CHAPTER 3

Schwartz and smoothing algebras

The standard algebra of operators discussed in the previous chapter is not really representative, in its global behaviour, of the algebra of pseudodifferential operators on a compact manifold. Of course this can be attributed to the non-compactness of $\mathbb{R}^n$. However, as we shall see below in the discussion of the isotropic algebra, and then again in the later discussion of the scattering algebra, there are closely related global algebras of pseudodifferential operators on $\mathbb{R}^n$ which behave much more as in the compact case.

The 'non-compactness' of the algebra $\Psi_\infty^\infty(\mathbb{R}^n)$ is evidenced by the fact the the elements of the 'residual' algebra $\Psi_\infty^{-\infty}(\mathbb{R}^n)$ are not all compact as operators on $L^2(\mathbb{R}^n)$, or any other interesting space on which they act. In this chapter we consider a smaller algebra of operators in place of $\Psi_\infty^{-\infty}(\mathbb{R}^n)$. Namely

$$\text{(3.1)}\quad A \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n) \Longleftrightarrow A : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n),\\ A\phi(x) = \int_{\mathbb{R}^n} A(x,y)\phi(y)dy,\ A \in \mathcal{S}(\mathbb{R}^{2n}).$$

The notation here, as the residual part of the isotropic algebra – which has not yet been defined – is rather arbitrary but it seems better than introducing a notation which will be retired later; it might be better to think of $\Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ as the 'Schwartz algebra.'

After discussing this 'Schwartz algebra' at some length we will turn to the corresponding algebra of smoothing operators on a compact manifold (even with corners). This requires a brief introduction to manifolds, with which however I will assume some familiarity, including integration of densities. Then essentially all the results discussed here for operators on $\mathbb{R}^n$ are extended to the more general case, and indeed the Schwartz algebra itself is realized as one version of this generalization.

By definition then, $\Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ is the algebra which corresponds to the non-commutative product on $\mathcal{S}(\mathbb{R}^{2n})$ given by

$$A \circ B(x,y) = \int_{\mathbb{R}^n} A(x,z)B(z,y)dz. \tag{3.2}$$

The properties we discuss here have little direct relation to the 'microlocal' concepts which are discussed in the preceeding chapter. Rather they are more elementary, or at least familiar, results which are needed (and in particular are generalized) later in the discussion of global properties. This formula, (3.2) extends to smoothing operators on manifolds and gives $\mathcal{C}^\infty(M^2)$, where M is a compact manifold, the structure of a non-commutative algebra.

In the discussion of the semiclassical limit of smoothing operators at the end of this chapter the relationship between this non-commutative product and the commutative product on T^*M is discussed. This is used extensively later.

3.1. The residual algebra

The residual algebra in both the isotropic and scattering calculi, discussed below, has two important properties not shared by the residual algebra $\Psi_\infty^{-\infty}(\mathbb{R}^n)$, of which it is a subalgebra (and in fact in which it is an ideal). The first is that as operators on $L^2(\mathbb{R}^n)$ the residual isotropic operators are compact.

PROPOSITION 3.1. *Elements of $\Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ are characterized amongst continuous operators on $\mathcal{S}(\mathbb{R}^n)$ by the fact that they extend by continuity to define continuous linear maps*

$$A : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n). \tag{3.3}$$

In particular the image of a bounded subset of $L^2(\mathbb{R}^n)$ under an element of $\Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ is contained in a compact subset.

PROOF. The kernels of elements of $\Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ are in $\mathcal{S}(\mathbb{R}^{2n})$ so the mapping property (3.3) follows.

The norm $\sup_{|\alpha|\le 1} |\langle x\rangle^{n+1} D^\alpha u(x)|$ is continuous on $\mathcal{S}(\mathbb{R}^n)$. Thus if $S \subset L^2(\mathbb{R}^n)$ is bounded and $A \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ the continuity of $A : L^2(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$ implies that $A(S)$ is bounded with respect to this norm. The theorem of Arzela-Ascoli shows that any sequence in $A(S)$ has a strongly convergent subsequence in $\langle x\rangle^n \mathcal{C}_\infty^0(\mathbb{R}^n)$ and such a sequence converges in $L^2(\mathbb{R}^n)$. Thus $A(S)$ has compact closure in $L^2(\mathbb{R}^n)$ which means that A is compact. $\square$

The second important property of the residual algebra is that it is 'bi-ideal' or a 'corner' in the bounded operators on $L^2(\mathbb{R}^n)$. Note that it is not an ideal.

LEMMA 3.1. *If $A_1,\ A_2 \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$ and B is a bounded operator on $L^2(\mathbb{R}^n)$ then $A_1 B A_2 \in \Psi_{\text{iso}}^{-\infty}(\mathbb{R}^n)$.*

PROOF. The kernel of the composite $C = A_1 B A_2$ can be written as a distributional pairing

$$C(x,y) = \int_{\mathbb{R}^{2n}} B(x',y')A_1(x,x')A_2(y',y)dx'dy' = (B, A_1(x,\cdot)A_2(\cdot,y)) \in \mathcal{S}(\mathbb{R}^{2n}). \tag{3.4}$$

Thus the result follows from the continuity of the exterior product, $\mathcal{S}(\mathbb{R}^{2n}) \times \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}(\mathbb{R}^{4n})$. $\square$

In fact the same conclusion, with essentially the same proof, holds for any continuous linear operator B from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

3.2. The augmented residual algebra

Recall that a bounded operator is said to have finite rank if its range is finite dimensional. If we consider a bounded operator B on $L^2(\mathbb{R}^n)$ which is of finite rank then we may choose an orthonormal basis f_j, $j = 1, \dots, N$ of the range $BL^2(\mathbb{R}^n)$. The functionals $u \longmapsto \langle Bu, f_j\rangle$ are continuous and so define non-vanishing elements $g_j \in L^2(\mathbb{R}^n)$. It follows that the Schwartz kernel of B is

$$B = \sum_{j=1}^N f_j(x)\overline{g_j(y)}. \tag{3.5}$$

If $B \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ then the range must lie in $\mathcal{S}(\mathbb{R}^n)$ and similarly for the range of the adjoint, so the functions f_j are linearly dependent on some finite collection of functions $f_j' \in \mathcal{S}(\mathbb{R}^n)$ and similarly for the g_j. Thus it can be arranged that the f_j and g_j are in $\mathcal{S}(\mathbb{R}^n)$.

PROPOSITION 3.2. *If $A \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ then $\mathrm{Id}+A$ has, as an operator on $L^2(\mathbb{R}^n)$, finite dimensional null space and closed range which is the orthocomplement of the null space of $\mathrm{Id}+A^*$. There is an element $B \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ such that*

$$(\mathrm{Id}+A)(\mathrm{Id}+B) = \mathrm{Id}-\Pi_1,\ (\mathrm{Id}+B)(\mathrm{Id}+A) = \mathrm{Id}-\Pi_0 \tag{3.6}$$

where Π_0, $\Pi_1 \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ are the orthogonal projections onto the null spaces of $\mathrm{Id}+A$ and $\mathrm{Id}+A^*$ and furthermore, there is an element $A' \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ of rank equal to the dimension of the null space such that $\mathrm{Id}+A+sA'$ is an invertible operator on $L^2(\mathbb{R}^n)$ for all $s \neq 0$.

PROOF. Most of these properties are a direct consequence of the fact that A is compact as an operator on $L^2(\mathbb{R}^n)$.

We have shown, in Proposition 3.1 that each $A \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$ is compact. It follows that

$$N_0 = \mathrm{Nul}(\mathrm{Id}+A) \subset L^2(\mathbb{R}^n) \tag{3.7}$$

has compact unit ball. Indeed the unit ball, $B = \{u \in \mathrm{Nul}(\mathrm{Id}+A)\}$ satisfies $B = A(B)$, since $u = -Au$ on B. Thus B is closed (as the null space of a continuous operator) and precompact, hence compact. Any Hilbert space with a compact unit ball is finite dimensional, so $\mathrm{Nul}(\mathrm{Id}+A)$ is finite dimensional.

