Bu \in H^s(\mathbb{R}^n)$. If $G \in \Psi^0_\infty(\mathbb{R}^n)$ is a microlocal parametrix for B at $(\bar{x}, \bar{\xi})$ then $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}'(GB - \mathrm{Id})$ so by the first part $(\bar{x}, \bar{\xi}) \notin \mathrm{WF}_{s-m}(A(GB - \mathrm{Id})u)$ and on the other hand, $AGBu \in H^{s-m}(\mathbb{R}^n)$, so (5.104) follows. □

Now, we can state a relative version of Theorem 5.1:-

Theorem 5.2 (Hörmander's propagation theorem, L^2, local version). *Suppose $P \in \Psi^1_\infty(M)$ has real principal symbol, that $c : [a,b] \longrightarrow \Sigma_m(P)$ is an interval of a null bicharacteristic curve (meaning $c_*(\frac{d}{dt}) = H_p$) and that $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ satisfies*

$$c([a,b]) \cap \mathrm{WF}_{\frac{1}{2}}(Pu) = \emptyset\ \textit{(eventually } c([a,b]) \cap \mathrm{WF}_0(Pu) = \emptyset) \tag{5.105}$$

then

$$\begin{cases} \textit{either} & c([a,b]) \cap \mathrm{WF}_0(u) = \emptyset \\ \textit{or} & c([a,b]) \subset \mathrm{WF}_0(u). \end{cases} \tag{5.106}$$

Proof that Theorem 5.1 follows from Theorem 5.2. The basic idea is to apply (5.101), remembering that there is not equality (in general) in (5.100) – the necessary uniformity here comes from the geometry so let us check that first.

Lemma 5.10. *First, we can act on P on the left with some elliptic operator with positive principal symbol, such as $\langle D\rangle^{-m+1}$ which changes the order of P to 1. This does not change $\Sigma(P)$ as the principal symbol changes from p to ap where $a > 0$, and only scales the Hamilton vector field on $\Sigma(P)$ since*

$$H_{ap} = aH_p + pH_a \tag{5.107}$$

and the second term vanishes on $\Sigma(P)$. Thus it suffices to consider the case $m = 1$.

If p is real and homogeneous of degree 1, Γ is an open conic neighbourhood of a bicharacteristic segment $c([a,b])$ such that dp and the canonical 1-form $\alpha = \xi\cdot dx$ are independent at $c(a)$ and γ is an open conic neighbourhood of $c(t)$ for some $t \in [a,b]$ then there is an open conic neighbourhood G of $c([a,b])$, $G \subset \Gamma$ such that $G \cap \Sigma(P)$ is a union of (null) bicharacteristic intervals $c^q(a_q, b_q))$ which intersect γ.

Proof. If dp and α are linearly dependent at a some point $(\bar{x}, \bar{\xi}) \in \Sigma(P)$ then $H_p = c\xi \cdot \partial_\xi$ is a multiple of the radial vector field at that point. By homogeneity the same must be true at $(\bar{x}, s\bar{\xi})$ for all $s > 0$ so the integral curve of H_p through

$(\bar{x},\bar{\xi})$ must be contained in the ray through that point. Thus the condition that dp and $\xi\cdot dx$ are linearly independent at $c(a)$ implies that this must be true on all points of $c([a,b])$ and hence in a neighbourhood of this interval.

Thus it follows that H_p and $\xi\cdot\partial_\xi$ are linearly independent near $c([a,b])$. Since p is homogeneous of degree 1, H_p is homogeneous of degree 0. It follows that there are local coordinates $\Xi\neq 0$ homogeneous of degree 1 and y_k, homogeneous of degree 0, in a neighbourhood of $c([a,b])$ in terms of which $H_p=\partial_{y_1}$. These can be obtained by integrating along H_p to solve

$$H_py_1=1,\ H_py_k=0,\ k>1,\ H_p\Xi=0 \tag{5.108}$$

with appropriate initial conditions on a conic hypersurface transversal to H_p. Then the integral curves, including $c([a,b])$ must just be the y_1 lines for which the conclusion is obvious, noting that ∂_{y_1} must be tangent to $\Sigma(P)$. □

Now, returning to the proof note that we are assuming that Theorem 5.2 has been proved for *all* first order pseudodifferential operators with real principal symbol. Suppose we have the same set up but assume that

$$c([a,b])\cap \mathrm{WF}_{s+\frac12}(Pu)=\emptyset \text{ (eventually } c([a,b])\cap\mathrm{WF}_s(Pu)=\emptyset) \tag{5.109}$$

in place of (5.96). Then we can simply choose a globally invertible elliptic operator of order s, say $Q_s=\langle D\rangle^s$ and rewrite the equation as

$$P_sv=Q_sf,\ P_s=Q_sPQ_{-s},\ v=Q_su \tag{5.110}$$

Then (5.109) implies that

$$c([a,b])\cap\mathrm{WF}_{\frac12}(P_sv)=\emptyset \tag{5.111}$$

and $P_s\in\Psi^0_\infty(\mathbb{R}^n)$ is another operator with real principal symbol – in fact the same as before, so we get (5.97) which means that for each s we have the alternatives

$$\begin{cases}\text{either} & c([a,b])\cap\mathrm{WF}_s(u)=\emptyset\\ \text{or} & c([a,b])\subset\mathrm{WF}_s(u).\end{cases} \tag{5.112}$$

Now the hypothesis in (5.96) implies (5.109) for each s and hence for each s we have the alternatives (5.112). Of course if the second condition holds for any one s then it holds for all larger s and in particular implies that the second case in (5.97) (but for the compact interval) holds. So, what we really need to show is that if the first case in (5.112) holds for all s then

$$c([a,b])\cap\mathrm{WF}(u)=\emptyset. \tag{5.113}$$

This is where we need to get some uniformity. However, consider nearby points and bicharacteristics. Our assumption is that for some $t\in[a,b]$, $c(t)\notin\mathrm{WF}(u)$ – otherwise we are in the second case. Since the set $\mathrm{WF}(u)$ is closed and conic, this implies that some open cone γ containing $c(t)$ is also disjoint from $\mathrm{WF}(u)$. Thus it follows that $\gamma\cap\mathrm{WF}_s(u)=\emptyset$ for all s. This is where the geometry comes in to show that there is a fixed open conic neighbourhood G of $c([a,b])$ such that

$$G\cap\mathrm{WF}_s(u)=\emptyset\ \forall\ s\in\mathbb{R}. \tag{5.114}$$

Namely we can take G to be a small neighbourhood as in Lemma 5.10. Since one point on each of the null bicharacteristic intervals forming $G\cap\Sigma(P)$ meets a point of γ, the first alternative in (5.112) must hold for all these intervals, for all s. That is,

$$G \cap \mathrm{WF}_s(u) = \emptyset \ \forall\ s. \tag{5.115}$$

Now (5.101) applies to show that $G \cap \mathrm{WF}(u) = \emptyset$ so in particular we are in the first case in (5.97) and the theorem follows. $\square$

Finally we further simplify Theorem 5.2 to a purely local statement.

Proposition 5.9. *Under the hypotheses of Theorem 5.2 if $t \in (a, b)$ and $\mathrm{WF}_0(u) \cap c((t \pm \epsilon)) = \emptyset$ for some $\epsilon > 0$ then $c(t) \notin \mathrm{WF}_0(u)$.*

Derivation of Theorem 5.2 from Proposition 5.9. The dicotomy in (5.97) amounts to the statement that if $c(t) \notin \mathrm{WF}_0(u)$ for some $t \in [a, b]$ then $C = \{t' \in [a, b]; c(t) \in \mathrm{WF}_0(u)\}$ must be empty. Since $\mathrm{WF}(u)$ is closed, C is also closed. Applying the Proposition to $\sup(C \cap [a, t))$ shows that it cannot be in C and neither can $\inf(C \cap (t, b])$ so both these sets must be empty and hence C itself must be empty. $\square$

5.15. Proof of Proposition 5.9

Before we finally get down to the analysis let me note some more simiplifications. We can actually assume that $c(t) = a = 0$ and that the interval is $[0, \delta]$ for some $\delta > 0$. Indeed this is just changing the parameter in the case of the positive sign. In the case of the negative sign reversing the sign of P leaves the hypotheses unchanged but reverses the parameter along the integral curve. Thus our hypotheses are that

$$c([0, \delta]) \cap \mathrm{WF}_{\frac{1}{2}}(Pu) = \emptyset \text{ (eventually just } c([0, \delta]) \cap \mathrm{WF}_0(Pu) = \emptyset) \text{ and } \\ c((0, \delta]) \cap \mathrm{WF}_0(u) = 0 \tag{5.116}$$

and we wish to conclude that

$$c(0) \notin \mathrm{WF}_0(u). \tag{5.117}$$

We can also assume that

$$c(0) \notin \mathrm{WF}_{-\frac{1}{2}}(u). \tag{5.118}$$

In fact, if (5.118) does not hold, then there is in fact some $s < -\frac{1}{2}$ such that $c(0) \notin \mathrm{WF}_s(u)$ but $c(0) \in \mathrm{WF}_t(u)$ for some $t \leq \max(-\frac{1}{2}, s + \frac{1}{2})$. Indeed, u itself is in some Sobolev space. Now we can apply the argument used earlier to deduce (5.112) from (5.97). Namely, replace P by $\langle D\rangle^{s+\frac{1}{2}} P \langle D\rangle^{-s-\frac{1}{2}}$ and u by $u' = \langle D\rangle^{s+\frac{1}{2}} u$. Then (5.118) is satisfied by u' and if the argument to prove (5.117) works, we conclude that $c(0) \notin \mathrm{WF}_s(u)$ which is a contradiction. Thus, proving that (5.117) follows form (5.116) and (5.117) suffices to prove everything.

Okay, now to the construction. What we will first do is find a 'test' operator $A \in \Psi^0_\infty(\mathbb{R}^n)$ which has

$$\mathrm{WF}'(A) \subset N(c(0)),\ A^* = A \tag{5.119}$$

for a preassigned conic neighbourhood $N(c(0))$ of the point of interest. Then we want in addition to arrange that for a preassigned conic neigbourhood $N(c(\delta/2))$,

$$\begin{gathered}\frac{1}{i}(AP - P^*A) = B^2 + E_0 + E_1,\\ B \in \Psi^0_\infty(\mathbb{R}^n),\ B^* = B \text{ is elliptic at } c(0),\\ E_0 \in \Psi^0_\infty(\mathbb{R}^n),\ \mathrm{WF}'(E_0) \subset N(c(\frac{\delta}{2}))\\ \text{and } E_1 \in \Psi^{-1}_\infty(\mathbb{R}^n).\end{gathered} \tag{5.120}$$

Before checking that we can arrange (5.120) let me comment on why it will help! In fact there is a flaw in the following argument which will be sorted out below. Given (5.120) let us apply the identity to u and then take the L^2 pairing with u which would give

$$-2Im\langle u, APu\rangle = -i\langle u, APu\rangle + i\langle APu, u\rangle = \|Bu\|^2 + \langle u, E_0\rangle + \langle u, E_1u\rangle. \tag{5.121}$$

where I have illegally integrated by parts, which is part of the flaw in the argument. Anyway, the idea is that APu is smooth – at least it would be if we assumed that $N(c(0)) \cap \mathrm{WF}(Pu) = \emptyset$ – so the left side is finite. Similarly by the third line of (5.120), $\mathrm{WF}'(E_0)$ is confined to a region where u is known to be well-behaved and the order of E_1 allows us to use (5.118). So with a little luck we can show, and indeed we will, that

$$Bu \in L^2(\mathbb{R}^n) \Longrightarrow c(0) \notin \mathrm{WF}_0(u) \tag{5.122}$$

which is what we are after. The problems with this argument are of the same nature that are met in discussions of elliptic regularity and the niceties are discussed below.