Now, let $R_1 = \mathrm{Ran}(\mathrm{Id}+A)$ be the range of $\mathrm{Id}+A$; we wish to show that this is a closed subspace of $L^2(\mathbb{R}^n)$. Let $f_k \to f$ be a sequence in R_1, converging in $L^2(\mathbb{R}^n)$. For each k there exists a unique $u_k \in L^2(\mathbb{R}^n)$ with $u_k \perp N_0$ and $(\mathrm{Id}+A)u_k = f_k$. We wish to show that $u_k \to u$. First we show that $\|u_k\|$ is bounded. If not, then along a subsequent $v_j = u_{k(j)}$, $\|v_j\| \to \infty$. Set $w_j = v_j/\|v_j\|$. Using the compactness of A, $w_j = -Aw_j + f_{k(j)}/\|v_j\|$ must have a convergent subsequence, $w_j \to w$. Then $(\mathrm{Id}+A)w = 0$ but $w \perp N_0$ and $\|w\| = 1$ which are contradictory. Thus the sequence u_k is bounded in $L^2(\mathbb{R}^n)$. Then again $u_k = -Au_k + f_k$ has a convergent subsequence with limit u which is a solution of $(\mathrm{Id}+A)u = f$; hence R_1 is closed. The orthocomplement of the range of a bounded operator is always the null space of its adjoint, so R_1 has a finite-dimensional complement $N_1 = \mathrm{Nul}(\mathrm{Id}+A^*)$. The same argument applies to $\mathrm{Id}+A^*$ so gives the orthogonal decompositions

$$\begin{aligned} L^2(\mathbb{R}^n) &= N_0 \oplus R_0,\ N_0 = \mathrm{Nul}(\mathrm{Id}+A),\ R_0 = \mathrm{Ran}(\mathrm{Id}+A^*) \\ L^2(\mathbb{R}^n) &= N_1 \oplus R_1,\ N_1 = \mathrm{Nul}(\mathrm{Id}+A^*),\ R_1 = \mathrm{Ran}(\mathrm{Id}+A). \end{aligned} \tag{3.8}$$

Thus we have shown that $\mathrm{Id}+A$ induces a continuous bijection $\tilde{A} : R_0 \longrightarrow R_1$. From the closed graph theorem the inverse is a bounded operator $\tilde{B} : R_1 \longrightarrow R_0$. In this case continuity also follows from the argument above.[1] Thus $\tilde{B}$ is the generalized inverse of $\mathrm{Id}+A$ in the sense that $B = \tilde{B} - \mathrm{Id}$ satisfies (3.6). It only remains to show that $B \in \Psi_{\mathrm{iso}}^{-\infty}(\mathbb{R}^n)$. This follows from (3.6), the identities in which

[1]We need to show that $\|\tilde{B}f\|$ is bounded when $f \in R_1$ and $\|f\| = 1$. This is just the boundedness of $u \in R_0$ when $f = (\mathrm{Id}+A)u$ is bounded in R_1.

show that

$$B = -A - AB - \Pi_1,\ -B = A + BA + \Pi_0 \implies B = -A + A^2 + ABA - \Pi_1 + A\Pi_0. \tag{3.9}$$

All terms here are in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$; for ABA this follows from Proposition 3.1.

It remains to show the existence of the finite rank perturbation A'. This is equivalent to the vanishing of the index, that is

$$\operatorname{Ind}(\operatorname{Id}+A) = \dim \operatorname{Nul}(\operatorname{Id}+A) - \dim \operatorname{Nul}(\operatorname{Id}+A^*) = 0. \tag{3.10}$$

Indeed, let f_j and g_j, $j = 1, \dots, N$, be respective bases of the two finite dimensional spaces $\operatorname{Nul}(\operatorname{Id}+A)$ and $\operatorname{Nul}(\operatorname{Id}+A^*)$. Then

$$A' = \sum_{j=1}^{N} g_j(x)\overline{f_j(y)} \tag{3.11}$$

is an isomorphism of N_0 onto N_1 which vanishes on R_0. Thus $\operatorname{Id}+A + sA'$ is the direct sum of $\operatorname{Id}+A$ as an operator from R_0 to R_1 and sA' as an operator from N_0 to N_1, invertible when $s \neq 0$.

There is a very simple proof[2] of the equality (3.10) if we use the trace functional discussed in Section 3.5 below; this however is logically suspect as we use (although not crucially) approximation by finite rank operators in the discussion of the trace and this in turn might appear to use the present result via the discussion of ellipticity and the harmonic oscillator. Even though this is not really the case we give a clearly independent, but less elegant proof.

Consider the one-parameter family of operators $\operatorname{Id}+tA$, $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. We shall see that the index, the difference in dimension between $\operatorname{Nul}(\operatorname{Id}+tA)$ and $\operatorname{Nul}(\operatorname{Id}+tA^*)$ is locally constant. To see this it is enough to consider a general A near the point $t = 1$. Consider the pieces of A with respect to the decompositions $L^2(\mathbb{R}^n) = N_i \oplus R_i$, $i = 0, 1$, of domain and range. Thus A is the sum of four terms which we write as a 2×2 matrix

$$A = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}.$$

Since $\operatorname{Id}+A$ has only one term in such a decomposition, $\tilde{A}$ in the lower right, the solution of the equation $(\operatorname{Id}+tA)u = f$ can be written

$$(t-1)A_{00}u_0 + (t-1)A_{01}u_\perp = f_1,\ (t-1)A_{10}u_0 + (A' + (t-1)A_{11})u_\perp = f_\perp \tag{3.12}$$

Since $\tilde{A}$ is invertible, for $t - 1$ small enough the second equation can be solved uniquely for $u_\perp$. Inserted into the first equation this gives

$$\begin{gathered} G(t)u_0 = f_1 + H(t)f_\perp, \\ G(t) = (t-1)A_{00} - (t-1)^2 A_{01}(A' + (t-1)A_{11})^{-1}A_{10}, \\ H(t) = -(t-1)A_{01}(A' + (t-1)A_{11})^{-1}. \end{gathered} \tag{3.13}$$

[2]Namely the trace of a finite rank projection, such as either Π_0 or Π_1, is its rank, hence the dimension of the space onto which it projects. From the identity satisfied by the generalized inverse we see that

$$\operatorname{Ind}(\operatorname{Id}+A) = \operatorname{Tr}(\Pi_0) - \operatorname{Tr}(\Pi_1) = \operatorname{Tr}\left((\operatorname{Id}+B)(\operatorname{Id}+A) - (\operatorname{Id}+A)(\operatorname{Id}+B)\right) = \operatorname{Tr}([B, A]) = 0$$

from the basic property of the trace.

The null space is therefore isomorphic to the null space of $G(t)$ and a complement to the range is isomorphic to a complement to the range of $G(t)$. Since $G(t)$ is a finite rank operator acting from N_0 to N_1 the difference of these dimensions is constant in t, namely equal to $\dim N_0 - \dim N_1$, near $t = 1$ where it is defined.

This argument can be applied to tA so the index is actually constant in $t \in [0, 1]$ and since it certainly vanishes at $t = 0$ it vanishes for all t. In fact, as we shall note below, $\mathrm{Id} + tA$ is invertible outside a discrete set of $t \in \mathbb{C}$. □

Corollary 3.1. *If* $\mathrm{Id} + A$, $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *is injective or surjective on* $L^2(\mathbb{R}^n)$, *in particular if it is invertible as a bounded operator, then it has an inverse of the form* $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

Corollary 3.2. *If* $A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *then as an operator on* $\mathcal{S}(\mathbb{R}^n)$ *or* $\mathcal{S}'(\mathbb{R}^n)$, $\mathrm{Id} + A$ *is Fredholm in the sense that its null space is finite dimensional and its range is closed with a finite dimensional complement.*

Proof. This follows from the existence of the generalized inverse of the form $\mathrm{Id} + B$, $B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. □

3.3. Exponential and logarithm

Proposition 3.3. *The exponential*

$$\exp(A) = \sum_j \frac{1}{j!} A^j : \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.14}$$

is a globally defined, entire, function with range containing a neighbourhood of the identity and with inverse on such a neighbourhood given by the analytic function

$$\log(\mathrm{Id} + A) = \sum_j \frac{(-1)^j}{j} A^j,\ \ A \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n),\ \ \|A\|_{L^2} < 1 \tag{3.15}$$

3.4. The residual group

By definition, $\mathcal{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is the set (if you want to be concrete you can think of them as operators on $L^2(\mathbb{R}^n)$) of invertible operators in $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. If we identify this topologically with $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ then, as follows from Corollary 3.1, $\mathcal{G}^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is open. We will think of it as an infinite-dimensional manifold modeled, of course, on the linear space $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \simeq \mathcal{S}(\mathbb{R}^{2n})$. Since I have no desire to get too deeply into the general theory of such Fréchet manifolds I will keep the discussion as elementary as possible.

The dual space of $\mathcal{S}(\mathbb{R}^p)$ is $\mathcal{S}'(\mathbb{R}^p)$. If we want to think of $\mathcal{S}(\mathbb{R}^p)$ as a manifold we need to consider smooth functions and forms on it. In the finite-dimensional case, the exterior bundles are the antisymmetric parts of the tensor powers of the dual. Since we are in infinite dimensions the tensor power needs to be completed and the usual choice is the 'projective' tensor product. In our case this is something quite simple, namely the k-fold completed tensor power of $\mathcal{S}'(\mathbb{R}^p)$ is just $\mathcal{S}'(\mathbb{R}^{kp})$. Thus we set

$$\begin{multline}\Lambda^k \mathcal{S}(\mathbb{R}^p) = \{u \in \mathcal{S}'(\mathbb{R}^{kp});\ \text{for any permutation} \\ e,\ u(x_{e(1)}, \dots x_{e(h)}) = \mathrm{sgn}(e) u(x_1, \dots x_k)\}.\end{multline} \tag{3.16}$$

In view of this it is enough for us to consider smooth functions on open sets $F \subset \mathcal{S}(\mathbb{R}^p)$ with values in $\mathcal{S}'(\mathbb{R}^p)$ for general p. Thus

$$v : F \longrightarrow \mathcal{S}'(\mathbb{R}^p),\ F \subset \mathcal{S}(\mathbb{R}^n) \text{ open} \tag{3.17}$$

is continuously differentiable on F if there exists a continuous map

$$v' : F \longrightarrow \mathcal{S}'(\mathbb{R}^{n+p}) \text{ and each } u \in F \text{ has a neighbourhood } U$$
$$\text{such that for each } N \ \exists\, M \ \text{ with}$$
$$\|v(u+u') - v(u) - v'(u;u')\|_N \le C\|u'\|_M^2,\ \forall\ u, u+u' \in U.$$

Then, as usual we define smoothness as infinite differentiability by iterating this definition. The smoothness of v in this sense certainly implies that if $f : X \longrightarrow \mathcal{S}(\mathbb{R}^n)$ is a smooth from a finite dimensional manifold then $v \circ F$ is smooth.

Thus we define the notion of a smooth *form* on $F \subset \mathcal{S}(\mathbb{R}^n)$, an open set, as a smooth map

$$\alpha : F \to \Lambda^k \mathcal{S}(\mathbb{R}^p) \subset \mathcal{S}'(\mathbb{R}^{kp}). \tag{3.18}$$

In particular we know what smooth forms are on $\mathcal{G}^{-\infty}_{\text{iso}}(\mathbb{R}^n)$.

The de Rham differential acts on forms as usual. If $v : F \to \mathbb{C}$ is a function then its differential at $f \in F$ is $dv : F \longrightarrow \mathcal{S}'(\mathbb{R}^n) = \Lambda^1 \mathcal{S}(\mathbb{R}^n)$, just the derivative. As in the finite-dimensional case d extends to forms by enforcing the condition that $dv = 0$ for constant forms and the distribution identity over exterior products

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta. \tag{3.19}$$

3.5. Traces on the residual algebra

The algebras we are studying are topological algebras, so it makes sense to consider continuous linear functionals on them. The most important of these is the *trace*. To remind you what it is we consider first its properties for matrix algebras.

Let $M(N;\mathbb{C})$ denote the algebra of $N \times N$ complex matrices. We can simply define

$$\text{Tr} : M(N;\mathbb{C}) \to \mathbb{C}, \quad \text{Tr}(A) = \sum_{i=1}^N A_{ii} \tag{3.20}$$

as the sum of the diagonal entries. The fundamental property of this functional is that

$$\text{Tr}([A,B]) = 0 \ \forall\ A, B \in M(N;\mathbb{C}). \tag{3.21}$$

To check this it is only necessary to write down the definition of the composition in the algebra. Thus

$$(AB)_{ij} = \sum_{k=1}^N A_{ik} B_{kj}.$$

It follows that

$$\begin{aligned} \text{Tr}(AB) &= \sum_{i=1}^N (AB)_{ii} = \sum_{i,k=1}^N A_{ik} B_{ki} \\ &= \sum_{k=1}^N \sum_{i=1}^N B_{ki} A_{ik} = \sum_{k=1}^N (BA)_{kk} = \text{Tr}(BA) \end{aligned}$$

which is just (3.21).

Of course any multiple of Tr has the same property (3.21) but the normalization condition

$$\mathrm{Tr}(\mathrm{Id}) = N \tag{3.22}$$

distinguishes it from its multiples. In fact (3.21) and (3.22) together distinguish $\mathrm{Tr} \in M(N;\mathbb{C})'$ as a point in the N^2 dimensional linear space which is the dual of $M(N;\mathbb{C})$.

LEMMA 3.2. *If $F : M(N;\mathbb{C}) \to \mathbb{C}$ is a linear functional satisfying* (3.21) *and $B \in M(N;\mathbb{C})$ is any matrix such that $F(B) \neq 0$ then $F(A) = \frac{F(B)}{\mathrm{Tr}(B)} \mathrm{Tr}(A)$.*

PROOF. Consider the basis of $M(N;\mathbb{C})$ given by the elementary matrices E_{jk}, where E_{jk} has jk-th entry 1 and all others zero. Thus

$$E_{jk}E_{pq} = \delta_{kp}E_{jq}.$$

If $j \neq k$ it follows that

$$E_{jj}E_{jk} = E_{jk},\ E_{jk}E_{jj} = 0.$$

Thus

$$F([E_{jj}, E_{jk}]) = F(E_{jk}) = 0 \text{ if } j \neq k.$$

On the other hand, for any i and j

$$E_{ji}E_{ij} = E_{jj},\ E_{ij}E_{ji} = E_{ii}$$

so

$$F(E_{jj}) = F(E_{11})\ \forall\ j.$$

Since the E_{jk} are a basis,

$$\begin{aligned} F(A) &= F(\sum_{j,k=1}^{N} A_{ij}E_{ij}) \\ &= \sum_{j,l=1}^{N} A_{jj}F(E_{ij}) \\ &= F(E_{11})\sum_{j=1}^{N} A_{jj} = F(E_{11})\,\mathrm{Tr}(A). \end{aligned}$$

This proves the lemma. □

For the isotropic smoothing algebra we have a similar result.

PROPOSITION 3.4. *If $F : \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \simeq \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathbb{C}$ is a continuous linear functional satisfying*

$$F([A,B]) = 0\ \forall\ A, B \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \tag{3.23}$$

then F is a constant multiple of the functional

$$\mathrm{Tr}(A) = \int_{\mathbb{R}^n} A(x,x)dx. \tag{3.24}$$

PROOF. Recall that $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) \subset \Psi^{\infty}_{\text{iso}}(\mathbb{R}^n)$ is an ideal so $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ and $B \in \Psi^{\infty}_{\text{iso}}(\mathbb{R}^n)$ implies that AB, $BA \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ and it follows that the equality $F(AB) = F(BA)$, or $F([A,B]) = 0$, is meaningful. To see that it holds we just use the continuity of F. We know that if $B \in \Psi^{\infty}_{\text{iso}}(\mathbb{R}^n)$ then there is a sequence $B_n \to B$ in the topology of $\Psi^{m}_{\text{iso}}(\mathbb{R}^n)$ for some m. Since this implies $AB_n \to AB$, $B_nA \to BA$ in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ we see that

$$F([A,B]) = \lim_{n\to\infty} F([A,B_n]) = 0.$$

We use this identity to prove (3.24). Take $B = x_j$ or D_j, $j = 1,\dots,n$. Thus for any $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$

$$F([A,x_j]) = F([A,D_j]) = 0.$$

Now consider F as a distribution acting on the kernel $A \in \mathcal{S}(\mathbb{R}^{2n})$. Since the kernel of $[A,x_j]$ is $A(x,y)(y_j - x_j)$ and the kernel of (A,D_j) is $-(D_{y_j} + D_{x_j})A(x,y)$ we conclude that, as an element of $\mathcal{S}'(\mathbb{R}^{2n})$, F satisfies

$$(x_j - y_j)F(x,y) = 0,\ (D_{x_j} + D_{y_j})F(x,y) = 0.$$

If we make the linear change of variables to $p_i = \frac{x_i+y_i}{2}$, $q_i = x_i - y_i$ and set $\tilde{F}(p,q) = F(x,y)$ these conditions become

$$D_{q_i}\tilde{F} = 0,\ p_i\tilde{F} = 0,\ i = 1,\dots,N.$$

As we know from Lemmas 1.2 and 1.3, this implies that $\tilde{F} = c\delta(p)$ so

$$F(x,y) = c\delta(x-y)$$

as a distribution. Clearly $\delta(x-y)$ gives the functional Tr defined by (3.24), so the proposition is proved. $\square$

We still need to justify the use of the same notation, Tr, for these two functionals. However, if $L \subset \mathcal{S}(\mathbb{R}^n)$ is any finite dimensional subspace we may choose an orthonal basis $\varphi_i \in L$, $i = 1,\dots,l$,

$$\int_{\mathbb{R}^n} |\varphi_i(x)|^2 dx = 0,\ \int_{\mathbb{R}^n} \varphi_i(x)\overline{\varphi_j}(x)dx = 0,\ i \neq j.$$

Then if a_{ij} is an $l \times l$ matrix,

$$A = \sum_{i,j=1}^{\ell} a_{ij}\varphi_i(x)\overline{\varphi_j(y)} \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n).$$

From (3.24) we see that

$$\begin{aligned}
\operatorname{Tr}(A) &= \sum_{ij} a_{ij}\operatorname{Tr}(\varphi_i\bar{\varphi}_j) \\
&= \sum_{ij} a_{ij}\int_{\mathbb{R}^n} \varphi_i(x)\overline{\varphi_j}(x)dx \\
&= \sum_{i=1}^{n} a_{ii} = \operatorname{Tr}(a).
\end{aligned}$$

Thus the two notions of trace coincide. In any case this already follows, up to a constant, from the uniqueness in Lemma 3.2.