So, let us now see that we can arrange (5.120). First recall that we have normalized P to be of order 1 with real principal symbol. So

$$P^* = P + iQ,\ Q \in \Psi^0_\infty(\mathbb{R}^n),\ Q = Q^*.$$

Thus the left side of the desired identity in (5.120) can be written

$$-i[A, P] + QA \in \Psi^0_\infty(\mathbb{R}^n),\ \sigma_0(-i[A, P] + QA) = -H_pa + qa \tag{5.123}$$

where q is the principal symbol of q etc. Since E_1 in (5.120) can include any terms of order -1 we just need to arrange the principal symbol identity

$$-H_pa + qa = b^2 + e. \tag{5.124}$$

Notice that p is by assumption a function which is homogeneous of degree 1 so the vector field H_p is homogeneous of degree 0. We can further assume that

$$H_p \neq 0 \text{ on } c([0, \delta]). \tag{5.125}$$

Indeed, if $H_p = 0$ at $c(0)$ then the whole integral curve through $c(0)$ consists of the point and the result is trivial. So we can assume that $H_p \neq 0$ at $c(0)$ and then (5.125) follows by shrinking δ. As noted above we can now introduce coordinates t $s \in \mathbb{R}^{2n-2}$ and $\Theta > 0$, homogeneous respectively of degrees 0, 0 and 1, in terms of which $H_p = \frac{\partial}{\partial t}$, $c(0) = (0, 1)$ so the integral curve is just $(t, 0, 1)$ and the differential equation (5.124) only involves the t variable and the s variables as parameters (ξ_j disappears because of the assumed homogeneity)

$$-\frac{d}{dt}a + qa = b^2 + e. \tag{5.126}$$

So, simply choose $b = \phi(t)\phi(|s|)$ for some cut-off function $\phi(x) \in \mathcal{C}_c^\infty(\mathbb{R})$ which is 1 near 0 and has small support in $|x| \le \delta'$ which will be chosen small. Then solve

$$-\frac{d}{dt}\tilde{a} + q\tilde{a} = b^2 \Longrightarrow \tilde{a}(t,s) = -\phi^2(|s|)e^{-Q(t,s)}\int_{-\infty}^t e^{Q(t',s)}\phi^2(t',s)dt' \tag{5.127}$$

where Q is a primitive of q. Integrating from $t << 0$ ensures that the support of a' is confined to $|s| \le \delta'$ and $t \ge -\delta'$. Now simply choose a function $\psi \in \mathcal{C}^\infty(\mathbb{R})$ which is equal to 1 in $t < \frac{1}{2}\delta - \delta'$ and equal to 0 in $t > \frac{1}{2}\delta + \delta'$. Then setting $a(t,s) = \psi(t)\tilde{a}(t,s)$ gives a solution of (5.126) with the desired support properties. Namely if we simply cut a and b off in Θ near zero to make them into smooth symbols and select operators B and A self-adjoint and with these principal symbols then (5.120) follows where the supports behave as we wish when δ' is made small.

So, what is the problem with the derivation of (5.121). For one thing the integration by parts, but for another the pairing which we do not know to make sense. In particular the norm $\|Bu\|$ which we wish to show to be finite certainly has to be for this argument to be possible. The solution to these problems is simply to regularize the operators.

So, now choose a sequence $\mu_n(\mathbb{R})$ where the variable will be Θ. We want

$$\mu_n \in \mathcal{C}_c^\infty(\mathbb{R}),\ \mu_n \text{ bounded in } S^0(\mathbb{R}) \text{ and } \mu_n \to 1 \in S^\epsilon(\mathbb{R})\ \forall\ \epsilon > 0. \tag{5.128}$$

This is easily arranged, for instance taking $\mu \in \mathcal{C}_c^\infty(\mathbb{R})$ equal to 1 near 0 and setting $\mu_n(\Theta) = \mu(\Theta/n)$. Since we have arranged that the homogeneous variable Θ is annihilated by $H_p = \frac{d}{dt}$ we can simply multiply through the equation and get a similar family of solutions to (5.124)

$$-H_p a_n + q a_n = b_n^2 + e_n \tag{5.129}$$

where all terms are bounded in $S^0_\infty(\mathbb{R}^n;\mathbb{R}^n)$ (and have compact support in the base variables). Now if we take operators A_n, B_n with these full symbols, and then their self-adjoint parts, we conclude that A_n, $B_n \in \Psi^0_\infty(\mathbb{R}^n)$ have left symbols bounded in S^0 and we get a sequence of solutions to the identity (5.120) with uniformity. Let's check that we know precisely what this means. Namely for all $\epsilon > 0$,

$$\begin{gathered}A_n \text{ is bounded in } \Psi^0_\infty(\mathbb{R}^n),\ A_n \to A \text{ in } \Psi^\epsilon_\infty(\mathbb{R}^n),\ \mathrm{WF}'(A_n) \subset N(c(\delta)) \text{ is uniform},\\ \frac{1}{i}(A_nP - P^*A_n) = B_n^2 + E_{0,n} + E_{1,n},\\ B_n^* = B_n \in \Psi^0_\infty(\mathbb{R}^n) \text{ is bounded},\ B_n \to B \text{ in } \Psi^\epsilon_\infty(\mathbb{R}^n),\ \Psi^0_\infty(\mathbb{R}^n) \ni B \text{ is elliptic at } c(0),\\ E_{0,n} \in \Psi^0_\infty(\mathbb{R}^n) \text{ is bounded},\ \mathrm{WF}'(E_{0,n}) \subset N(c(\frac{\delta}{2})) \text{ is uniform}\\ \text{and } E_{1,n} \in \Psi^{-1}_\infty(\mathbb{R}^n) \text{ is bounded.}\end{gathered} \tag{5.130}$$

where the boundedness of the sequences means that the symbols estimates on the left symbols have fixed constants independent of n and uniformity of the essential support conditions means that for instance

$$\begin{gathered}q \notin N(c(\frac{\delta}{2})) \Longrightarrow\ \exists\ R \in \Psi^0_\infty(\mathbb{R}^n) \text{ elliptic at } q\\ \text{such that } RE_{0,n} \text{ is bounded in } \Psi^{-\infty}_\infty(\mathbb{R}^n).\end{gathered} \tag{5.131}$$

All this follows from our choice of symbols.

I leave as an exercise the effect of the uniformity statement on the essential support.

LEMMA 5.11. *Suppose A_n is bounded in $\Psi^m_\infty(\mathbb{R}^n)$ for some m and that*

$$\mathrm{WF}'(A_n) \subset G \text{ uniformly} \tag{5.132}$$

for a closed cone G in the sense of (5.131). *Then if $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ is such that*

$$\mathrm{WF}_m(u) \cap G = \emptyset \text{ then } A_n u \text{ is bounded in } L^2(\mathbb{R}^n). \tag{5.133}$$

Now we are in a position to finish! For finite n all the operators in the identity in (5.130) are smoothing so we can apply the operators to u and pair with u. Then the integration by parts used to arrive at (5.121) is really justified in giving

$$-2Im\langle u, A_nPu\rangle = -i\langle u, A_nPu\rangle + i\langle A_nPu, u\rangle = \|B_nu\|^2 + \langle u, E_{0,n}u_n\rangle + \langle u, E_{1,n}u\rangle. \tag{5.134}$$

We have arranged that $\mathrm{WF}'(A_n)$ is uniformly concentrated near (the cone over) $c([0, \frac{\delta}{2}])$ and, from (5.116), that $\mathrm{WF}_{\frac{1}{2}}(Pu)$ does not meet such a set. Thus Lemma 5.11 shows us that A_nPu is bounded in $H^{\frac{1}{2}}(\mathbb{R}^n)$. Since we know that $\mathrm{WF}_{-\frac{1}{2}}(u)$ does not meet $c([0,\delta])$ we conclude (always taking the parameter δ' determining the size of the supports small enough) that

$$|\langle u, A_nPu\rangle| \text{ is bounded} \tag{5.135}$$

as $n \to \infty$. Similarly $|\langle u, E_{0,n}\rangle|$ is bounded since $E_{0,n}$ is bounded in $\Psi^0_\infty(\mathbb{R}^n)$ and has essential support uniformly in the region where u is known to be in $L^2(\mathbb{R}^n)$ and $|\langle u, E_{1,n}\rangle|$ is bounded since $E_{1,n}$ is uniformly of order -1 and has essential support (uniformly) in the region where u is known to be in $H^{-\frac{1}{2}}(\mathbb{R}^n)$. Thus indeed, $\|B_nu\|_{L^2}$ is bounded. Thus B_nu is bounded in $L^2(\mathbb{R}^n)$, hence has a weakly convergent subsequence, but this must converge to Bu when paired with test functions. Thus in fact $Bu \in L^2(\mathbb{R}^n)$ and (5.117) follows.

5.16. Hörmander's propagation theorem

There are still some global issues to settle. Theorem 5.1, which has been proved above, can be immediately globalized and microlocalized at the same time. It is also coordinate invariant – see the discussion in Chapter 6, so can be transferred to any manifold as follows.

THEOREM 5.3. *If $P \in \Psi^m(M)$ has real principal symbol and is properly supported then for any distribution $u \in \mathcal{C}^{-\infty}(M)$,*

$$\mathrm{WF}(u) \setminus \mathrm{WF}(Pu) \subset \Sigma(P) \tag{5.136}$$

is a union of maximally extended null bicharacteristics in $\Sigma(P) \setminus \mathrm{WF}(Pu)$.

Some consequences of this in relation to the wave equation are discussed below, and extension of it in Chapter 7.

As already noted, the strengthened assumption on the regularity of Pu in (5.96) is not necessary to deduce (5.97), or correspondingly (5.116) for (5.117). This is not important in the proof of Theorem 5.1 since we are making a much stronger assumption on the regularity of Pu anyway. However, to get the more refined version of Theorem 5.2, as stated 'eventually' we only need to prove (5.117) using the corresponding form of (5.116). This in turn involves a more careful choice of $\phi(x)$ using the following sort of division result.

LEMMA 5.12. *There exist a function $\phi \in \mathcal{C}^\infty(\mathbb{R})$ with support in $[0,\infty)$ which is strictly positive in $(0,\infty)$ and such that for any $0 < f \in \mathcal{C}^\infty(\mathbb{R})$,*

$$\int_{-\infty}^{t} f(t')\phi^2(t')dt' = \phi(t)a(t),\ a \in \mathcal{C}^\infty(\mathbb{R}),\ \operatorname{supp}(a) \subset [0,\infty). \tag{5.137}$$

PROOF. This is true for $\phi = \exp(-1/t)$ in $t > 0$, $\phi(t) = 0$ in $t \le 0$. Indeed the integral is then bounded by

$$|\int_{-\infty}^{t} f(t')\exp(-2/t')dt'| \le C\exp(-2/t),\ t \le 1. \tag{5.138}$$

This shows that $a(t)$, defined as the quotient for $t > 0$ and 0 for $t < 0$ is bounded by $C\phi(t)$. A similar argument show that each of the derivatives are also uniformly bounded by $t^{-N}\phi(t)$ and is therefore also bounded. □

Taking ϕ to be such a function in the discussion above (near the lower bound of its support) allows the symbol a defined by integration, and then a_n, to be decomposed as

$$a_n = b_n g_n + a'_n \tag{5.139}$$

where a'_n is uniformly supported in $t < \delta'/10$ and g_n is also a uniformly bounded sequence of symbols of order 0. This results in a similar decomposition for the operators

$$A_n = B_n G_n + A'_n + R'_n \tag{5.140}$$

where R'_n is uniformly of order -1, G_n is uniformly of order 0 and A'_n, also uniformly of order 0 is uniformly supported in the region where we already know that $u \in L^2(\mathbb{R}^n)$. The previous estimate (5.135) on the left side of (5.134) can then be replaced by

$$|\langle u, A_nPu\rangle| \le |\langle B_nu, G_nPu\rangle| + |\langle u, A'_nPu\rangle| + |\langle u, R'_nPu\rangle| \le C\|B_nu\| + C' \tag{5.141}$$

using only the 'eventual' estimate in (5.116) to control the third term. The other terms in (5.134) behave as before which results in an estimate

$$\|B_nu\|^2 \le C'\|B_nu\| + C'' \tag{5.142}$$

which still implies that $\|B_nu\|$ is bounded, so the argument can be completed as before. This then proves the 'eventual' form of Theorem 5.2 and hence, after reinterpretation, Theorem 5.3.

5.17. Elementary calculus of wavefront sets

We want to achieve a reasonable understanding, in terms of wavefront sets, of three fundamental operations. These are

$$\text{Pull-back: } F^*u \tag{5.143}$$

$$\text{Push-forward: } F_*u \text{ and} \tag{5.144}$$

$$\text{Multiplication: } u_1 \cdot u_2. \tag{5.145}$$

In order to begin to analyze these three operations we shall first introduce and discuss some other more "elementary" operations:

$$\text{Pairing: } (u,v) \longrightarrow \langle u,v\rangle = \int u(x)\overline{v(x)}dx \tag{5.146}$$

$$\text{Projection: } u(x,y) \longmapsto \int u(x,y)dy \tag{5.147}$$

$$\text{Restriction: } u(x,y) \longmapsto u(x,0) \tag{5.148}$$

$$\text{Exterior product: } (u,v) \longmapsto (u\boxtimes v)(x,y) = u(x)v(y) \tag{5.149}$$

$$\text{Invariance: } F^*u, \text{ for } F \text{ a diffeomorphism.} \tag{5.150}$$

Here (5.148) and (5.150) are special cases of (5.143), (5.147) of (5.144) and (5.149) is a combination of (5.145) and (5.143). Conversely the three fundamental operations can be expressed in terms of these elementary ones. We can give direct definitions of the latter which we then use to analyze the former. We shall start with the pairing in (5.146).

5.18. Pairing

We know how to 'pair' a distribution and a $\mathcal{C}^\infty$ function. If both are $\mathcal{C}^\infty$ and have compact supports then

$$\langle u_1,u_2\rangle = \int u_1(x)\overline{u_2(x)}dx \tag{5.151}$$

and in general this pairing extends by continuity to either $\mathcal{C}_c^{-\infty}(\mathbb{R}^n)\times\mathcal{C}^\infty(\mathbb{R}^n)$ or $\mathcal{C}^\infty(\mathbb{R}^n)\times\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ Suppose *both* u_1 and u_2 are distributions, when can we pair them?

Proposition 5.10. *Suppose u_1, $u_2\in\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ satisfy*

$$\mathrm{WF}(u_1)\cap\mathrm{WF}(u_2)=\emptyset \tag{5.152}$$

then if $A\in\Psi^0_\infty(\mathbb{R}^n)$ has

$$\mathrm{WF}(u_1)\cap\mathrm{WF}'(A)=\emptyset,\ \mathrm{WF}(u_2)\cap\mathrm{WF}'(\mathrm{Id}-A)=\emptyset \tag{5.153}$$

the bilinear form

$$\langle u_1,u_2\rangle = \langle Au_1,u_2\rangle + \langle u_1,(\mathrm{Id}-A^*)u_2\rangle \tag{5.154}$$

is independent of the choice of A.

Notice that A satisfying (5.153) does indeed exist, just choose $a\in S^0_\infty(\mathbb{R}^n;\mathbb{R}^n)$ to be identically 1 on $\mathrm{WF}(u_2)$, but to have $\operatorname{cone\,supp}(a)\cap\mathrm{WF}(u_1)=\emptyset$, possible because of (5.152), and set $A=q_L(a)$.

Proof. Of course (5.154) makes sense because Au_1, $(\mathrm{Id}-A^*)u_2\in\mathcal{C}^\infty(\mathbb{R}^n)$ by microlocality and the fact that $\mathrm{WF}'(A)=\mathrm{WF}'(A^*)$. To prove that this definition is independent of the choice of A, suppose A' also satisfies (5.153). Set

$$\langle u_1,u_2\rangle' = \langle A'u_1,u_2\rangle + \langle u_1,(\mathrm{Id}-A')^*u_2\rangle. \tag{5.155}$$

Then

$$\mathrm{WF}'(A-A')\cap\mathrm{WF}(u_1)=\mathrm{WF}'((A-A')^*)\cap\mathrm{WF}(u_2)=\emptyset. \tag{5.156}$$

The difference can be written

$$\langle u_1,u_2\rangle' - \langle u_1,u_2\rangle = \langle (A-A')u_1,u_2\rangle - \langle u_1,(A-A')^*u_2\rangle. \tag{5.157}$$

Naturally we expect this to be zero, but this is not quite obvious since u_1 and u_2 are both distributions. We need an approximation argument to finish the proof.

Choose $B \in \Psi^0_\infty(\mathbb{R}^n)$ with

$$\begin{gathered}\mathrm{WF}'(B) \cap \mathrm{WF}(u_1) = \mathrm{WF}'(B) \cap \mathrm{WF}(u_2) = \emptyset \\ \mathrm{WF}'(\mathrm{Id} - B) \cap \mathrm{WF}(A - A') = \emptyset\end{gathered} \tag{5.158}$$

If $v_n \longrightarrow u_2$, in $\mathcal{C}^{-\infty}_c(\mathbb{R}^n)$, $v_n \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ then

$$w_n = \phi\big[(\mathrm{Id} - B)\, v_n + Bu_2\big] \longrightarrow u_2 \tag{5.159}$$

if $\phi \equiv 1$ in a neighbourhood of $\operatorname{supp}(u_2)$, $\phi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. Here $Bu_2 \in \mathcal{C}^\infty(\mathbb{R}^n)$, so

$$(A - A')w_n = (A - A')\phi(\mathrm{Id} - B) \cdot v_n + (A - A')\phi B u_2 \longrightarrow (A - A')u_2 \text{ in } \mathcal{C}^\infty(\mathbb{R}^n), \tag{5.160}$$

since $(A - A')\phi(\mathrm{Id} - B) \in \Psi^{-\infty}_\infty(\mathbb{R}^n)$. Thus

$$\begin{aligned}\langle (A - A')\, u_1, u_2\rangle &\longrightarrow \langle (A - A')\, u_1, u_2\rangle \\ \langle u_1, (A - A')^*\, w_n\rangle &\longrightarrow \langle u_1, (A - A')^*\, u_2\rangle,\end{aligned}$$

since $w_n \longrightarrow u_2$ in $\mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ and $(A - A')^* w_n \longrightarrow (A - A')^*\, u_2$ in $\mathcal{C}^\infty(\mathbb{R}^n)$. Thus

$$\langle u_1, u_2\rangle' - \langle u_1, u_2\rangle = \lim_{n\to\infty}\big[\langle (A - A')\, u_1, w_n\rangle - \langle u_1, (A - A')^*\, w_n\big] = 0. \tag{5.161}$$

□

Here we are using the *complex* pairing. If we define the real pairing by

$$(u_1, u_2) = \langle u_1, \overline{u}_2\rangle \tag{5.162}$$

then we find

PROPOSITION 5.11. *If u_1, $u_2 \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ satisfy*

$$(x, \xi) \in \mathrm{WF}(u_1) \Longrightarrow (x, -\xi) \notin \mathrm{WF}(u_2) \tag{5.163}$$

then the real pairing, defined by

$$(u_1, u_2) = (Au_1, u_2) + (u_1, (\mathrm{Id} - A^t)u_2), \tag{5.164}$$

where A satisfies (5.153)*, is independent of A.*

PROOF. Notice that

$$\mathrm{WF}(\overline{u}) = \big\{(x, -\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0); (x, \xi) \in \mathrm{WF}(u)\big\}. \tag{5.165}$$

We can write (5.163), using (5.162), as

$$(u_1, u_2) = \langle Au_1, \overline{u}_2\rangle + \langle u_1, \overline{(\mathrm{Id} - A^t)u_2}\rangle. \tag{5.166}$$

Since, by definition, $\overline{A^t u_2} = A^*\overline{u}_2$,

$$(u_1, u_2) = \langle Au_1, \overline{u}_2\rangle + \langle u_1, (\mathrm{Id} - A^*)\overline{u}_2\rangle = \langle u_1, \overline{u}_2\rangle \tag{5.167}$$

is defined by (5.154), since (5.163) translates to (5.152). □

5.19. Multiplication of distributions

The pairing result (5.164) can be used to define the *product* of two distributions under the same hypotheses, (5.163).

PROPOSITION 5.12. *If* $u_1,\ u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *satisfy*

$$(x,\xi) \in \mathrm{WF}(u_1) \Longrightarrow (x,-\xi) \notin \mathrm{WF}(u_2) \tag{5.168}$$

then the product of u_1 *and* $u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *is well-defined by*

$$u_1 u_2(\phi) = (u_1, \phi u_2) = (\phi u_1, u_2)\ \forall\ \phi \in \mathcal{C}_c^{\infty}(\mathbb{R}^n) \tag{5.169}$$

using (5.164).