3.6. Fredholm determinant

For $N \times N$ matrices, the determinant is a multiplicative polynomial map

$$\det : M(N;\mathbb{C}) \longrightarrow \mathbb{C},\ \det(AB) = \det(A)\det(B),\ \det(\mathrm{Id}) = 1. \tag{3.25}$$

It is not quite determined by these conditions, since $\det(A)^k$ also satisfies then. The fundamental property of the determinant is that it defines the group of invertible elements

$$\mathrm{GL}(N,\mathbb{C}) = \{A \in M(N;\mathbb{C}); \det(A) \neq 0\}. \tag{3.26}$$

A reminder of a direct definition is given in Problem 4.7.

The Fredholm determinant is an extension of this definition to a function on the ring $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. This can be done in several ways using the density of finite rank operators, as shown in Corollary 4.2. We proceed by generalizing the formula relating the determinant to the trace. Thus, for any smooth curve with values in $\mathrm{GL}(N;\mathbb{C})$ for any N,

$$\frac{d}{ds}\det(A_s) = \det(A_s)\,\mathrm{tr}(A_s^{-1}\frac{A_s}{ds}). \tag{3.27}$$

In particular if (3.25) is augmented by the normalization condition

$$\frac{d}{ds}\det(\mathrm{Id} + sA)\big|_{s=0} = \mathrm{tr}(A)\ \forall\ A \in M(N;\mathbb{C}) \tag{3.28}$$

then it is determined.

A branch of the logarithm can be introduced along any curve, smoothly in the parameter, and then (3.27) can be rewritten

$$d\log\det(A) = \mathrm{tr}(A^{-1}dA). \tag{3.29}$$

Here $\mathrm{GL}(N;\mathbb{C})$ is regarded as a subset of the linear space $M(N;\mathbb{C})$ and dA is the canonical identification, at the point A, of the tangent space to $M(N,\mathbb{C})$ with $M(N,\mathbb{C})$ itself. This just arises from the fact that $M(N,\mathbb{C})$ is a linear space. Thus $dA(\frac{d}{ds}(A + sB)\big|_{s=0} = B$. This allows the expression on the right in (3.29) to be interpreted as a smooth 1-form on the manifold $\mathrm{GL}(N;\mathbb{C})$. Note that it is independent of the local choice of logarithm.

To define the Fredholm determinant we shall extend the 1-form

$$\alpha = \mathrm{Tr}(A^{-1}dA) \tag{3.30}$$

to the group $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \hookrightarrow \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Here dA has essentially the same meaning as before, given that Id is fixed. Thus at any point $A = \mathrm{Id} + B \in \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ it is the identification of the tangent space with $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ using the linear structure:

$$dA(\frac{d}{ds}(\mathrm{Id} + B + sE)\big|_{s=0}) = E,\ E \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n).$$

Since dA takes values in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, the trace functional in (3.30) is well defined.

The 1-form α is closed. In the finite-dimensional case this follows from (3.29). For (3.30) we can compute directly. Since $d(dA) = 0$, essentially by definition, and

$$dA^{-1} = -A^{-1}dAA^{-1} \tag{3.31}$$

we see that

$$d\alpha = -\,\mathrm{Tr}(A^{-1}(dA)A^{-1}(dA)) = 0. \tag{3.32}$$

Here we have used the trace identity, and the antisymmetry of the implicit wedge product in (3.32), to conlcude that $d\alpha = 0$. For a more detailed discussion of this point see Problem 4.8.

From the fact that $d\alpha = 0$ we can be confident that there is, locally near any point of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$, a function f such that $df = \alpha$; then we will define the Fredholm determinant by $\det_{\mathrm{Fr}}(A) = \exp(f)$. To define $\det_{\mathrm{Fr}}$ globally we need to see that this is well defined.

Lemma 3.3. *For any smooth closed curve $\gamma : \mathbb{S}^1 \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ the integral*

$$\int_\gamma \alpha = \int_{\mathbb{S}^1} \gamma^* \alpha \in 2\pi i \mathbb{Z}. \tag{3.33}$$

That is, α defines an integral cohomology class, $[\frac{\alpha}{2\pi i}] \in H^1(G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n); \mathbb{Z})$.

Proof. This is where we use the approximability by finite rank operators. If π_N is the orthogonal projection onto the span of the eigenspaces of the smallest N eigenvalues of the harmonic oscillator then we know from Section 4.3 that $\pi_N E \pi_N \to E$ in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ for any element. In fact it follows that for the smooth curve that $\gamma(s) = \mathrm{Id} + E(s)$ and $E_N(s) = \pi_N E(s) \pi_N$ converges uniformly with all s derivatives. Thus, for some N_0 and all $N > N_0$, $\mathrm{Id} + E_N(s)$ is a smooth curve in $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ and hence $\gamma_N(s) = \mathrm{Id}_N + E_N(s)$ is a smooth curve in $\mathrm{GL}(N; \mathbb{C})$. Clearly

$$\int_{\gamma_N} \alpha \longrightarrow \int_\gamma \alpha \text{ as } N \to \infty, \tag{3.34}$$

and for finite N it follows from the identity of the trace with the matrix trace (see Section 3.5) that $\int_N \gamma_N^* \alpha$ is the variation of $\arg \log \det(\gamma_N)$ around the curve. This gives (3.33). $\square$

Now, once we have (3.33) and the connectedness of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ we may define

$$\det_{\mathrm{Fr}}(A) = \exp(\int_\gamma \alpha), \ \gamma : [0,1] \longrightarrow G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n), \ \gamma(0) = \mathrm{Id}, \ \gamma(1) = A. \tag{3.35}$$

Indeed, Lemma 3.3 shows that this is independent of the path chosen from the identity to A. Notice that the connectedness of $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ follows from the connectedness of the $\mathrm{GL}(N, \mathbb{C})$ and the density argument above.

The same arguments and results apply to $G^{-2n-\epsilon}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ using the fact that the trace functional extends continuously to $\Psi^{-2n-\epsilon}_{\infty-\mathrm{iso}}(\mathbb{R}^n)$ for any $\epsilon > 0$.

Proposition 3.5. *The Fredholm determinant, defined by* (3.35) *on $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ (or $G^{-2n-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)$ for $\epsilon > 0$) and to be zero on the complement in* $\mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ *(or* $\mathrm{Id} + \Psi^{-2n-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)$) *is an entire function satisfying*

$$\det_{\mathrm{Fr}}(AB) = \det_{\mathrm{Fr}}(A) \det_{\mathrm{Fr}}(B), \ A, B \in \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \\ (\mathit{or}\ \mathrm{Id} + \Psi^{-2n-\epsilon}_{\mathrm{iso}}(\mathbb{R}^n)), \ \det_{\mathrm{Fr}}(\mathrm{Id}) = 1. \tag{3.36}$$

Proof. We start with the multiplicative property of $\det_{\mathrm{Fr}}$ on $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$. Thus is $\gamma_1(s)$ is a smooth curve from Id to A_1 and $\gamma_2(s)$ is a smooth curve from Id to A_2 then $\gamma(s) = \gamma_1(s)\gamma_2(s)$ is a smooth curve from Id to $A_1 A_2$. Consider the differential on this curve. Since

$$\frac{d(A_1(s)A_2(s))}{ds} = \frac{dA_1(s)}{ds} A_2(s) + A_1(s) \frac{dA_2(s)}{ds}$$

the 1-form becomes

$$\gamma^*(s)\alpha(s) = \operatorname{Tr}(A_2(s)^{-1}\frac{dA_2(s)}{ds}) + \operatorname{Tr}(A_2(s)^{-1}A_1(s)^{-1}\frac{dA_2(s)}{ds}A_2(s)). \tag{3.37}$$

In the second term on the right we can use the trace identity, since $\operatorname{Tr}(GA) = \operatorname{Tr}(AG)$ if $G \in \Psi^{\mathbb{Z}}_{\text{iso}}(\mathbb{R}^n)$ and $A \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Thus (3.37) becomes

$$\gamma^*(s)\alpha(s) = \gamma_1^*\alpha + \gamma_2^*\alpha.$$

Inserting this into the definition of $\det_{\text{Fr}}$ gives (3.36) when both factors are in $G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Of course if either factor is not invertible, then so is the product and hence both $\det_{\text{Fr}}(AB)$ and at least one of $\det_{\text{Fr}}(A)$ and $\det_{\text{Fr}}(B)$ vanishes. Thus (3.36) holds in general when $\det_{\text{Fr}}$ is extended to be zero on the non-invertible elements.