PROOF. We only need to observe that if $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ and $A \in \Psi_\infty^m(\mathbb{R}^n)$ has $\mathrm{WF}'(A) \cap \mathrm{WF}(u) = \emptyset$ then for any fixed $\psi \in \mathcal{C}_c^{\infty}(\mathbb{R}^n)$

$$\|\psi A \phi u\|_{C^k} \le C\|\phi\|_{C^p} \quad p = k + N \tag{5.170}$$

for some N, depending on m. This implies the continuity of $\phi \longmapsto u_1 u_2(\phi)$ defined by (5.169). $\square$

5.20. Projection

Here we write $\mathbb{R}_z^n = \mathbb{R}_x^p \times \mathbb{R}_y^k$ and define a continuous linear map, which we write rather formally as an integral

$$\mathcal{C}_c^{-\infty}(\mathbb{R}^n) \ni u \longmapsto \int u(x,y)dy \in \mathcal{C}_c^{-\infty}(\mathbb{R}^p) \tag{5.171}$$

by pairing. If $\phi \in \mathcal{C}^{\infty}(\mathbb{R}^p)$ then

$$\pi_1^* \phi \in \mathcal{C}^{\infty}(\mathbb{R}^n), \quad \pi_1 :\ \mathbb{R}^n \ni (x,y) \longmapsto x \in \mathbb{R}^p \tag{5.172}$$

and for $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ we define the formal 'integral' in (5.171) by

$$(\int u(x,y)dy, \phi) = ((\pi_1)_* u, \phi) := u(\pi_1^* \phi). \tag{5.173}$$

In this sense we see that the projection is dual to pull-back (on functions) under π_1, so is "push-forward under π_1," a special case of (5.144). The support of the projection satisfies

$$\mathrm{supp}\left((\pi_1)_* u\right) \subset \pi_1\left(\mathrm{supp}(u)\right)\ \forall\ u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n), \tag{5.174}$$

as follows by duality from

$$\mathrm{supp}(\pi_1^* \phi) \subset \pi_1^{-1}\left(\mathrm{supp}\,\phi\right). \tag{5.175}$$

PROPOSITION 5.13. *Let* $\pi_1 : \mathbb{R}^{p+k} \longrightarrow \mathbb{R}^p$ *be projection, then for every* $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^{p+k})$

$$\begin{aligned} \mathrm{WF}\left((\pi_1)_* u\right) \subset \big\{&(x,\xi) \in \mathbb{R}^p \times (\mathbb{R}^p \backslash 0)\,; \\ &\exists\ y \in \mathbb{R}^k \text{ with } (x,y,\xi,0) \in \mathrm{WF}(u)\big\}. \end{aligned} \tag{5.176}$$

PROOF. First notice that

$$(\pi_1)_* :\ \mathcal{C}_c^{\infty}(\mathbb{R}^n) \longrightarrow \mathcal{C}_c^{\infty}(\mathbb{R}^p). \tag{5.177}$$

Combining this with (5.174) we see that

$$\mathrm{sing\,supp}\left((\pi_1)_* u\right) \subset \pi_1\left(\mathrm{sing\,supp}\, u\right) \tag{5.178}$$

which is at least consistent with Proposition 5.13. To prove the proposition in full let me restate the local characterization of the wavefront set, in terms of the Fourier transform:

LEMMA 5.13. *Suppose* $K \subset\subset \mathbb{R}^n$ *and* $\Gamma \subset \mathbb{R}^n \setminus 0$ *is a* closed *cone, then*

$$u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n),\ \mathrm{WF}(u) \cap (K \times \Gamma) = \emptyset,\ A \in \Psi_\infty^m(\mathbb{R}^n),\ \mathrm{WF}'(A) \subset K \times \Gamma \\ \Longrightarrow Au \in \mathcal{S}(\mathbb{R}^n). \tag{5.179}$$

In particular

$$u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n),\ \mathrm{WF}(u) \cap (K \times \Gamma) = \emptyset,\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n), \mathrm{supp}(\phi) \subset K \\ \Longrightarrow \widehat{\phi u}(\xi) \text{ is rapidly decreasing in } \Gamma. \tag{5.180}$$

Conversely suppose $\Gamma \subset \mathbb{R}^n \setminus 0$ *is a closed cone and* $u \in \mathcal{S}'(\mathbb{R}^n)$ *is such that for some* $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$\widehat{\phi u}(\xi) \text{ is rapidly decreasing in } \Gamma \tag{5.181}$$

then

$$\mathrm{WF}(u) \cap \{x \in \mathbb{R}^n; \phi(x) \neq 0\} \times \mathrm{int}(\Gamma) = \emptyset. \tag{5.182}$$

With these local tools at our disposal, let us attack (5.176). We need to show that

$$(\overline{x}, \overline{\xi}) \in \mathbb{R}^p \times (\mathbb{R}^p \setminus 0) \text{ s.t. } (\overline{x}, y, \overline{\xi}, 0) \notin \mathrm{WF}(u)\ \forall\ y \in \mathbb{R}^n \\ \Longrightarrow (\overline{x}, \overline{\xi}) \notin \mathrm{WF}\left((\pi_1)_* u\right). \tag{5.183}$$

Notice that, $\mathrm{WF}(u)$ being conic and $\pi(\mathrm{WF}(u))$ being compact, $\mathrm{WF}(u) \cap (\mathbb{R}^n \times S^{n-1})$ is compact. The hypothesis (5.183) is the statement that

$$\{\overline{x}\} \times \mathbb{R}^k \times \mathbb{S}^{n-1} \times \{0\} \cap \mathrm{WF}(u) = \emptyset. \tag{5.184}$$

Thus $\overline{x}$ has an open neighbourhood, W, in $\mathbb{R}^p$, and $(\overline{\xi}, 0)$ a conic neighbourhood γ_1 in $(\mathbb{R}^n \setminus 0)$ such that

$$(W \times \mathbb{R}^k \times \gamma_1) \cap \mathrm{WF}(u) = \emptyset. \tag{5.185}$$

Now if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^p)$ is chosen to have support in W

$$\widehat{(\pi_1^* \phi) u}(\xi, \eta) \text{ is rapidly decreasing in } \gamma_1. \tag{5.186}$$

Set $v = \phi(\pi_1)_* u$. From the definition of projection and the identity

$$v = \phi(\pi_1)_* u = (\pi_1)_*[(\pi_1^* \phi) u], \tag{5.187}$$

we have

$$\widehat{v}(\xi) = v(e^{-ix\cdot\xi}) = (\widehat{(\pi_1^* \phi) u})(\xi, 0). \tag{5.188}$$

Now (5.186) shows that $\widehat{v}(\xi)$ is rapidly decreasing in $\gamma_1 \cap (\mathbb{R}^p \times \{0\})$, which is a cone around $\overline{\xi}$ in $\mathbb{R}^p$. Since $v = \phi(\pi_1)_* u$ this shows that $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}\left((\pi_1)_* u\right)$, as claimed. □

Before going on to talk about the other operations, let me note a corollary of this which is useful and, even more, helps to explain what is going on:

COROLLARY 5.1. *If* $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *and*

$$\mathrm{WF}(u) \cap \{(x, y, \xi, 0); x \in \mathbb{R}^p, y \in \mathbb{R}^k, \xi \in \mathbb{R}^p \setminus 0\} = \emptyset \tag{5.189}$$

then $(\pi_1)_*(u) \in \mathcal{C}_c^\infty(\mathbb{R}^n)$.

PROOF. Indeed, (5.176) says $\mathrm{WF}\left((\pi_1)_* u\right) = \emptyset$. $\square$

Here, the vectors $(x, y, \xi, 0)$ are the ones "normal" (as we shall see, really *conormal*) to the surfaces over which we are integrating. Thus Lemma 5.13 and Corollary 5.1 both state that the only singularities that survive integration are the ones which are conormal to the surface along which we integrating; the ones even partially in the direction of integration are wiped out. This in particular fits with the fact that if we integrate in *all variables* then there are no singularities left.

5.21. Restriction

Next we wish to consider the *restriction* of a distribution to a subspace

$$\mathcal{C}_c^{-\infty}(\mathbb{R}^n) \ni u \longmapsto u \restriction \{y = 0\} \in \mathcal{C}_c^{-\infty}(\mathbb{R}^p). \tag{5.190}$$

This is *not* always defined, i.e. no reasonable map (5.190) exists for all distributions. However under an appropriate condition on the wavefront set we can interpret (5.190) in terms of pairing, using our definition of products. Thus let

$$\iota : \mathbb{R}^p \ni x \longmapsto (x, 0) \in \mathbb{R}^n \tag{5.191}$$

be the *inclusion* map. We want to think of $u \restriction \{y = 0\}$ as $\iota^* u$. If $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then for any $\phi' \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ the identity

$$\iota^* u(\iota^* \phi') = u\left(\phi' \delta(y)\right) \tag{5.192}$$

holds.

The restriction map $\iota^* : \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}_c^\infty(\mathbb{R}^p)$ is surjective. If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ satisfies the condition

$$\mathrm{WF}(u) \cap \left\{(x, 0, 0, \eta); x \in \mathbb{R}^p, \eta \in \mathbb{R}^{n-p}\right\} = \emptyset \tag{5.193}$$

then we can interpret the pairing

$$\begin{gathered} \iota^* u(\phi) = u\left(\phi' \delta(y)\right) \ \forall \ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^p) \\ \text{where } \phi' \in \mathcal{C}_c^\infty(\mathbb{R}^n) \text{ and } \iota^* \phi' = \phi \end{gathered} \tag{5.194}$$

to define $\iota^* u$. Indeed, the right side makes sense by Proposition 5.12.

Thus we have directly proved the first part of

PROPOSITION 5.14. *Set* $\mathcal{R} = \left\{u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n); (5.193) \text{ *holds*}\right\}$ *then* (5.194) *defines a linear* restriction *map* $\iota^* : \mathcal{R} \longrightarrow \mathcal{C}_c^{-\infty}(\mathbb{R}^p)$ *and*

$$\mathrm{WF}(\iota^* u) \subset \left\{(x, \xi) \in \mathbb{R}^p \times (\mathbb{R}^p \setminus 0);\ \exists\ \eta \in \mathbb{R}^n \text{ with } (x, 0, \xi, \eta) \in \mathrm{WF}(u)\right\}. \tag{5.195}$$

PROOF. First note that (5.193) means *precisely* that

$$\hat{u}(\xi, \eta) \text{ is rapidly decreasing in a cone around } \{0\} \times \mathbb{R}^k \backslash 0. \tag{5.196}$$

When $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ taking Fourier transforms in (5.192) gives

$$\widehat{\iota^* u}(\xi) = \frac{1}{(2\pi)^k} \int \hat{u}(\xi, \eta) d\eta. \tag{5.197}$$

In general (5.196) ensures that the integral in (5.197) converges, it will then hold by continuity.