Thus it remains to establish the smoothness. That $\det_{\text{Fr}}(A)$ is smooth in any real parameters in which $A \in G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ depends, or indeed is holomorphic in holomorphic parameters, follows from the definition since α clearly depends smoothly, or holomorphically, on parameters. In fact the same follows if holomorphy is examined as a function of E, $A = \operatorname{Id} + E$, for $E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Thus it is only smoothness across the non-invertibles that is at issue. To prove this we use the multiplicativity just established.

If $A = \operatorname{Id} + E$ is not invertible, $E \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ then it has a generalized inverse $\operatorname{Id} + E'$ as in Proposition 4.3. Since A has index zero, we may actually replace E' by $E' + E''$, where E'' is an invertible linear map from the orthocomplement of the range of A to its null space. Then $\operatorname{Id} + E' + E'' \in G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$ and $(\operatorname{Id} + E' + E'')A = \operatorname{Id} - \Pi_0$. To prove the smoothness of $\det_{\text{Fr}}$ on a neighbourhood of A it is enough to prove the smoothness on a neighbourhood of $\operatorname{Id} - \Pi_0$ since $\operatorname{Id} + E' + E''$ maps a neighbourhood of the first to a neighbourhood of the second and $\det_{\text{Fr}}$ is multiplicative. Thus consider $\det_{\text{Fr}}$ on a set $\operatorname{Id} - \Pi_0 + E$ where E is near 0 in $\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$, in particular we may assume that $\operatorname{Id} + E \in G^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Thus

$$\det_{\text{Fr}}(\operatorname{Id} + E - \Pi_0) = \det(\operatorname{Id} + E)\det(\operatorname{Id} - \Pi_0 + (G_E - \operatorname{Id})\Pi_0)$$

were $G_E = (\operatorname{Id} + E)^{-1}$ depends holomorphically on E. Thus it suffices to prove the smoothness of $\det_{\text{Fr}}(\operatorname{Id} - \Pi_0 + H\Pi_0)$ where $H \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$

Consider the deformation $H_s = \Pi_0 H \Pi_0 + s(\operatorname{Id} - \Pi_0)H\Pi_0$, $s \in [0,1]$. If $\operatorname{Id} - \Pi_0 + H_s$ is invertible for one value of s it is invertible for all, since its range is always the range of $\operatorname{Id} - \Pi_0$ plus the range of $\Pi_0 H \Pi_0$. It follows that $\det_{\text{Fr}}(\operatorname{Id} - \Pi_0 + H_s)$ is smooth in s; in fact it is constant. If the family is not invertible this follows immediately and if it is invertible then

$$\begin{aligned}\frac{d\det_{\text{Fr}}(\operatorname{Id} - \Pi_0 + H_s)}{ds}& \\ &= \det_{\text{Fr}}(\operatorname{Id} - \Pi_0 + H_s)\operatorname{Tr}\left((\operatorname{Id} - \Pi_0 + H_s)^{-1}(\operatorname{Id} - Pi_0)H\Pi_0)\right) = 0\end{aligned}$$

since the argument of the trace is finite rank and off-diagonal with respect to the decomposition by Π_0.

Thus finally it is enough to consider the smoothness of $\det_{\text{Fr}}(\operatorname{Id} - \Pi_0 + \Pi_0 H \Pi_0)$ as a function of $H \in \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n)$. Since this is just $\det(\Pi_0 H \Pi_0)$, interpreted as a finite rank map on the range of Π_0 the result follows from the finite dimensional case. $\square$

3.7. Fredholm alternative

Since we have shown that $\det_{\mathrm{Fr}} : \mathrm{Id} + \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ is an entire function, we see that $G^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is the complement of a (singular) holomorphic hypersurface, namely the surface $\{\mathrm{Id} + E; \det_{\mathrm{Fr}}(\mathrm{Id} + E) = 0\}$. This has the following consequence, which is sometimes call the 'Fredholm alternative' and also part of 'analytic Fredholm theory'.

Lemma 3.4. *If $\Omega \subset \mathbb{C}$ is an open, connected set and $A : \Omega \longrightarrow \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$ is a holomorphic function then either $\mathrm{Id} + A(z)$ is invertible on all but a discrete subset of Ω and $(\mathrm{Id} + A(z))$ is meromorphic on Ω with all residues of finite rank, or else it is invertible at no point of Ω.*

Proof. Of course the point here is that $\det_{\mathrm{Fr}}(\mathrm{Id} + A(z))$ is a holomorphic function on Ω. Thus, either $\det_{\mathrm{Fr}}(A(z)) = 0$ is a discrete set, $D \subset \Omega$ or else $\det_{\mathrm{Fr}}(\mathrm{Id} + A(z)) \equiv 0$ on Ω; this uses the connectedness of Ω. Since this corresponds exactly to the invertibility of $\mathrm{Id} + A(z)$ the main part of the lemma is proved. It remains only to show that, in the former case, $(\mathrm{Id} + A(z))^{-1}$ is meromorphic. Thus consider a point $p \in D$. Thus the claim is that near p

$$(\mathrm{Id} + A(z))^{-1} = \mathrm{Id} + E(z) + \sum_{j=1}^{N} z^{-j} E_j,\ E_j \in \Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) \text{ of finite rank} \tag{3.38}$$

and where $E(z)$ is locally holomorphic with values in $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n)$.

If N is sufficiently large and Π_N is the projection onto the first N eigenspaces of the harmonic oscillator then $B(z) = \mathrm{Id} + E(z) - \Pi_N E(z) \Pi_N$ is invertible near p with the inverse being of the form $\mathrm{Id} + F(z)$ with $F(z)$ locally holomorphic. Now

$$\begin{aligned}(\mathrm{Id} + F(z))(\mathrm{Id} + E(z)) &= \mathrm{Id} + (\mathrm{Id} + F(z))\Pi_N E(z) \Pi_N \\ &= (\mathrm{Id} - \Pi_N) + \Pi_N M(z) \Pi_N + (\mathrm{Id} - \Pi_N) M'(z) \Pi_N.\end{aligned}$$

It follows that this is invertible if and only if $M(z)$ is invertible as a matrix on the range of Π_N. Since it must be invertible near, but not at, p, its inverse is a meromorphic matrix $K(z)$. It follows that the inverse of the product above can be written

$$\mathrm{Id} - \Pi_N + \Pi_N K(z) \Pi_N - (\mathrm{Id} - \Pi_N) M'(z) \Pi_N K(z) \Pi_N. \tag{3.39}$$

This is meromorphic and has finite rank residues, so it follows that the same is true of $A(z)^{-1}$. $\square$

3.8. Manifolds and functions

Here is a version of the standard definition of a manifold (with corners). First let M be a Hausdorff topological space. That is, we already have the 'topology' of open subsets of M, closed under arbitrary intersections and finite unions. We then know which real-functions on M are continuous – namely those $f : M \longrightarrow \mathbb{R}$ such that $f^{-1}(a, b) \subset M$ is open for every $a < b$. The Hausdorff condition is that these continuous functions separate points, so if $p_1 \neq p_2$ are two points in M then there is a continuous function f on M such that $f(p_1) \neq f(p_2)$. We also assume that M is second countable, that the topology has a countable basis – there is a countable collection of open subsets such that every open subset is a union of these particular open subsets.

A $\mathcal{C}^\infty$ structure on M can be taken to be a subset $\mathcal{C}^\infty(M) \subset \mathcal{C}^0(M)$ of the space of continuous functions which has the following properties. First, it is a subalgebra. Second it generates (product) coordinate systems. That is there is a countable open cover of M by subsets U_i for each of which there are n elements $f_{i,j} \in \mathcal{C}^\infty(M)$ such that $F_i = (f_{1,1}, \dots, f_{i,n})$ restricts to U_i to give a topological isomorphism

$$F_i\big|_{U_i} : U_i \longrightarrow [0,1)^k \times (-1,1)^{n-k} \subset \mathbb{R}^n \tag{3.40}$$

and such that if $g \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ has support in $(-1,1)^n$ then

$$g' = \begin{cases} F_i^* g & \text{on } U_i \\ 0 & \text{on } M \setminus U_i \end{cases} \in \mathcal{C}^\infty(M), \tag{3.41}$$

and that these functions form an ideal in $\mathcal{C}^\infty(M)$. Thirdly we require that $\mathcal{C}^\infty(M)$ is maximal in the sense that if $g : M \longmapsto \mathbb{R}$ and for each i, $g\big|_{U_i} = F_i^* h_i$ for some $h_i \in \mathcal{C}^\infty((-1)^n)$ then $g \in \mathcal{C}^\infty(M)$.

In fact I would call a manifold as defined in the preceeding paragraph a t-manifold. It has various problems. One is that I have not insisted that the local dimension n is not fixed. This is not a serious problem, but it means that M may be up to even a countable union of compoents, each of which is a connected manifold, in the same sense, and hence has fixed dimension. Often this is required anyway, at at least it is how most people think – that a manifold is connected. Apart from that there are more serious problems with the boundary when k, which is the local boundary codimension, takes the value 2 or greater. This is not really imortant here but I usually insist on an additional condition, that the boundary faces be embedded. This is actually a combinatorial condition and means that each boundary hypersurface, defined as the closure of a component of the set of boundary points of 'codimension one' (meaning the union of the the inverse images of the subsets, in the coordinate patches, of $[0,1)^k \times (-1,1)^{n-k}$ where exactly one of the first k variables vanishes), is embedded. One way of thinking about this is that some neighbourhood of each point in the closure of such a boundary point meets the component of the codimension one boundary in a connected set.

A map between manifolds, $f : M \longrightarrow N$ is smooth if and only if the composite $u \circ f \in \mathcal{C}^\infty(M)$ for every $y \in \mathcal{C}^\infty(N)$. It is usual to write this as a pull-back map

$$f^* : \mathcal{C}^\infty(N) \longrightarrow \mathcal{C}^\infty(M),\ f^* u = u \circ f. \tag{3.42}$$

The discussion above is not a good way to learn about manifolds – I am assuming you will look things up somewhere if you don't know about them. The only real virtue of this definition is that it is short. [3]

3.9. Tangent and cotangent bundles

From one manifold we can make others. The most basic examples of this is the passage to a boundary face of a manifold with corners and taking products of manifolds. A more sophisticated example, blow up, is discussed briefly below and we have already described to compactification of Euclidean space to a ball. However the most frequently encountered 'derived' manifold below is the cotangent

[3] In case you, gentle reader, really want to learn the elementary theory of manifolds for yourself and are unable to pick up an appropriate book I have added (or will add) lots of 'problems' to guide, or remind, you a little.

bundle. Once again the approach I give here is not really introductory, its main virtue is brevity.

On Euclidean space of a smooth function near a point, $\bar{z}$, can always be decomposed in terms of coordinate functions

$$f(z) = f(\bar{z}) + \sum_{j=1}^{n} f_j(z)(z_j - \bar{z}_j) \tag{3.43}$$

where the coefficient functions f_j are smooth near $\bar{z}$. The f_j are not determined by this Taylor expansion but their values at $\bar{z}$, namely the derivatives of f at $\bar{z}$, are determined. We can capture these derivatives, collectively, as elements of the vector space

$$\mathcal{J}(\bar{z})/\mathcal{J}(\bar{z})^2,\ \mathcal{J}(\bar{z}) = \{f \in \mathcal{C}^\infty(\mathbb{R}^n); f(\bar{z}) = 0\},\ \mathcal{J}(\bar{z})^2 = \{\sum_{\text{finite}} f_i g_i,\ f_i,\ g_i \in \mathcal{J}(\bar{z})\}. \tag{3.44}$$

Thus $f(z) - f(\bar{z}) \in \mathcal{J}(\bar{z})$ and $\mathcal{J}(\bar{z})/\mathcal{J}(\bar{z})^2$ is an n-dimensional vector space. In fact it is only necessary for f to be defined and smooth in some neighbourhood of $\bar{z}$ for this to be well defined since if ϕ is a cutoff, supported sufficiently close to $\bar{z}$ and equal to 1 in some neighbourhood, then the class of $f\phi - f(\bar{z})$ in $\mathcal{J}(\bar{z})/\mathcal{J}(\bar{z})^2$ is independent of the choice of ϕ. Of course this is the deRham differential. Moreover the discussion extends immediately to smooth manifold and defines

$$df(p) \in T_p^*M = \mathcal{J}(p)/\mathcal{J}(p)^2, \tag{3.45}$$

the cotangent space at each point $p \in M$. This is a vector space of dimension n which is spanned by the differentials of any coordinate system in a neighbourhood of p.

The union of the cotangent fibres has a natural structure as a manifold

$$T^*M = \bigcup_{p \in M} T_p^*M \xrightarrow{\pi} M. \tag{3.46}$$

Namely a coordinate system on an open set $U \subset M$ gives a global coordinate system on the open subset $\pi^{-1}(U)$ identifying it (by definition smoothly) with $U \times \mathbb{R}^n$.

The tangent bundle can be defined as the dual of T^*M or directly in terms of vector fields; taking the first approach

$$T_pM = \{v : T_p^*M \longrightarrow \mathbb{R},\ \text{linear}\},\ TM = \bigcup_{p \in M} T_pM \xrightarrow{\pi} M. \tag{3.47}$$

Coordinate systems on M again give coordinate systems on TM.

3.10. Integration and densities

There is no natural notion equivalent to the Lebesgue integral on a manifold, the problem being that the 'measure' part is changes by a positive smooth multiple under coordinate transformations, namely by the Jacobian determinant. It is therefore necessary either to make a choice of 'density' or else to include the density in the integrand, and integrate only densities. The latter approach is taken here and this requires the introduction of the density bundle, which is a simple example of a trivial line bundle which is not canonically trivial.

PROBLEM 3.1. Show that the smooth functions on $\mathbb{R}^n \setminus \{0\}$ which are 'positively' homogeneous of some complex degree s, meaning the satify

$$f(rz) = r^s f(z),\ \forall\ r > 0,\ z \in \mathbb{R}^n \setminus \{0\} \tag{3.48}$$

(where r^s is the standard branch) is a trivial, but not canonically trivial, line bundle over $\mathbb{S}^{n-1}$, except in the case $s = 0$ when it *is* canonically trivial.

At each point of a manifold consider the 1-dimensional, real, vector space of totally antisymmetric absolutely homogeneous n-multilinear functions

(3.49)

$$\Omega_p M = \{\nu : T_pM \times \cdots \times T_pM \longrightarrow \mathbb{R},\ \nu(v_{e(1)}, \dots, v_{e(n)}) = \operatorname{sgn} e\nu(v_1, \dots, v_n),\ \nu(tv_1, \dots, v_n) = |t|\nu(v_1, \dots, v_n),\ t \in$$

where $v_i \in T_pM$, $i = 1, \dots, n$ are arbitrary and e is any permutation. It is straightforward to check that this is a linear space (it seems a little strange if view of the absolute value of t in the last identity but it is true). If z_i are local coordinates in a neighbourhood of p then the differentials dz_i define a density

$$\nu(v_1, \dots, v_n) = |\det dz_i(v_j)|. \tag{3.50}$$

This is the local coordinate representative of Lebesgue measure at the point.

As for the tangent bundle above, the union of the fibres Ω_p form a manifold,

$$\Omega M = \bigcup_{p \in M} \Omega_p M \xrightarrow{\pi} M. \tag{3.51}$$

A section of ΩM, meaning a smooth map $\nu : M \longrightarrow \Omega M$ such that $\pi\nu = \mathrm{Id}_M$, is by definition a *smooth density* on M. The linear space of such sections is denoted $\mathcal{C}^\infty(M; \Omega)$ and the behaviour of integrals under coordinate transformation reduces directly to the existence of a well defined integral:

$$\int_M : \mathcal{C}^\infty(M; \Omega) \longrightarrow \mathbb{R}. \tag{3.52}$$

Checking that this is well-defined reduces to the usual change-of-variable formula fo Lebesgue (or Riemann) integral in local coordinates.

3.11. Smoothing operators

Now, we come to the point of interest in this chapter. If M is a compact manifold then the algebra of smoothing operators on M behaves in very much the same was as the Schwartz algebra on $\mathbb{R}^n$. In fact it is isomorphic to it as an algebra (if the dimension of M is positive) although there is no natural isomorphism. As we shall see later, the smoothing operators form the residual part of the pseudodifferential algebra on a manifold and are important for that reason. However they also play a crucial role in the index theorem as presented here.

By definition we can take a smoothing operator to be an integral operator with smooth kernel:-

$$A : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M),\ Au(z) = \int_M A(z, z')u(z'),\ A \in \mathcal{C}^\infty(M^2; \pi_R^*\Omega). \tag{3.53}$$

Here $\pi :_R M^2 \ni (z, z') \longmapsto z' \in M$ is the 'right' projection. Thus A, the kernel (where we use the same letter for kernel and operator because they determine each other and so to use a separate notation is rather wasteful) is just a smooth function on M^2 which 'carries along with it' a smooth denisty on the right factor of M. If one prefers to do so, one can simply choose a positive denisty $0 < \nu \in \mathcal{C}^\infty(M; \Omega)$

and then the kernel becomes $A = A'\nu(z')$ where $A' \in \mathcal{C}^\infty(M^2)$. I prefer the more invariant approach of hiding the density in the kernel.