We actually apply (5.197) to a localized version of u; if $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^p)$ then

$$\widehat{\psi \iota^*(u)}(\xi) = (2\pi)^{-k} \int \widehat{\psi}(\xi) \widehat{u}(\xi, \eta) d\eta. \tag{5.198}$$

Thus suppose $(\overline{x},\overline{\xi}) \in \mathbb{R}^p \times (\mathbb{R}^p\backslash 0)$ is such that $(\overline{x},0,\overline{\xi},\eta) \notin \mathrm{WF}(u)$ for *any* η. If ψ has support close to $\overline{x}$ and $\zeta \in \mathcal{C}^\infty_c(\mathbb{R}^{n-p})$ has support close to 0 this means

$$\widehat{\psi\zeta u}(\xi,\eta) \text{ is rapidly decreasing in a cone around each } (\overline{\xi},\eta). \tag{5.199}$$

We also have rapid decrease around $(0,\eta)$ from (5.196) (make sure you understand this point) as

$$\widehat{\psi\zeta u}(\xi,\eta) \text{ is rapidly decreasing in } \gamma \times \mathbb{R}^p \tag{5.200}$$

for a cone, γ, around $\overline{\xi}$. From (5.197)

$$\widehat{\psi\iota^*(\zeta u)}(\xi) \text{ is rapidly decreasing in } \gamma. \tag{5.201}$$

Thus $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(\iota^*(\zeta u))$. Of course if we choose $\zeta(y) = 1$ near 0, $\iota^*(\zeta u) = \iota^*(u)$ so $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(u)$, provided $(\overline{x},0,\overline{\xi},\eta) \notin \mathrm{WF}(u)$, for all η. This is what (5.195) says. □

Try to picture what is going on here. We can restate the main conclusion of Proposition 5.14 as follows.

Take $\mathrm{WF}(u) \cap \big\{(x,0,\xi,\eta) \in \mathbb{R}^p \times \{0\} \times (\mathbb{R}^n\backslash 0)\big\}$ and let Z denote projection *off* the η variable:

$$\mathbb{R}^p \times \{0\} \times \mathbb{R}^p \times \mathbb{R}^k \xrightarrow{Z} \mathbb{R}^p \times \mathbb{R}^p \tag{5.202}$$

then

$$\mathrm{WF}(\iota^* u) \subset Z(\mathrm{WF}(u) \cap \{y=0\}). \tag{5.203}$$

We will want to think more about these operations later.

5.22. Exterior product

This is maybe the easiest of the elementary operators. It is always defined

$$(u_1 \boxtimes u_2)(\phi) = u_1\left(u_2(\phi(x,\cdot)\right) = u_2(u_1(\phi(\cdot,y)). \tag{5.204}$$

Moreover we can easily compute the Fourier transform:

$$\widehat{u_1 \boxtimes u_2}(\xi,\eta) = \hat{u}_1(\xi)\hat{u}_2(\eta). \tag{5.205}$$

Proposition 5.15. *The (exterior) product*

$$\mathcal{C}^{-\infty}_c(\mathbb{R}^p) \times \mathcal{C}^{-\infty}_c(\mathbb{R}^k) \longleftarrow \mathcal{C}^{-\infty}_c(\mathbb{R}^{p+k}) \tag{5.206}$$

is a bilinear map such that

$$\begin{aligned}\mathrm{WF}(u_1 \boxtimes u_2) \subset {} & [(\mathrm{supp}(u_1) \times \{0\}) \times \mathrm{WF}(u_2)] \\ & \cup [\mathrm{WF}(u_1) \times (\mathrm{supp}(u_2) \times \{0\})] \cup [\mathrm{WF}(u_1) \times \mathrm{WF}(u_2)].\end{aligned} \tag{5.207}$$

Proof. We can localize near any point $(\overline{x},\overline{y})$ with $\phi_1(x)\phi_2(y)$, where ϕ_1 is supported near $\overline{x}$ and ϕ_2 is supported near $\overline{y}$. Thus we only need examine the decay of

$$\widehat{\phi_1 u_1 \boxtimes \phi_2 u_2} = \widehat{\phi_1 u_1}(\xi) \cdot \widehat{\phi_2 u_2}(\eta). \tag{5.208}$$

Notice that if $\widehat{\phi_1 u_1}(\xi)$ is rapidly decreasing around $\overline{\xi} \neq 0$ then the product is rapidly decreasing around *any* $(\overline{\xi},\eta)$. This gives (5.207). □

5.23. Diffeomorphisms

We next turn to the question of the extension of F^*, where $F : \Omega_1 \longrightarrow \Omega_2$ is a $\mathcal{C}^\infty$ map, from $\mathcal{C}^\infty(\Omega_2)$ to some elements of $\mathcal{C}^{-\infty}(\Omega_2)$. The simplest example of pull-back is that of transformation by a diffeomorphism.

We have already noted how pseudodifferential operators behave under a diffeomorphism: $F : \Omega_1 \longrightarrow \Omega_2$ between open sets of $\mathbb{R}^n$. Suppose $A \in \Psi^m_\infty(\mathbb{R}^n)$ has Schwartz kernel of *compact* support in $\Omega_1 \times \Omega_1$ then we define

$$A_F : \mathcal{C}^\infty_c(\Omega_2) \longrightarrow \mathcal{C}^\infty_c(\Omega_2) \tag{5.209}$$

by $A_F = G^* \cdot A \cdot F^*$, $G = F^{-1}$. In § 5.4 we showed that $A_F \in \Psi^m_\infty(\mathbb{R}^n)$. In fact we showed much more, namely we computed a (very complicated) formula for the full symbols. Recall the definition of the *cotangent bundle* of $\mathbb{R}^n$

$$T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n \tag{5.210}$$

identified as pairs of points $(\overline{x}, \overline{\xi})$, where $\overline{x} \in \mathbb{R}^n$ and

$$\overline{\xi} = df(\overline{x}) \text{ for some } f \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{5.211}$$

The *differential* $df(\overline{x})$ of f at $\overline{x} \in \mathbb{R}^n$ is just the equivalence class of $f(x) - f(\overline{x}) \in \mathcal{I}_{\overline{x}}$ modulo $\mathcal{I}^2_{\overline{x}}$. Here

$$\begin{cases} \mathcal{I}_{\overline{x}} = \left\{ g \in \mathcal{C}^\infty(\mathbb{R}^n);\ g(\overline{x}) = 0 \right\} \\ \mathcal{I}^2_{\overline{x}} = \left\{ \sum\limits_{\text{finite}} g_i h_i,\ g_i, h_i \in \mathcal{I}_{\overline{x}} \right\}. \end{cases} \tag{5.212}$$

The identification of $\overline{\xi}$, given by (5.210) and (5.211), with a point in $\mathbb{R}^n$ is obtained using Taylor's formula. Thus if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$

$$f(x) = f(\overline{x}) + \sum_{i=1}^n \frac{\partial f}{\partial x_j}(\overline{x})(x - \overline{x})_j + \sum_{i,j=1} g_{ij}(x) x_i x_j. \tag{5.213}$$

The double sum here is in $\mathcal{I}^2_{\overline{x}}$, so the residue class of $f(x) - f(\overline{x})$ in $\mathcal{I}_{\overline{x}} / \mathcal{I}^2_{\overline{x}}$ is the same as that of

$$\sum_{i=1}^n \frac{\partial f}{\partial x_j}(\overline{x})(x - \overline{x})_j. \tag{5.214}$$

That is, $d(x - \overline{x})_j = dx_j$, $j = 1, \dots, n$ form a basis for $T^*_{\overline{x}}\mathbb{R}^n$ and in terms of this basis

$$df(\overline{x}) = \sum_{i=1}^n \frac{\partial f}{\partial x_j}(\overline{x}) dx_j. \tag{5.215}$$

Thus the entries of $\overline{\xi}$ are just $\left(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n}\right)$ for some f. Another way of saying this is that the *linear functions* $\xi \cdot x = \xi_1 x_1 + \xi_2 x_2 \cdots \xi_n x_n$ have differentials spanning $T^*_x\mathbb{R}^n$.

So suppose $F : \Omega_1 \longrightarrow \Omega_2$ is a $\mathcal{C}^\infty$ map. Then

$$F^* : T^*_{\overline{y}}\Omega_2 \longrightarrow T^*_{\overline{x}}\Omega_1,\ \overline{y} = F(\overline{x}) \tag{5.216}$$

is defined by $F^*df(\overline{y}) = d(F^*f)(\overline{x})$ since $F^* : \mathcal{I}_{\overline{y}} \longrightarrow \mathcal{I}_{\overline{x}}$, $F^* : \mathcal{I}^2_{\overline{y}} \longrightarrow \mathcal{I}^2_{\overline{x}}$. In coordinates $F(x) = y \Longrightarrow$

$$\frac{\partial}{\partial x_j}(F^*f(x)) = \frac{\partial}{\partial y}f(F(x)) = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(y)\frac{\partial F_k}{\partial x_j} \tag{5.217}$$

i.e. $F^*(\eta \cdot dy) = \xi \cdot dx$ if

$$\xi_j = \sum_{k=1}^n \frac{\partial F_k}{\partial x_j}(x) \cdot \eta_k. \tag{5.218}$$

Of course if F is a diffeomorphism then the Jacobian matrix $\frac{\partial F}{\partial x}$ is invertible and (5.218) is a linear isomorphism. In this case

$$\begin{gathered} F^* : T^*_{\Omega_2}\mathbb{R}^n \longleftrightarrow T^*_{\Omega_1}\mathbb{R}^n \\ (x,\xi) \longleftrightarrow (F(x),\eta) \end{gathered} \tag{5.219}$$

with ξ and η connected by (5.218). Thus $(F^*)^* : \mathcal{C}^\infty(T^*\Omega_1) \longrightarrow \mathcal{C}^\infty(T^*\Omega_2)$.

Proposition 5.16. *If $F : \Omega_1 \longrightarrow \Omega_2$ is a diffeomorphism of open sets of $\mathbb{R}^n$ and $A \in \Psi^m_\infty(\mathbb{R}^n)$ has Schwartz kernel with compact support in $\Omega_1 \times \Omega_2$ then*

$$\sigma_m(A_F) = (F^*)^*\sigma_m(A) \tag{5.220}$$

and

$$F^*\left(\mathrm{WF}'(A_F)\right) = \mathrm{WF}'(A). \tag{5.221}$$

It follows that symbol $\sigma_m(A)$ of A is well-defined as an element of $S^{m-[1]}_\infty(T^*\mathbb{R}^n)$ independent of coordinates and $\mathrm{WF}'(A) \subset T^*\mathbb{R}^n\backslash 0$ is a well-defined closed conic set, independent of coordinates. The elliptic set and the characteristic set Σ_m are therefore also well-defined complementary conic subsets of $T^*\Omega\backslash 0$.

Proof. Look at the formulae. □

The main use we make of this invariance result is the freedom it gives us to *choose* local coordinates adapted to a particular problem. It also suggests that there should be neater ways to write various formulae, e.g. the wavefront sets of push-forward and pull-backs.