PROPOSITION 3.6. *The smoothing operators on a compact manifold form an algebra, denoted $\Psi^{-\infty}(M)$, under operator composition.*

PROOF. Indeed if A and B are smoothing operators on M with kernels having the same names then, by Fubini's theorem,

(3.54)
$$(AB)u(x) = A(Bu)(z) = \int_M A(z,z'')(Bu)(z'') = \int_M A(z,z'')\int_M B(z'',z')u(z')Mso$$
$$(AB)(z,z') = \int_M A(z,z'')B(z'',z').$$

Thus this formula defines an associative algebra structure (because composition of operators is associative) on $\Psi^{-\infty}(M) = \mathcal{C}^\infty(M^2;\pi_R^*\Omega)$ as claimed. □

A moments thought will show that this argument, and the composition law , carry over perfectly well to any compact manifold with corners. This more general case is interesting in part because of the subalgebras (but not ideals) that then arise in $\Psi^{-\infty}(M)$.

PROPOSITION 3.7. *If M is a compact manifold with corners and $H \subset M$ is a boundary face then the subspace of $\Psi^{-\infty}(M)$ consisting of kernels which vanish to order k at $H \times M$ and $M \times H$ is a subalgebra.*

The case of $k = \infty$ and $H = \partial\mathbb{B}^n$ for a ball is of particular interest since if the ball is interpreted as the radial compactification $\overline{\mathbb{R}^n}$ of $\mathbb{R}^n$, then

$$\Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n) = \{A \in \Psi^{-\infty}(\overline{\mathbb{R}^n}); A \equiv 0 \text{ at } (\partial\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}) \cup (\overline{\mathbb{R}^n} \times \partial\overline{\mathbb{R}^n}).\} \tag{3.55}$$

Here $\equiv$ stands for equality in Taylor series.

PROBLEM 3.2. Prove the equality in (3.55). Let me use the notation

$$\dot{\mathcal{C}}^\infty(M) = \{u \in \mathcal{C}^\infty(M); u \equiv 0 \text{ at } \partial M\} \subset \mathcal{C}^\infty(M)$$

for the space of smooth functions on a manifold with corners which vanish to infinite order at each boundary point. Then the identity (3.55) becomes

$$\dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^n} \times \overline{\mathbb{R}^n}) = \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n})$$

under radial compactification. First check the single space version

$$\dot{\mathcal{C}}^\infty(\overline{\mathbb{R}^n}) = \mathcal{S}(\mathbb{R}^n) \tag{3.56}$$

and then generalize (or use a clever argument) to pass to (3.2).

We remark on some related simple properties of smoothing operators. If $U \subset M$ is a coordinate neighbourhood, with coordinate map $F : U \longrightarrow U' \subset \mathbb{R}^n$ and ψ, $\psi' \in \mathcal{C}^\infty(M)$ has $\operatorname{supp}(\psi) \cup \operatorname{supp}(\psi') \subset U$ then

$$\begin{gathered} A_{\psi,\psi',F} : \mathcal{S}(\mathbb{R}^n) \ni f \longmapsto (F^{-1})^*\left(\psi A(F^*((F^{-1})^*\psi' \cdot f))\right) \in \mathcal{S}(\mathbb{R}^n) \\ \text{is an element of } \Psi^{-\infty}_{\text{iso}}(\mathbb{R}^n). \end{gathered} \tag{3.57}$$

Indeed, the kernel of $A_{\psi,F}$ is

(3.58)
$$(F^{-1})^*\psi(z)((F^{-1})^*\times(F^{-1})^*A)(z,z')(F^{-1})^*\psi'(z') = B|dz'|,\ B \in \mathcal{C}^\infty_c(\mathbb{R}^{2n}) \subset \mathcal{S}(\mathbb{R}^{2n}).$$

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Extension of the results above for the residual isotropic algebra on Euclidean space to smoothing operators on compact manifolds.

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3.12. Semiclassical limit algebra

Now we next want to extend the discussion of semiclassical smoothing operators on $\mathbb{R}^n$, in §2.19 , to smoothing operators on compact manifolds; later we will extend this to pseudodifferential operators. Initially at least let M be a compact manifold without boundary. Let $\Delta \subset M^2$ be the diagonal,

$$\Delta = \{(z,z) \in M^2; z \in M\}. \tag{3.59}$$

Definition 3.1. *An element of $\Psi_{\mathrm{sl}}^{-\infty}(M)$, the space of semiclassical families of smoothing operators on a compact manifold (without boundary) M, is a smooth family of smoothing operators $A_\epsilon \in \mathcal{C}^\infty((0,1] \times M^2; \pi_L^*\Omega)$ such that as $\epsilon \downarrow 0$ the kernel satisfies the two conditions:*

$$\begin{gathered} A_\epsilon \phi(z,z') \in \dot{\mathcal{C}}^\infty([0,1) \times M^2; \pi_R^*\Omega) \ \textit{if}\ \phi \in \mathcal{C}^\infty(M^2),\ \operatorname{supp}(\phi) \cap \Delta = \emptyset. \\ \textit{For a covering of } M \textit{ by coordinate systems } F_j : U_j \longrightarrow U_j' \\ \textit{and any elements } \psi_j, \psi_j' \in \mathcal{C}^\infty(M),\ \operatorname{supp}(\psi) \cup \operatorname{supp}(\psi_j') \subset U_j, \\ (A_\epsilon)_{\psi_j,\psi_j',F_j} \in \Psi_{\mathrm{sl}}^{-\infty}(\mathbb{R}^n). \end{gathered} \tag{3.60}$$

This is just supposed to say that $A_\epsilon \in \Psi_{\mathrm{sl}}^{-\infty}(M)$ reduces to a semiclassical family on $\mathbb{R}^n$ in local coordinates. We do not really need quite as much as in the second part of the defintion, which involves *all* pairs of smooth functions ψ_j, ψ_j' with compact support in a covering by coordinate patches. There is an equivalent and more geometric characterizations of the kernels of these semiclassical families below.

For the moment we note the following more useful description of the local behaviour of these operators.

Proposition 3.8. *On a compact manifold M,*

$$\{A \in \mathcal{C}^\infty([0,1]_\epsilon \times M^2; \pi_L^*\Omega); A \equiv 0 \ \textit{at}\ \{\epsilon = 0\}\} \subset \Psi_{\mathrm{sl}}^{-\infty}(M). \tag{3.61}$$

If $F : U \longrightarrow U' \in \mathbb{R}^n$ is a coordinate patch on M and $A \in \Psi_{\mathrm{sl}}^{-\infty}(\mathbb{R}^n)$ has kernel with support in $[0,1]_\epsilon \times K \times K$, $K \subset U'$ compact then

$$\begin{gathered} A_F \in \Psi_{\mathrm{sl}}^{-\infty}(M) \ \textit{where} \\ (A_F)_\epsilon : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M),\ (A_F u) = F^*(A(F^{-1})^* u). \end{gathered} \tag{3.62}$$

Moreover any element of $\Psi_{\mathrm{sl}}^{-\infty}(M)$ is the sum of a family of the first type and a finite sum, over any covering by coordinate patches, of operators as in (3.62)*.*

Proof. For the moment, see the proof of the corresponding theorem for pseudodifferential operators, Lemmas 6.1 and 6.2. The present result is is a bit easier; I will move the proof here and change it a bit. □

We can capture the 'semiclassical symbol' by oscillatory testing.

Lemma 3.5. *If $A_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M)$ then there exists a function $\sigma_{\mathrm{sl}}(A_\epsilon) \in \mathcal{S}(T^*M)$ such that whenever $f : M \longrightarrow \mathbb{R}$ and $\psi \in \mathcal{C}^\infty(M)$ are such that $df \neq 0$ on $\operatorname{supp}(\psi)$ then*

$$A_\epsilon e^{-if/\epsilon}\psi = e^{-if/\epsilon}b,\ b \in \mathcal{C}^\infty([0,1]\times M),\ b\big|_{\epsilon=0} = \sigma_{\mathrm{sl}}(A_\epsilon)\circ df. \tag{3.63}$$

I need to define $\mathcal{S}(T^*M)$ first!

Proof. Do the local, Euclidean, and then patch. $\square$

Proposition 3.9. *The semiclassical symbol of an element of $\Psi^{-\infty}_{\mathrm{sl}}(M)$ is determined by* (3.63) *and gives a short exact, multiplicative, sequence*

$$0 \longrightarrow \epsilon\Psi^{-\infty}_{\mathrm{sl}}(M) \longrightarrow \Psi^{-\infty}_{\mathrm{sl}}(M) \longrightarrow \mathcal{S}(T^*M) \longrightarrow 0 \tag{3.64}$$

Later, after discussing pseudodifferential operators on manifolds, we will also discuss semiclassical families of pseudodifferential operators, generalizing the discussion here. However there is one case which is very elementary. Namely the identity operator can be considered as a semiclassical family, even though it is independent of the parameter ϵ. By *fiat* its semiclassical symbol is declared to be the constant function 1 on the cotangent bundle. This is consistent with the multiplicativity of the semiclassical symbol, since of course for any family $A_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M)$,

$$\sigma_{\mathrm{sl}}(A_\epsilon) = \sigma_{\mathrm{sl}}(\operatorname{Id}\circ A_\epsilon) = 1 \times \sigma_{\mathrm{sl}}(A_\epsilon). \tag{3.65}$$

We can also immediately allow the algebra $\Psi^{-\infty}_{\mathrm{sl}}(M)$ to be 'valued in matrices', just by taking matrices of operators; we will denote this algebra as $\Psi^{-\infty}_{\mathrm{sl}}(M;\mathbb{C}^N)$ since the act on N-vectors of smooth functions on M. The symbol is then also valued in matrices.

Proposition 3.10. *If $a \in \mathcal{S}(T^*M; M(N,\mathbb{C}))$ is such that $\operatorname{Id}_{N\times N} - a$ is invertible at every point of T^*M then any semiclassical family $A_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M;\mathbb{C}^N)$ with $\sigma_{\mathrm{sl}}(A_\epsilon) = a$ is such that $\operatorname{Id} - A_\epsilon$ is invertible for small $\epsilon > 0$ with inverse of the form $\operatorname{Id} - B_\epsilon$ for some $B_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M;\mathbb{C}^N)$.*

3.13. Submanifolds and blow up

A brief description of blow up of a submanifold, enough to introduce the semiclassical resolution of $[0,1]\times M^2$ in the next section.

3.14. Resolution of semiclassical kernels

3.15. Quantization of projections

Proposition 3.11. *If $a \in \mathcal{S}(T^*M; M(N,\mathbb{C}))$ is such that for a constant projection $\pi_0 \in M(N,\mathbb{C})$, i.e. such that $\pi_0^2 = \pi_0$, $\pi_0 + a$ is a smooth family of projections, $(\pi_0 + a)^2 = \pi_0 + a$ then there exists a semiclassical family $A_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M;\mathbb{C}^N)$ such that $\sigma_{\mathrm{sl}}(A_\epsilon) = a$ and such that*

$$(\pi_0 + A_\epsilon)^2 = \pi_0 + A_\epsilon \tag{3.66}$$

is a semiclassical family of projections.

Proof. Just 'quantizing' a by choosing a semiclassical family $A'_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M;\mathbb{C}^n)$ with $\sigma_{\mathrm{sl}}(A'_\epsilon) = a$ ensures that

$$(\pi_0 + A'_\epsilon)^2 - (\pi_o + A'_\epsilon) = \epsilon E^{(1)}_\epsilon,\ E^{(1)}_\epsilon \in \Psi^{-\infty}_{\mathrm{sl}}(M;\mathbb{C}^N). \tag{3.67}$$

We proceed to show, inductively, that there is a series of 'correction terms' $A^{(j)} \in \Psi_{\mathrm{sl}}^{-\infty}(M;\mathbb{C}^N)$ such that for all l,
(3.68)

$$(\pi_0 + A'_\epsilon + \sum_{k=1}^{l} \epsilon^k A_\epsilon^{(k)})^2 - (\pi_o + A'_\epsilon \sum_{k=1}^{l} \epsilon^k A_\epsilon^{(k)}) = \epsilon^{l+1} E_\epsilon^{(l+1)},\ E_\epsilon^{(l)} \in \Psi_{\mathrm{sl}}^{-\infty}(M;\mathbb{C}^N).$$

Composing on the left and on the right with $\pi_o + A'_\epsilon \sum\limits_{k=1}^{l} \epsilon^k A_\epsilon^{(k)}$ and using the associativity of the product it follows that

$$\pi_0 \sigma_{\mathrm{sl}}(E_\epsilon^{(l+1)}) = \sigma_{\mathrm{sl}}(E_\epsilon^{(l+1)})\pi_0. \tag{3.69}$$

This in turn means that if $A_\epsilon^{(l+1)} \in \Psi_{\mathrm{sl}}^{-\infty}(M;\mathbb{C}^N)$ satisfies

$$\sigma_{\mathrm{sl}}(A_\epsilon^{(l+1)}) = (2\pi_0 - \mathrm{Id})\sigma_{\mathrm{sl}}(E_\epsilon^{(l+1)}) \tag{3.70}$$

then the next identity, (3.68), for $l+1$, holds.

Now, if A''_ϵ is an asymptotic sum of the series then

$$(\pi_0 + A''_\epsilon)^2 - \pi_0 + A''_\epsilon \in \{A \in \mathcal{C}^\infty([0,1];\Psi^{-\infty}(M;\mathbb{C}^N); A \equiv 0 \text{ at } \{\epsilon = 0\}\}. \tag{3.71}$$

To correct this family of 'projections to infinite order' $P'_\epsilon = \pi_0 + A''_\epsilon$ to a true projection we may use the holomorphic calculus of smoothing operators. Thus, the family

$$Q(s) = s^{-1}(\mathrm{Id} - P') + (s-1)^{-1}P',\ s \in \mathbb{C} \setminus \{0,1\} \tag{3.72}$$

satisfies the 'resolvent identity' to infinite order in ϵ :
(3.73)

$$(s\,\mathrm{Id} - P')Q(s) = (s(\mathrm{Id} - P') - (1-s)P')\,(Q(s) =$$
$$(\mathrm{Id} - P')^2 + (P')^2 + s^{-1}(s-1)(\mathrm{Id} - P')P' + (s-1)^{-1}sP'(\mathrm{Id} - P') = \mathrm{Id} + R(s)$$

where $R_\epsilon(s)$ is a family of smoothing operators vanishing to infinite order at $\epsilon = 0$ and depending holomorphically on $s \in \mathbb{C} \setminus \{0,1\}$. Thus in any region $|s| \ge \delta$, $|1-s| \ge \delta$, that is away from $s = 0$ and $s = 1$, $R(s)$ has uniformly small norm as $\epsilon \to 0$. It follows that $(\mathrm{Id} + R(s))^{-1} = \mathrm{Id} + M(s)$ exists in this region, for $\epsilon > 0$ small, and $M(s)$ is a holomorphic family of smoothing operators vanishing to infinite order at $\epsilon = 0$.

Thus the resolvent exists in this region and

$$(s\,\mathrm{Id} - P')^{-1} = Q(s) + M'(s) \tag{3.74}$$

where $M'(s)$ is another holomorphic family of smoothing operators vanishing to infinite order at $\epsilon = 0$.

To 'correct' P' to a family of projections we simply define

$$P = \frac{1}{2\pi i} \oint_{|1-s| = \frac{1}{2}} (s - P'(s))^{-1} ds. \tag{3.75}$$

From the decomposition (3.74) and (3.72) we see immediately that

$$P = P' + M,\ M = \frac{1}{2\pi i} \oint_{|1-s| = \frac{1}{2}} M(s) ds \in \epsilon^\infty \Psi_{\mathrm{sl}}^{-\infty}(M). \tag{3.76}$$

Moreover it follows from (3.75) that P is a projection. First, using Cauchy's theorem, we can shift the contour away from $s = 1$ a little, to $|s-1| = \gamma$ for some $\gamma > 0$, small. Then

$$P^2 = \frac{1}{2\pi i} \oint_{|1-s|=\frac{1}{2}} \frac{1}{2\pi i} \oint_{|1-t|=\frac{1}{2}+\gamma} (t - P'(t))^{-1}(s - P'(s))^{-1} ds dt. \tag{3.77}$$

The resolvent identity

$$(t - P'(t))^{-1}(s - P'(s))^{-1} = (s-t)^{-1}\left((t - P'(t))^{-1} - (s - P'(s))^{-1}\right) \tag{3.78}$$

allows the integral to be split into two. In the first double integral there are no singularities in s within $|1-s| \le \frac{1}{2}$ since $|1-t| = \frac{1}{2} + \gamma$, so by Cauchy's theorem this evaluates to zero. In the remaining term the t integral can be evaluated by residues, with the only singular point being at $t = s$ so

$$\begin{aligned} P^2 &= -\frac{1}{2\pi i} \oint_{|1-s|=\frac{1}{2}} \frac{1}{2\pi i} \oint_{|1-t|=\frac{1}{2}+\gamma} (s-t)^{-1}(s - P'(s))^{-1} ds dt \\ &= \frac{1}{2\pi i} \oint_{|1-s|=\frac{1}{2}} (s - P'(s))^{-1} ds = P. \end{aligned} \tag{3.79}$$

Thus P is a semiclassical quantization of the projection-valued symbol to a family of projections. $\square$

We will show below that this same argument works in other contexts.