Proposition 5.17. *If $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ has $\operatorname{supp}(u) \subset \Omega_2$ and $F : \Omega_1 \longrightarrow \Omega_2$ is a diffeomorphism then*

$$\mathrm{WF}(F^*u) \subset \big\{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash 0); (F(x),\eta) \in \mathrm{WF}(u), \eta_j = \sum_i \frac{\partial F_i}{\partial x_j}(x)\xi_i\big\}. \tag{5.222}$$

Proof. Just use the standard definition

$$\mathrm{WF}(F^*u) = \bigcap\big\{\Sigma(A);\ A(F^*u) \in \mathcal{C}^\infty\big\}. \tag{5.223}$$

To test the wavefront set of F^*u it suffices to consider A's with kernels supported in $\Omega_1 \times \Omega_1$ since $\operatorname{supp}(F^*u \Subset \Omega_1$ and for a general pseudodifferential operator A'

there exists A with kernel supported in Ω_1 such that $A'u - Au \in \mathcal{C}^\infty(\mathbb{R}^n)$. Then $AF^*u \in \mathcal{C}^\infty_c(\Omega_1) \Longleftrightarrow A_F u \in \mathcal{C}^\infty_c(\Omega_2)$. Thus

$$\operatorname{WF}(F^*u) = \bigcap \big\{\Sigma(A);\ A_F u \in \mathcal{C}^\infty\big\} \tag{5.224}$$

$$= \bigcap \big\{F^*(\Sigma(A_F)); A_F u \in \mathcal{C}^\infty\big\} \tag{5.225}$$

$$= F^* \operatorname{WF}(u) \tag{5.226}$$

since, for u, it is enough to consider operators with kernels supported in $\Omega_2 \times \Omega_2$. $\square$

5.24. Products

Although we have discussed the definition of the product of two distributions we have not yet analyzed the wavefront set of the result.

Proposition 5.18. *If $u_1,\ u_2 \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ are such that*

$$(x,\xi) \in \operatorname{WF}(u_1) \Longrightarrow (x,-\xi) \notin \operatorname{WF}(u_2) \tag{5.227}$$

then the product $u_1u_2 \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$, defined by Proposition 5.12 satisfies

$$\begin{aligned} \operatorname{WF}(u_1u_2) \subset & \big\{(x,\xi); x \in \operatorname{supp}(u_1) \textit{ and } (x,\xi) \in \operatorname{WF}(u_2)\big\} \\ & \cup \big\{(x,\xi); x \in \operatorname{supp}(u_2) \textit{ and } (x,\xi) \in \operatorname{WF}(u_1)\big\} \\ & \cup \big\{(x,\xi); \xi = \eta_1 + \eta_2, (x,\eta_i) \in \operatorname{WF}(u_i), i = 1,2\big\}. \end{aligned} \tag{5.228}$$

Proof. We can represent the product in terms of three 'elementary' operations.

$$u_1u_2(x) = \iota^*\big[F^*(u_1 \boxtimes u_2)\big] \tag{5.229}$$

where $F : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is the linear transformation

$$F(x,y) = (x+y, x-y) \tag{5.230}$$

and $\iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n}$ is inclusion as the first factor. Thus (5.229) corresponds to the 'informal' notation

$$u_1u_2(x) = u_1(x+y)u_2(x-y) \restriction \{y = 0\} \tag{5.231}$$

and will follow by continuity once we analyse the wavefront set properties.

We know from Proposition 5.15 that

$$\begin{aligned} \operatorname{WF}(u_1 \boxtimes u_2) \subset & \big\{(X,Y,\Xi,H); X \in \operatorname{supp}(u_1), \Xi = 0, (Y,H) \in \operatorname{WF}(u_2)\big\} \\ & \cup \big\{(X,Y,\Xi,H);\ (X,\Xi) \in \operatorname{WF}(u_1),\ Y \in \operatorname{supp}(u_2), H = 0\big\} \\ & \cup \big\{(X,Y,\Xi,H);\ (X,\Xi) \in \operatorname{WF}(u_1),\ (Y,H) \in \operatorname{WF}(u_2)\big\}. \end{aligned} \tag{5.232}$$

Since F is a diffeomorphism, by Proposition 5.17,

$$\begin{aligned} \operatorname{WF}(F^*(u_1 \boxtimes u_2)) = \big\{(x,y,\xi,\eta); (F^t(x,y),\Xi,H) \in \operatorname{WF}(u_1 \boxtimes u_2), \\ (\xi,\eta) = A^t(\Xi,H)\big\}. \end{aligned}$$

where F^t is the transpose of F as a linear map. In fact $F^t = F$, so

$$\begin{aligned} & \operatorname{WF}(F^*(u_1 \boxtimes u_2)) \subset \\ & \quad \big\{(x,y,\xi,\eta); x+y \in \operatorname{supp}(u_1),\ \xi+\eta = 0, (x-y, \tfrac{1}{2}(\xi-\eta)) \in \operatorname{WF}(u_2)\big\} \\ & \quad \cup \big\{(x,y,\xi,\eta); (x+y, \tfrac{1}{2}(\xi+\eta)) \in \operatorname{WF}(u_1),\ (x-y, \tfrac{1}{2}(\xi-\eta)) \in \operatorname{WF}(u_2)\big\} \end{aligned}$$

and so using Proposition 5.14

$$\begin{aligned}\mathrm{WF}(F^*(u_1 \boxtimes u_2)) \restriction \{y=0\} \\ &\subset \big\{(x,0,\xi,-\xi); x \in \mathrm{supp}(u_1), (x,\xi) \in \mathrm{WF}(u_2)\big\} \\ &\cup \big\{(x,0,\xi,\eta); (x \in \mathrm{supp}(u_2), (x,\xi) \in \mathrm{WF}(u_2)\big\} \\ &\cup \big\{(x,0,\xi,\eta); (x, \frac{1}{2}(\xi+\eta)) \in \mathrm{WF}(u_2), (x, \frac{1}{2}(\xi-\eta)) \in \mathrm{WF}(u_1)\big\}\end{aligned}$$

Notice that

(5.233)
$$(x,0,0,\eta) \in \mathrm{WF}\left(F^*(u_1 \boxtimes u_2)\right) \Longrightarrow (x, \frac{1}{2}\eta) \in \mathrm{WF}(u_1) \text{ and } (x, \frac{1}{2}\eta)\, \mathrm{WF}(u_2)$$

which introduces the assumption under which $u_1 u_2$ is defined. Finally then we see that

(5.234)
$$\begin{aligned}\mathrm{WF}(u_1u_2) \subset &\big\{(x,\xi); x \in \mathrm{supp}(u_1), (x,\xi) \in \mathrm{WF}(u_2)\big\} \\ &\cup\big\{(x,\xi); x \in \mathrm{supp}(u_2), (x,\xi) \in \mathrm{WF}(u_1)\big\} \\ &\cup\big\{(x,\xi); (x,\eta_1) \in \mathrm{WF}(u_1), (x,\eta_2) \in \mathrm{WF}(u_2) \text{ and } \xi = \eta_1 + \eta_2\big\}.\end{aligned}$$

which is another way of writing the conclusion of Proposition 5.18. □

5.25. Pull-back

Now let us consider a general $\mathcal{C}^\infty$ map

(5.235)
$$F : \Omega_1 \longrightarrow \Omega_2,\ \Omega_1 \subset \mathbb{R}^n, \Omega_2 \subset \mathbb{R}^m.$$

Thus even the dimension of domain and range spaces can be different. When can we *define* F^*u, for $u \in \mathcal{C}_c^{-\infty}(\Omega_2)$ and what can we say about $\mathrm{WF}(F^*u)$? For a general map F it is not possible to give a sensible, i.e. consistent, definition of F^*u for all distributions $u \in \mathcal{C}^{-\infty}(\Omega_2)$.

For smooth functions we have defined

(5.236)
$$F^* : \mathcal{C}_c^\infty(\Omega_2) \longrightarrow \mathcal{C}^\infty(\Omega_1)$$

but in general $F^*\phi$ does not have compact support, even if ϕ does. We therefore impose the condition that F be *proper*

(5.237)
$$F^{-1}(K) \Subset \Omega_2 \ \forall\ K \Subset \Omega_2,$$

(mostly just for convenience). In fact if we want to understand F^*u near $\overline{x}_1 \in \Omega_1$ we only need to consider u near $F(\overline{x}_1) \in \Omega_2$.

The problem is that the map (5.235) may be rather complicated. However *any* smooth map can be decomposed into a product of simpler maps, which we can analyze locally. Set

(5.238)
$$\mathrm{graph}(F) = \big\{(x,y) \in \Omega_1 \times \Omega_2;\ y = F(x)\big\} \xrightarrow{\iota_F} \Omega_1 \times \Omega_2.$$

This is *always* an embedded submanifold of $\Omega_1 \times \Omega_2$ the functions $y_i - F_i(x)$, $i = 1, \dots, N$ are independent defining functions for $\mathrm{graph}(F)$ and $x_1, \dots, x_n$ are coordinates on it. Now we can write

(5.239)
$$F = \pi_2 \circ \iota_F \circ g$$

where $g : \Omega_1 \longleftrightarrow \operatorname{graph}(F)$ is the diffeomorphism onto its range $x \longmapsto (x, F(x))$. This decomposes F as a projection, an inclusion and a diffeomorphism. Now consider

$$F^*\phi = g^* \cdot \iota_F^* \cdot \pi_2^*\phi \tag{5.240}$$

i.e. $F^*\phi$ is obtained by pulling ϕ back from Ω_2 to $\Omega_1 \times \Omega_2$, restricting to $\operatorname{graph}(F)$ and then introducing the x_i as coordinates. We have directly discussed $(\pi_2^*\phi)$ but we can actually write it as

$$\pi_2^*\phi = 1 \boxtimes \phi(y), \tag{5.241}$$

so the result we have proved can be applied to it. So let us see what writing (5.240) as

$$F^*\phi = g^* \circ \iota_F^*(1 \boxtimes \phi) \tag{5.242}$$

tells us. If $u \in \mathcal{C}_c^{-\infty}(\Omega_2)$ then

$$\mathrm{WF}(1 \boxtimes u) \subset \big\{(x, y, 0, \eta); (y, \eta) \in \mathrm{WF}(u)\big\} \tag{5.243}$$

by Proposition 5.15. So we have to discuss $\iota_F^*(1 \boxtimes u)$, i.e. restriction to $y = F(x)$. We can do this by making a diffeomorphism:

$$T_F(x, y) = (x, y + F(x)) \tag{5.244}$$

so that $T_F^{-1}(\operatorname{graph}(F)) = \{(x, 0)\}$. Notice that $g \circ T_F = \pi_1$, so

$$F^*\phi = \iota^*_{\{y=0\}}\left(T_F^*(1 \boxtimes u)\right). \tag{5.245}$$

Now from Proposition 5.17 we know that

$$\begin{aligned} \mathrm{WF}(T_F^*(1 \boxtimes u)) &= T_F^*(\mathrm{WF}(1 \boxtimes u)) \\ &= \big\{(X, Y, \Xi, H); (X, Y + F(X), \xi, \eta) \in \mathrm{WF}(1 \boxtimes u), \\ &\qquad\qquad \eta = H, \xi_i = \Xi_i + \Sigma \frac{\partial F_j}{\partial x_i} H_j\big\} \end{aligned} \tag{5.246}$$

i.e.

$$\mathrm{WF}(T_F^*(1 \boxtimes u)) = \big\{(x, y, \xi, \eta); \xi_i = \sum_j \frac{\partial F_j}{\partial x_j}(x)\eta_j, (F(x), \eta) \in \mathrm{WF}(u)\big\}. \tag{5.247}$$

So consider our existence condition for restriction to $y = 0$, that $\xi \neq 0$ on $\mathrm{WF}(T_F^*(1 \boxtimes u))$ i.e.

$$(F(x), \eta) \in \mathrm{WF}(u) \Longrightarrow \sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j \neq 0. \tag{5.248}$$

If (5.248) holds then, from (5.246) and Proposition 5.14

$$\mathrm{WF}(F^*u) \subset \big\{(x, \xi);\ \exists\ (F(x), \eta) \in \mathrm{WF}(u) \text{ and } \xi_j = \sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j\big\}. \tag{5.249}$$

We can reinterpret (5.248) and (5.249) more geometrically. The differential of F gives a map

$$\begin{gathered} F^* : T^*_{F(x)}\Omega_2 \longrightarrow T^*_x\Omega_1\ \forall\ x \in \Omega_1 \\ (F(x), \eta) \longmapsto (x, \xi) \text{ where } \xi_i = \Sigma \frac{\partial F_j}{\partial x_i}\eta_j. \end{gathered} \tag{5.250}$$

Thus (5.248) can be restated as:

$$\begin{aligned}\forall\ x \in \Omega_1,\ \text{the null space of } F_x^* : T^*_{F(x)}\Omega_2 \longrightarrow T^*_x\Omega_1 \\ \text{does not meet } \operatorname{WF}(u)\end{aligned} \tag{5.251}$$

and then (5.249) becomes

$$\operatorname{WF}(F^*u) \subset \bigcup_{x\in\Omega_1} F_x^*[\operatorname{WF}(u) \cap T^*_{F(x)}\Omega_2] = F^*(\operatorname{WF}(u)) \tag{5.252}$$

(proved we are a little careful in that F^* is *not* a map; it is a *relation* between $T^*\Omega_2$ and $T^*\Omega_1$) and in this sense (5.251) holds. Notice that (5.249) is a sensible "consequence" of (5.251), since otherwise $\operatorname{WF}(F^*u)$ would contain some zero directions.

Proposition 5.19. *If $F : \Omega_1 \longrightarrow \Omega_2$ is a proper $\mathcal{C}^\infty$ map then F^* extends (by continuity) from $\mathcal{C}_c^\infty(\Omega_2)$ to*

$$\left\{u \in \mathcal{C}_c^{-\infty}(\Omega_2); F^*(\operatorname{WF}(u)) \cap (\Omega_1 \times 0) = \emptyset \text{ in } T^*\Omega_1\right\} \tag{5.253}$$

and (5.252) *holds.*

5.26. The operation F_*

Next we will look at the dual operation, that of push-forward. Notice the basic properties of pull-back:

$$\text{Maps } \mathcal{C}_c^\infty \text{ to } \mathcal{C}_c^\infty (\text{if } F \text{ is proper}) \tag{5.254}$$

$$\textit{Not} \text{ always defined on distributions.} \tag{5.255}$$

Dually we find

Proposition 5.20. *If $F : \Omega_1 \longrightarrow \Omega_2$ is a $\mathcal{C}^\infty$ map of an open subset of $\mathbb{R}^n$ into an open subset of $\mathbb{R}^n$ then for any $u \in \mathcal{C}_c^{-\infty}(\Omega_1)$*

$$F_*(u)(\phi) = u(F^*\phi) \tag{5.256}$$

is a distribution of compact support and

$$F_* :\ \mathcal{C}_c^{-\infty}(\Omega_1) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_2) \tag{5.257}$$

has the property:

$$\begin{aligned}\operatorname{WF}(F_*u) \subset \big\{(y,\eta); y \in F(\operatorname{supp}(u)), y = F(x), F_x^*\eta = 0\big\} \cup \\ \big\{(y,\eta); y = F(x), (x, F_x^*\eta) \in \operatorname{WF}(u)\big\}.\end{aligned} \tag{5.258}$$

Proof. Notice that the 'opposite ' of (5.254) and (5.255) hold, i.e. F_* is *always* defined but even if $u \in \mathcal{C}_c^\infty(\Omega_1)$ in general $F_*u \notin \mathcal{C}_c^\infty(\Omega_2)$. All we really have to prove is (5.258). As usual we look for a formula in terms of elementary operations. So suppose $u \in \mathcal{C}_c^\infty(\Omega_1)$

$$\begin{aligned}F_*u(\phi) &= u(F^*\phi) \quad \phi \in \mathcal{C}_c^\infty(\Omega_2) \\ &= \int u(x)\ \phi(F(x))\ dx \\ &= \int u(x)\delta(y - F(x))\ \phi(y)\ dydx.\end{aligned} \tag{5.259}$$

Thus, we see that

$$F_*u = \pi_* H^*(u \boxtimes \delta) \tag{5.260}$$

where $\delta = \delta(y) \in \mathcal{C}_c^{-\infty}(\mathbb{R}^m)$, H is the diffeomorphism

$$H(x,y) = (x, y - F(x)) \tag{5.261}$$

and $\pi : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$ is projection *off* the first factor.

Thus (5.260) is the desired decomposition into elementary operations, since $u \boxtimes \delta \in \mathcal{C}_c^{-\infty}(\mathbb{R}^{n+m})$, $\pi_* H^*(u \boxtimes \delta)$ is always defined and indeed the map is continuous, which actually proves (5.260).

So all we need to do is estimate the wavefront set using our earlier results. From Proposition 5.15 it follows that

$$\begin{aligned}
\mathrm{WF}(u \boxtimes \delta) \subset & \big\{(x,0,\xi,\eta); x \in \operatorname{supp}(u), \xi = 0\big\} \cup \big\{(x,0,\xi,0); (x,\xi) \in \mathrm{WF}(u)\big\} \\
& \cup \big\{(x,0,\xi,\eta); (x,\xi) \in \mathrm{WF}(u)\big\} \\
= & \big\{(x,0,\xi,\eta); x \in \operatorname{supp}(u), \xi = 0\big\} \cup \big\{(x,0,\xi,\eta); (x,\xi) \in \mathrm{WF}(u)\big\}.
\end{aligned} \tag{5.262}$$

Then consider what happens under H^*. This is a diffeomorphism so the wavefront set transforms under the pull-back:

$$\begin{aligned}
& \mathrm{WF}(H^*(u \boxtimes \delta)) = \mathrm{WF}(u(x)\delta(y - F(x)) \\
& = \big\{(x, F(x), \Xi, \eta); \Xi_i = \xi_i - \sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j, (x,0,\xi,\eta) \in \mathrm{WF}(u \boxtimes \delta)\big\} \\
& = \big\{(x, F(x), \Xi, \eta); x \in \operatorname{supp}(u), \Xi_i = -\sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j)\big\} \\
& \quad \cup \big\{(x, F(x), \Xi, \eta); \eta \in \mathbb{R}^m, (x,\xi) \in \mathrm{WF}(u)), \Xi_i = \xi_i - \sum_j \frac{\partial F_i}{\partial x_j}\eta_j\big\}.
\end{aligned} \tag{5.263}$$

Finally recall the behaviour of wavefront sets under projection, to see that

$$\begin{aligned}
& \mathrm{WF}(F_* u) \subset \big\{(y,\eta);\ \exists\ (x,y,0,\eta) \in \mathrm{WF}(H^*(u \boxtimes \delta))\big\} \\
& = \big\{(y,\eta); y = F(x) \text{ for some } x \in \operatorname{supp}(u) \text{ and} \\
& \qquad \sum_j \frac{\partial F_j}{\partial x_i}\eta_j = 0,\ i = 1,\dots,n\big\} \\
& \cup \big\{(y,\eta); y = F(x) \text{ for some } (x,\xi) \in \mathrm{WF}(u) \text{ and} \\
& \qquad \xi_i = \sum_j \frac{\partial F_i}{\partial x_i}\eta_j, i = 1,\dots,n)\big\}.
\end{aligned}$$

This says

$$\mathrm{WF}(F_* u) \subset \big\{(y,\eta); y \in F(\operatorname{supp}(u)) \text{ and } F_x^*(\eta) = 0\big\} \tag{5.264}$$

$$\cup \big\{(y,\eta); y = F(x) \text{ with } (x, F_x^*\eta) \in \mathrm{WF}(u)\big\} \tag{5.265}$$

which is just (5.258). □

As usual one should note that the two terms here are "really the same".

Now let us look at F_* as a linear map,

$$F_* :\ \mathcal{C}_c^\infty(\Omega_1) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_2). \tag{5.266}$$

As such it has a Schwartz kernel, indeed (5.260) is just the usual formula for an operator in terms of its kernel:

$$F_*u(y) = \int K(y,x)u(x)dx,\ K(y,x) = \delta(y - F(x)). \tag{5.267}$$

So consider the wavefront set of the kernel:

$$\mathrm{WF}(\delta(y - F(x)) = \mathrm{WF}(H^*\delta(y)) = \big\{(y,x;\eta,\xi); y = F(x), \xi = F_x^*\eta\big\}. \tag{5.268}$$

Now changing the order of the factors we can regard this as a subset

$$\mathrm{WF}'(K) = \big\{((y,\eta),(x,\xi)); y = F(x), \xi = F^*\eta\big\} \subset (\Omega_2 \times \mathbb{R}^m) \times (\Omega_1 \times \mathbb{R}^n). \tag{5.269}$$

As a subset of the product we can regard $\mathrm{WF}'(K)$ as a *relation*: if $\Gamma \subset \Omega_2 \times (\mathbb{R}^n \backslash 0)$ set

$$\mathrm{WF}'(K) \circ \Gamma = \\ \big\{(y,\eta) \in \Omega_2 \times (\mathbb{R}^m \backslash 0); \exists\ ((y,\eta)),(x,\xi)) \in \mathrm{WF}'(K) \text{ and } (x,\xi) \in \Gamma\big\}$$

Indeed with this definition

$$\mathrm{WF}(F_*u) \subset \mathrm{WF}'(K) \circ \mathrm{WF}(u), \quad K = \text{ kernel of } F_*. \tag{5.270}$$

5.27. Wavefront relation

One serious application of our results to date is:

Theorem 5.4. *Suppose $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$ are open and $A \in \mathcal{C}^{-\infty}(\Omega_1 \times \Omega_2)$ has proper support, in the sense that the two projections*

$$\begin{tikzcd} & \operatorname{supp}(A) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Omega_1 & & \Omega_2 \end{tikzcd} \tag{5.271}$$

are proper, then A defines a linear map

$$A : \mathcal{C}_c^\infty(\Omega_2) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_1) \tag{5.272}$$

and extends by continuity to a linear map

$$A : \big\{u \in \mathcal{C}_c^{-\infty}(X); \mathrm{WF}(u) \cap \big\{(y,\eta) \in \Omega_2 \times (\mathbb{R}^n \backslash 0); \tag{5.273}$$

$$\exists\ (x,0,y,-\eta) \in \mathrm{WF}(K)\big\} = \emptyset\big\} \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_1) \tag{5.274}$$

for which

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \circ \mathrm{WF}(u), \tag{5.275}$$

where

$$\begin{aligned} \mathrm{WF}'(A) = \big\{((x,\xi),(y,\eta)) \in & (\Omega_1 \times \mathbb{R}^n) \times (\Omega_2 \times \mathbb{R}^m); (\xi,\eta) \neq 0 \\ & \textit{and } (x,y,\xi,-\eta) \in \mathrm{WF}(K)\big\}. \end{aligned} \tag{5.276}$$

Proof. The action of the map A can be written in terms of its Schwartz kernel as

$$Au(x) = \int K(x,y)u(y)dy = (\pi_1)_*(K \cdot (1 \boxtimes u)). \tag{5.277}$$

Here $1 \boxtimes u$ is always defined and

$$\mathrm{WF}(1 \boxtimes u) \subset \big\{(x,y,0,\eta); (y,\eta) \in \mathrm{WF}(u)\big\}. \tag{5.278}$$

So the main question is, when is the product defined? Our sufficient condition for this is:

$$(x,y,\xi,\eta) \in \mathrm{WF}(K) \Longrightarrow (x,y,-\xi,-\eta) \notin \mathrm{WF}(1 \boxtimes u) \tag{5.279}$$

which is

$$(x,y,0,\eta) \in \mathrm{WF}(K) \Longrightarrow (x,y,0,-\eta) \notin \mathrm{WF}(1 \boxtimes u) \tag{5.280}$$

$$\text{i.e. } (y,-\eta) \notin \mathrm{WF}(u) \tag{5.281}$$

This of course is (5.274):

$$Au \text{ is defined (by continuity) if} \tag{5.282}$$

$$\big\{(y,\eta) \in \mathrm{WF}(u);\ \exists\ (x,0,y,-\eta) \in \mathrm{WF}(A)\big\} = \emptyset. \tag{5.283}$$

Then from our bound on the wavefront set of a product

$$\begin{aligned}
&\mathrm{WF}\,(K \cdot (1 \boxtimes u)) \subset \\
&\big\{(x,y,\xi,\eta); (\xi,\eta) = (\xi',\eta') + (0,\eta'') \text{ with} \\
&\qquad (x,y,\xi',\eta') \in \mathrm{WF}(K) \text{ and } (x,\eta'') \in \mathrm{WF}(u)\big\} \\
&\cup \big\{(x,y,\xi,\eta); (x,y,\xi,\eta) \in \mathrm{WF}(K),\ y \in \mathrm{supp}(u)\big\} \\
&\cup \big\{(x,y,0,\eta); (x,y) \in \mathrm{supp}(A) (y,\eta) \in \mathrm{WF}(u)\big\}.
\end{aligned} \tag{5.284}$$

This gives the bound

$$\mathrm{WF}\,(\pi_*(K \cdot (1 \boxtimes u))) \subset \big\{(x,\xi); (x,y,\xi,0) \in \mathrm{WF}(K \cdot (1 \boxtimes u)) \text{ for some } y\big\} \tag{5.285}$$

$$\subset \mathrm{WF}'(A) \circ \mathrm{WF}(u). \tag{5.286}$$

□

5.28. Applications

Having proved this rather general theorem, let us note some examples and applications.

First, for pseudodifferential operators we know that

$$\mathrm{WF}'(A) \subset \{(x,x,\xi,\xi)\} \tag{5.287}$$

i.e. corresponds to the identity relation (which is a map). Then (5.275) is the microlocality of pseudodifferential operators. The next result also applies to all pseudodifferential operators.

Corollary 5.2. *If $K \in \mathcal{C}^{-\infty}(\Omega_1 \times \Omega_2)$ has proper support and*

$$\mathrm{WF}'(K) \cap \{(x,y,\xi,0)\} = \emptyset \tag{5.288}$$

then the operator with Schwartz kernel K defines a continuous linear map

$$A : \mathcal{C}_c^\infty(\Omega_2) \longrightarrow \mathcal{C}_c^\infty(\Omega_1). \tag{5.289}$$

If

$$\mathrm{WF}'(K) \cap \{(x,y,0,\eta)\} = \emptyset \tag{5.290}$$

then A extends by continuity to

$$A : \mathcal{C}_c^{-\infty}(\Omega_2) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_1). \tag{5.291}$$

Proof. Immediate from (5.272)-(5.291). □

5.29. Problems

PROBLEM 5.9. Show that the general definition (5.52) reduces to

$$\mathrm{WF}(u) = \bigcap \big\{ \Sigma_0(A);\ A \in \Psi^0_\infty(\mathbb{R}^n) \text{ and } Au \in \mathcal{C}^\infty(\mathbb{R}^n) \big\},\ u \in \mathcal{S}'(\mathbb{R}^n) \tag{5.292}$$

and prove the basic result of 'microlocal elliptic regularity:'

$$\begin{gathered} \text{If } u \in \mathcal{S}'(\mathbb{R}^n) \text{ and } A \in \Psi^m_\infty(\mathbb{R}^n) \text{ then} \\ \mathrm{WF}(u) \subset \Sigma(A) \cup \mathrm{WF}(Au). \end{gathered} \tag{5.293}$$

PROBLEM 5.10. Compute the wavefront set of the following distributions:

$$\begin{gathered} \delta(x) \in \mathcal{S}'(\mathbb{R}^n),\ |x| \in \mathcal{S}'(\mathbb{R}^n) \text{ and} \\ \chi_{\mathbb{B}^n}(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1. \end{cases} \end{gathered} \tag{5.294}$$

PROBLEM 5.11. Let $\Gamma \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ be an open cone and define

$$\mathcal{C}^{-\infty}_{c,\Gamma}(\mathbb{R}^n) = \big\{ u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n); Au \in \mathcal{C}^\infty(\mathbb{R}^n) \tag{5.295}$$

$$\forall\ A \in \Psi^0_\infty(\mathbb{R}^n) \text{ with } \mathrm{WF}'(A) \cap \Gamma = \emptyset \big\}. \tag{5.296}$$

Describe a complete topology on this space with respect to which $\mathcal{C}^\infty_c(\mathbb{R}^n)$ is a dense subspace.

PROBLEM 5.12. Show that, for any pseudodifferential operator $A \in \Psi^m_\infty(\mathbb{R}^n)$, $\mathrm{WF}'(A) = \mathrm{WF}'(A^*)$.

PROBLEM 5.13. Give an alternative proof to Lemma 5.5 along the following lines (rather than using Lemma 2.75). If $\sigma_L(A)$ is the left reduced symbol then for $\epsilon > 0$ small enough

$$b_0 = \gamma_\epsilon \big/ \sigma_L(A) \in S^{-m}_\infty(\mathbb{R}^n; \mathbb{R}^n). \tag{5.297}$$

If we choose $B_0 \in \Psi^{-m}_\infty(\mathbb{R}^n)$ with $\sigma_L(B_0) = b_0$ then

$$\mathrm{Id} - A \circ B_0 = G \in \Psi^0_\infty(\mathbb{R}^n) \tag{5.298}$$

has principal symbol

$$\sigma_0(G) = 1 - \sigma_L(A) \cdot b_0. \tag{5.299}$$

From (5.67)

$$\gamma_{\epsilon/4} \sigma_0(G) = \gamma_{\epsilon/4}. \tag{5.300}$$

Thus we conclude that if $\sigma_L(C) = \gamma_{\epsilon/4}$ then

$$G = (\mathrm{Id} - C)G + CG \text{ with } CG \in \Psi^{-1}_\infty(\mathbb{R}^n). \tag{5.301}$$

Thus (5.298) becomes

$$\mathrm{Id} - AB_0 = CG + R_1 \quad \mathrm{WF}'(R_1) \not\ni z. \tag{5.302}$$

Let $B_1 \sim \sum\limits_{j \ge 1} (CG)^j$, $B_1 \in \Psi^{-1}$ and set

$$B = B_0 (\mathrm{Id} + B_1) \in \Psi^{-m}_\infty(\mathbb{R}^n). \tag{5.303}$$

From (5.302)

$$AB = AB_0(I + B_1) \tag{5.304}$$

$$= (\mathrm{Id} - CG)\,(I + B_1) - R_1\,(\mathrm{Id} + B_1) \tag{5.305}$$

$$= \mathrm{Id} + R_2, \quad \mathrm{WF}'(R_2) \not\ni z. \tag{5.306}$$

Thus B is a right microlocal parametrix as desired. Write out the construction of a left parametrix using the same method, or by finding a right parametrix for the adjoint of A and then taking adjoints using Problem 5.12.

PROBLEM 5.14. Essential uniqueness of left and right parametrices.

PROBLEM 5.15. If $(\bar{x}, \bar{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ is a given point, construct a distribution $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ which has

$$\mathrm{WF}(u) = \left\{ (\bar{x}, t\bar{\xi}); t > 0 \right\} \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0). \tag{5.307}$$

PROBLEM 5.16. Suppose that $A \in \Psi_\infty^m(\mathbb{R}^n)$ has Schwartz kernel of compact support. If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ use the four 'elementary operations' (and an earlier result on the wavefront set of kernels) to investigate under what conditions

$$\kappa(x, y) = K_A(x, y)u(y) \text{ and then } \gamma(x) = (\pi_1)_* \kappa \tag{5.308}$$

make sense. What can you say about $\mathrm{WF}(\gamma)$?

PROBLEM 5.17. Consider the projection operation under $\pi_1 : \mathbb{R}^p \times \mathbb{R}^k \longrightarrow \mathbb{R}^p$. Show that $(\pi_1)_*$ can be extended to some distributions which do not have compact support, for example

$$\left\{ u \in \mathcal{S}'(\mathbb{R}^n); \operatorname{supp}(u) \cap K \times \mathbb{R}^k \text{ is compact for each } K \subset\subset \mathbb{R}^n \right\}. \tag{5.309}$$

PROBLEM 5.18. As an exercise, check the Jacobi identify for the Poisson bracket

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \ \forall\ f, g, h \in \mathcal{C}^\infty(\mathbb{R}^{2n}). \tag{5.310}$$

PROBLEM 5.19. The fact that (5.90) determines H_h uniquely is equivalent to the non-degeneracy of ω, that

$$\omega(v, w) = 0 \ \forall\ w \Longrightarrow v = 0. \tag{5.311}$$

Show that if ω is a non-degenerate form and (5.90) is used to define the Poisson bracket by

$$\{f, g\} = \omega(H_f, H_g) = dg(H_f) = H_f g \tag{5.312}$$

then the Jacobi identity (5.310) holds if and only if ω is closed as a 2-form.

PROBLEM 5.20. Check that a finite number of regions (5.94) cover the complement of a neighbourhood of 0 in $\mathbb{R}^n$ and that if a is smooth and has compact support in x then the estimates (5.95) is such neighbourhoods imply that $a \in S_{\mathrm{c}}^M(\mathbb{R}^n; \mathbb{R}^n)$ and conversely.